Linear Partial Differential Equations and Fourier Theory

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Colophon

All text was prepared using Leslie Lamport's LaTeX2e typesetting language[1], and written using Richard Stallman's EMACS editor[2]. Pictures were generated using William Chia-Wei Cheng's excellent TGIF object-oriented drawing program[3]. Plots were generated using Waterloo MAPLE[4]. Biographical information and portraits were obtained from the *Mathematical Biographies Index*[5] of the School of Mathematics and Statistics at the University of St Andrews, Scotland. Additional image manipulation and post-processing was done with GNU Image Manipulation Program (GIMP)[6].

This book was prepared entirely on the Linux operating system (initially RedHat[7], and later Ubuntu[8]).

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This is a work in progress. Updates and improvements are available at the author's website:

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[1] http://www.latex-project.org
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[6] http://www.gimp.org
[7] http://www.redhat.com
[8] http://www.ubuntu.com

Contents

Preface

These lecture notes are written with four principles in mind:

1. **You learn by doing, not by watching.** Because of this, most of the routine or technical aspects of proofs and examples have been left as `Practice Problems` (which have solutions at the back of the book) and as `Exercises` (which don't). Most of these really aren't that hard; indeed, it often actually *easier* to figure them out yourself than to pick through the details of someone else's explanation. It is also more fun. And it is definitely a better way to learn the material. I suggest you do as many of these exercises as you can. Don't cheat yourself.

2. **Pictures often communicate better than words.** Differential equations is a geometrically and physically motivated subject. Algebraic formulae are just a language used to communicate visual/physical ideas in lieu of pictures, and they generally make a poor substitute. I've included as many pictures as possible, and I suggest that you look at the pictures *before* trying to figure out the formulae; often, once you understand the picture, the formula becomes transparent.

3. **Learning proceeds from the concrete to the abstract.** Thus, I begin each discussion with a specific example or a low-dimensional formulation, and only later proceed to a more general/abstract idea. This introduces a lot of "redundancy" into the text, in the sense that later formulations subsume the earlier ones. So the exposition is not as "efficient" as it could be. This is a good thing. Efficiency makes for good reference books, but lousy texts.

4. **Make it simple, but not stupid.** Most mathematical ideas are really pretty intuitive, if properly presented, but are incomprehensible if they are poorly explained. The clever short cuts and high-density notation of professional mathematicians are quite confusing to the novice, and can make simple and natural ideas seem complicated and technical. Because of this, I've tried to explain things as clearly, completely, and simply as possible. This adds some bulk to the text, but will save many hours of confusion for the reader.

   However, I have avoided the pitfall of many applied mathematics texts, which equate 'simple' with 'stupid'. These books suppress important 'technical' details (e.g. the distinction between different forms of convergence, or the reason why one can formally differentiate an infinite series) because they are worried these things will 'confuse' students. In fact, this kind of pedagogical dishonesty has the opposite effect. Students end up *more* confused, because they know something is fishy, but they can't tell quite what. Smarter students know they are being misled, and quickly lose respect for the book, the instructor, or even the whole subject.

   Likewise, many books systematically avoid any 'abstract' perspective (e.g. eigenfunctions of linear differential operators on an infinite-dimensional function space), and instead present a smorgasbord of seemingly *ad hoc* solutions to special cases. This also cheats the student. Abstractions aren't there to make things esoteric and generate employment opportunities for pure mathematicians. The right abstractions provide simple yet powerful tools which help students understand a myriad of seemingly disparate special casss within a single unifying framework.

I have enough respect for the student to explicitly confront the technical issues and to introduce relevant abstractions. The important thing is to always connect these abstractions and technicalities back to the physical intuititions which drive the subject. I've also included "optional" sections (indicated with a "(∗)"), which aren't necessary to the main flow of ideas, but which may be interesting to some more advanced readers.

Suggested Twelve-Week Syllabus

Week 1: *Heat and Diffusion-related PDEs*

Lecture 1: §1.1-§1.2; §1.4 *Background*

Lecture 2: §2.1-§2.2 *Fourier's Law; The Heat Equation*

No seminar

Lecture 3: §2.3-§2.4 *Laplace Equation; Poisson's Equation*

Week 2: *Wave-related PDEs; Quantum Mechanics*

Lecture 1: §3.1 *Spherical Means*

Lecture 2: §3.2-§3.3 *Wave Equation; Telegraph equation*

Seminar: **Quiz #1** based on practice problems in §2.5.

Lecture 3: §4.1-§4.4 *The Schrödinger equation in quantum mechanics*

Week 3: *General Theory*

Lecture 1: §5.1 - §5.3 *Linear PDEs: homogeneous vs. nonhomogeneous*

Lecture 2: §6.1; §6.4, *Evolution equations & Initial Value Problems*

Seminar: **Quiz #2** based on practice problems in §3.4 and §4.7.

Lecture 3: §6.5(a) *Dirichlet Boundary conditions*; §6.5(b) *Neumann Boundary conditions*

Week 4: *Background to Fourier Theory*

Lecture 1: §6.5(b) *Neumann Boundary conditions*; §6.6 *Uniqueness of solutions*; §7.1 *Inner products*

Lecture 2: §7.2-§7.3 $\mathbf{L}^2$ *space; Orthogonality*

Seminar: **Quiz #3** based on practice problems in §5.4, 6.3 and §6.7.

Lecture 3: §7.4(a,b,c) $\mathbf{L}^2$ *convergence; Pointwise convergence; Uniform Convergence*

Week 5: *One-dimensional Fourier Series*

Lecture 1: §7.4(d) *Infinite Series*; §7.5 *Orthogonal bases*
§8.1 *Fourier (co/sine) Series: Definition and examples*

Lecture 2: §8.3(a,b,c,d,e) *Computing Fourier series of polynomials, piecewise linear and step functions*

Seminar: *Review/flex time*

Lecture 3: §11.1 *Heat Equation on line segment*

Week 6: *Fourier Solutions for BVPs in One dimension*

Lecture 1: §11.3-§11.2 *Poisson problem on line segment; Wave Equation on line segment*

Seminar: **Quiz #4** based on practice problems in §7.7 and §8.4.

Lecture 2: §12.1 *Laplace Equation on a Square*

Lecture 3: §10.1-§10.2 *Multidimensional Fourier Series* Review/flex time.

Reading week

Week 7: *Fourier Solutions for BVPs in Two or more dimensions; Separation of Variables*

Lecture 1: §12.2, §12.3(a) *Heat Equation, Poisson problem on a square*

Seminar: **Midterm Exam** based on problems in §2.5, §3.4, §4.7, §5.4, §6.3, §6.7, §7.7, §8.4, and §11.4.

Lecture 2: §12.3(b), §12.4 *Poisson problem and Wave equation on a square*

Lecture 3: §15.1-§15.2 *Separation of Variables —introduction*

Week 8: *BVP's in Polar Coordinates*

Lecture 1: §6.5(d); §9.1-§9.2 *Periodic Boundary Conditions; Real Fourier Series*

Seminar: **Quiz #5** based on practice problems in §10.3, §11.4 and §12.5.

Lecture 2: §14.1; §14.2(a,b,c,d) *Laplacian in Polar coordinates; Laplace Equation on (co)Disk*

Lecture 3: §14.3 *Bessel Functions*

Week 9: *BVP's in Polar Coordinates; Separation of Variables*

Lecture 1: §14.4-§14.6 *Heat, Poisson, and Wave equations in Polar coordinates*

Lecture 2: Legendre Polynomials; separation of variables; method of Frobenius (based on supplementary notes)

Seminar: **Quiz #6** based on practice problems in §12.5 and §14.9, and also Exercises 16.2 and 16.3(a,b) (p.239)

Lecture 3: §14.2(e); §16.1; §16.7; *Poisson Solution to Dirichlet problem on Disk; Impulse response functions; convolution*

Week 10: *Impulse Response Methods*

Lecture 1: §16.2 *Approximations of identity;*

Lecture 2: §16.3 *Gaussian Convolution Solution for Heat Equation*

Seminar: **Quiz #7** based on practice problems in §§14.9, and also Exercise 16.12 (p.258), Exercise 7.10 (p.111), and Exercise 7.16 (p.117).

Lecture 3: §16.4; §16.6 *Poisson Solution to Dirichlet problem on Half-plane; d'Alembert Solution to Wave Equation*

Week 11: *Fourier Transforms*

Lecture 1: §17.1 *One-dimensional Fourier Transforms*

Lecture 2: §17.2 *Properties of one-dimensional Fourier transform*

Seminar: **Quiz #8** based on practice problems in §16.8.

Lecture 3: §18.1 *Fourier Transform solution to Heat Equation*
§18.3 *Dirchlet problem on Half-plane*

Week 12: *Fourier Transform Solutions to PDEs*

Lecture 1: §17.3, §18.2(a) *Fourier Transform solution to Wave Equation*

Lecture 2: §18.2(b) *Poisson's Spherical Mean Solution; Huygen's Principle*

Seminar: **Quiz #9** based on practice problems in §17.6 and §18.6.

Lecture 3: Review/Flex time

The **Final Exam** will be based on *Practice Problems* in §2.5, §3.4, §4.7, §5.4, §6.3, §6.7, §7.7, §8.4, §10.3, §11.4, §12.5, §14.9, §16.8, §17.6 and §18.6.

I Motivating Examples & Major Applications

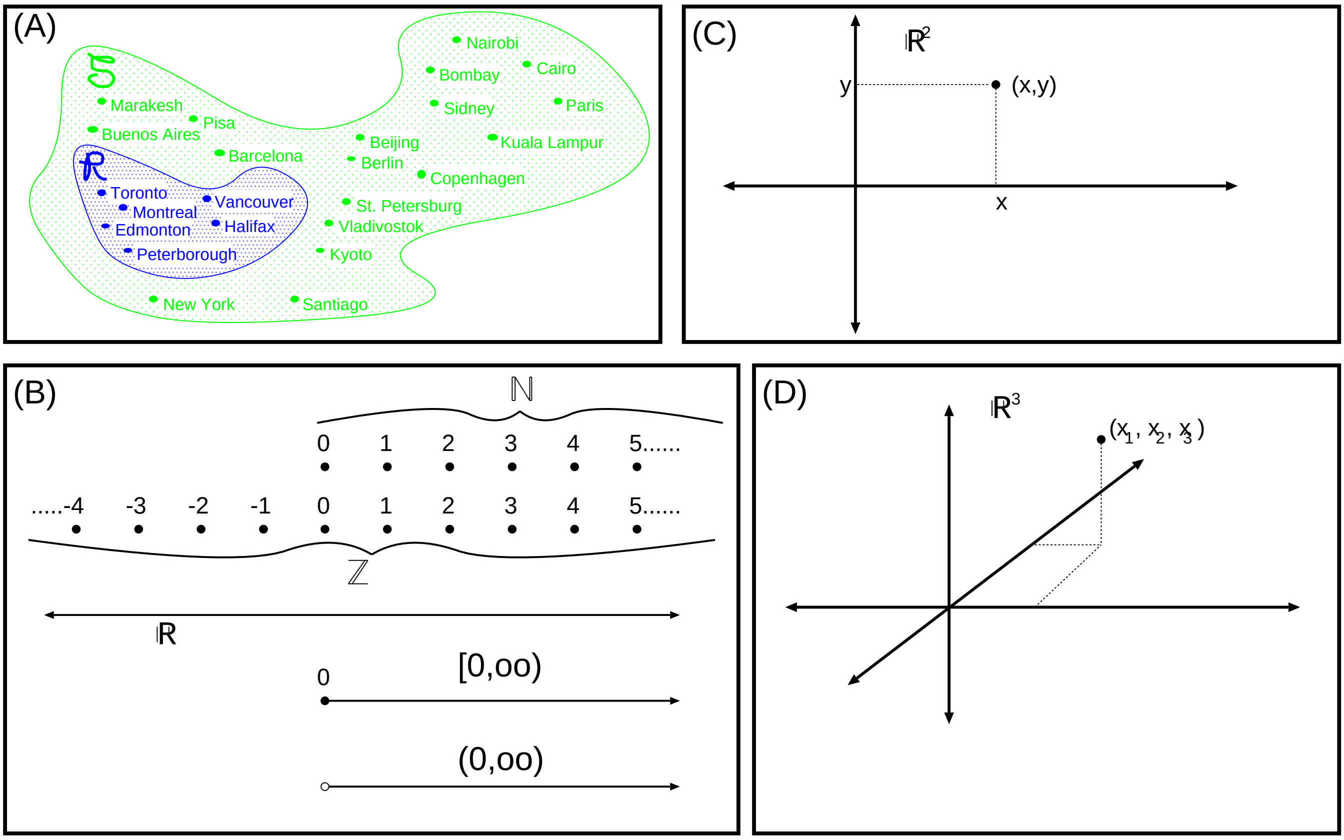


Figure 1.1: **(A)** $\mathcal{R}$ is a subset of $\mathcal{S}$ **(B)** Important Sets: $\mathbb{N}, \mathbb{Z}, \mathbb{R}, [0, \infty)$ and $(0, \infty)$. **(C)** $\mathbb{R}^2$ is two-dimensional space. **(D)** $\mathbb{R}^3$ is three-dimensional space.

1 Background

1.1 Sets and Functions

Sets: A **set** is a collection of objects. If $\mathcal{S}$ is a set, then the objects in $\mathcal{S}$ are called **elements** of $\mathcal{S}$; if s is an element of $\mathcal{S}$, we write "$s \in \mathcal{S}$. A **subset** of $\mathcal{S}$ is a smaller set $\mathcal{R}$ so that every element of $\mathcal{R}$ is also an element of $\mathcal{S}$. We indicate this by writing "$\mathcal{R} \subset \mathcal{S}$".

Sometimes we can explicitly list the elements in a set; we write "$\mathcal{S} = \{s_1, s_2, s_3, \ldots\}$".

Example 1.1:

(a) In Figure 1.1(A), $\mathcal{S}$ is the set of all cities in the world, so Peterborough $\in \mathcal{S}$. We might write $\mathcal{S} = \{$Peterborough, Toronto, Beijing, Kuala Lampur, Nairobi, Santiago, Pisa, Sidney, $\ldots\}$. If $\mathcal{R}$ is the set of all cities in Canada, then $\mathcal{R} \subset \mathcal{S}$.

(b) In Figure 1.1(B), the set of **natural numbers** is $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$. Thus, $5 \in \mathbb{N}$, but $\pi \notin \mathbb{N}$ and $-2 \notin \mathbb{N}$.

(c) In Figure 1.1(B), the set of **integers** is $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$. Thus, $5 \in \mathbb{Z}$ and $-2 \in \mathbb{Z}$, but $\pi \not\in \mathbb{Z}$ and $\frac{1}{2} \not\in \mathbb{Z}$. Observe that $\mathbb{N} \subset \mathbb{Z}$.

(d) In Figure 1.1(B), the set of **real numbers** is denoted $\mathbb{R}$. It is best visualised as an infinite line. Thus, $5 \in \mathbb{R}$, $-2 \in \mathbb{R}$, $\pi \in \mathbb{R}$ and $\frac{1}{2} \in \mathbb{R}$. Observe that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R}$.

(e) In Figure 1.1(B), the set of **nonnegative real numbers** is denoted $[0, \infty)$ It is best visualised as a half-infinite line, including zero. Observe that $[0, \infty) \subset \mathbb{R}$.

(f) In Figure 1.1(B), the set of **positive real numbers** is denoted $(0, \infty)$ It is best visualised as a half-infinite line, excluding zero. Observe that $(0, \infty) \subset [0, \infty) \subset \mathbb{R}$.

(g) Figure 1.1(C) depicts **two-dimensional space:** the set of all coordinate pairs (x, y), where x and y are real numbers. This set is denoted $\mathbb{R}^2$, and is best visualised as an infinite plane.

(h) Figure 1.1(D) depicts **three-dimensional space:** the set of all coordinate triples (x_1, x_2, x_3), where x_1, x_2, and x_3 are real numbers. This set is denoted $\mathbb{R}^3$, and is best visualised as an infinite void.

(i) If D is any natural number, then D**-dimensional space** is the set of all coordinate triples $(x_1, x_2, \ldots, x_D)$, where $x_1, \ldots, x_D$ are all real numbers. This set is denoted $\mathbb{R}^D$. It is hard to visualize when D is bigger than 3. ♢

Cartesian Products: If $\mathcal{S}$ and $\mathcal{T}$ are two sets, then their **Cartesian product** is the set of all pairs (s, t), where s is an element of $\mathcal{S}$, and t is an element of $\mathcal{T}$. We denote this set by $\mathcal{S} \times \mathcal{T}$.

Example 1.2:

(a) $\mathbb{R} \times \mathbb{R}$ is the set of all pairs (x, y), where x and y are real numbers. In other words, $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

(b) $\mathbb{R}^2 \times \mathbb{R}$ is the set of all pairs $(\mathbf{w}, z)$, where $\mathbf{w} \in \mathbb{R}^2$ and $z \in \mathbb{R}$. But if $\mathbf{w}$ is an element of $\mathbb{R}^2$, then $\mathbf{w} = (x, y)$ for some $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Thus, any element of $\mathbb{R}^2 \times \mathbb{R}$ is an object $\Big((x, y),\ z\Big)$. By suppressing the inner pair of brackets, we can write this as (x, y, z). In this way, we see that $\mathbb{R}^2 \times \mathbb{R}$ is the same as $\mathbb{R}^3$.

(c) In the same way, $\mathbb{R}^3 \times \mathbb{R}$ is the same as $\mathbb{R}^4$, once we write $\Big((x, y, z),\ t\Big)$ as (x, y, z, t). More generally, $\mathbb{R}^D \times \mathbb{R}$ is mathematically the same as $\mathbb{R}^{D+1}$.

Often, we use the final coordinate to store a 'time' variable, so it is useful to distinguish it, by writing $\Big((x, y, z),\ t\Big)$ as $(x, y, z;\ t)$. ♢

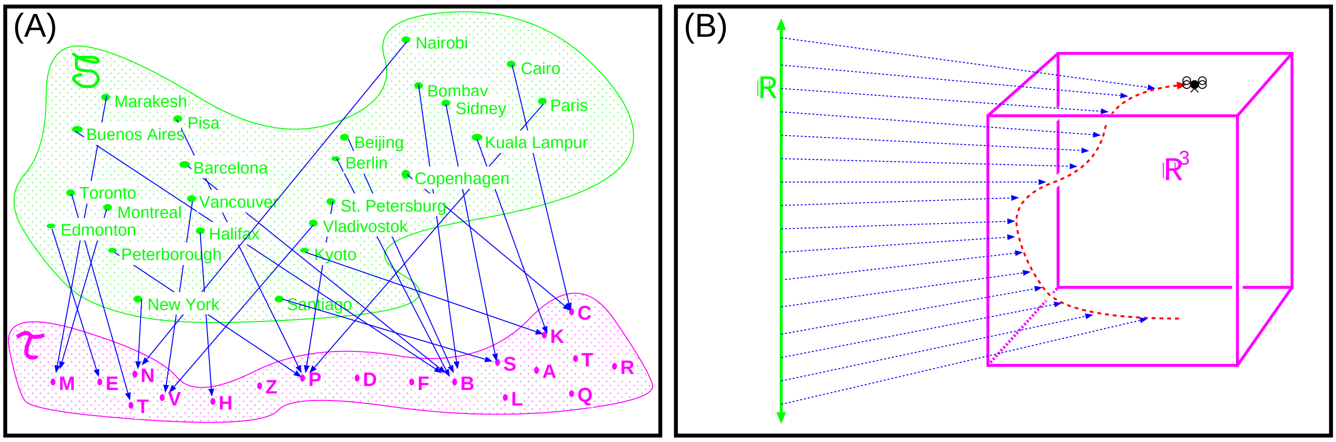


Figure 1.2: **(A)** $f(C)$ is the first letter of city C. **(B)** $\mathbf{p}(t)$ is the position of the fly at time t.

Functions: If $\mathcal{S}$ and $\mathcal{T}$ are sets, then a **function** from $\mathcal{S}$ to $\mathcal{T}$ is a rule which assigns a specific element of $\mathcal{T}$ to every element of $\mathcal{S}$. We indicate this by writing "$f : \mathcal{S} \longrightarrow \mathcal{T}$".

Example 1.3:

(a) In Figure 1.2(A), $\mathcal{S}$ is the cities in the world, and $\mathcal{T} = \{A, B, C, \ldots, Z\}$ is the letters of the alphabet, and f is the function which is the first letter in the name of each city. Thus $f(\text{Peterborough}) = P$, $f(\text{Santiago}) = S$, etc.

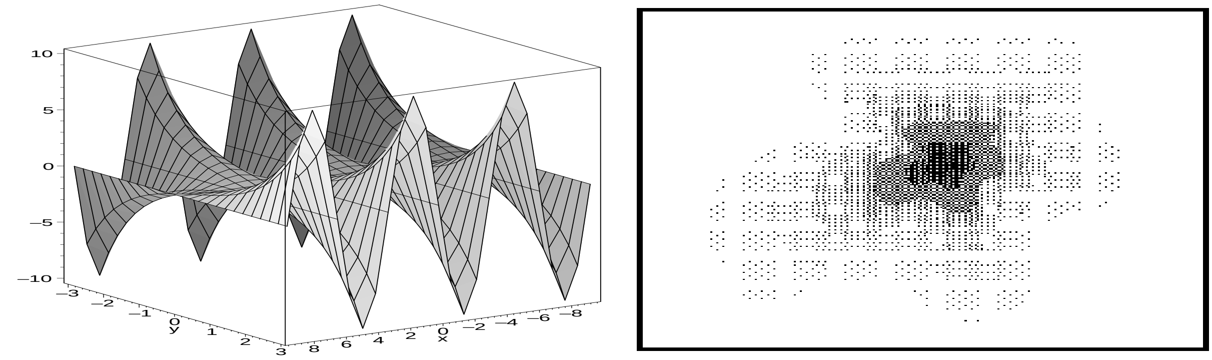
(b) if $\mathbb{R}$ is the set of real numbers, then $\sin : \mathbb{R} \longrightarrow \mathbb{R}$ is a function: $\sin(0) = 0$, $\sin(\pi/2) = 1$, etc. ◇

Two important classes of functions are **paths** and **fields**.

Paths: Imagine a fly buzzing around a room. Suppose you try to represent its trajectory as a curve through space. This defines a a function $\mathbf{p}$ from $\mathbb{R}$ into $\mathbb{R}^3$, where $\mathbb{R}$ represents *time*, and $\mathbb{R}^3$ represents the (three-dimensional) room, as shown in Figure 1.2(B). If $t \in \mathbb{R}$ is some moment in time, then $\mathbf{p}(t)$ is the position of the fly at time t. Since this $\mathbf{p}$ describes the path of the fly, we call $\mathbf{p}$ a **path**.

More generally, a **path** (or **trajectory** or **curve**) is a function $\mathbf{p} : \mathbb{R} \longrightarrow \mathbb{R}^D$, where D is any natural number. It describes the motion of an object through D-dimensional space. Thus, if $t \in \mathbb{R}$, then $\mathbf{p}(t)$ is the position of the object at time t.

Scalar Fields: Imagine a three-dimensional topographic map of Antarctica. The rugged surface of the map is obtained by assigning an altitude to every location on the continent. In other words, the map implicitly defines a function $\mathbf{h}$ from $\mathbb{R}^2$ (the Antarctic continent) to $\mathbb{R}$ (the set of altitudes, in metres above sea level). If $(x, y) \in \mathbb{R}^2$ is a location in Antarctica, then $\mathbf{h}(x, y)$ is the altitude at this location (and $\mathbf{h}(x, y) = 0$ means (x, y) is at sea level).



(A) **(B)**

Figure 1.3: **(A)** A height function describes a landscape. **(B)** A density distribution in $\mathbb{R}^2$.

This is an example of a **scalar field**. A scalar field is a function $u : \mathbb{R}^D \longrightarrow \mathbb{R}$; it assigns a numerical quantity to every point in D-dimensional space.

Example 1.4:

(a) In Figure 1.3(A), a landscape is represented by a **height function h** : $\mathbb{R}^2 \longrightarrow \mathbb{R}$.

(b) Figure 1.3(B) depicts a **concentration function** on a two-dimensional plane (eg. the concentration of bacteria on a petri dish). This is a function $\rho : \mathbb{R}^2 \longrightarrow [0, \infty)$ (where $\rho(x, y) = 0$ indicates zero bacteria at (x, y)).

(c) The **mass density** of a three-dimensional object is a function $\rho : \mathbb{R}^3 \longrightarrow [0, \infty)$ (where $\rho(x_1, x_2, x_3) = 0$ indicates vacuum).

(d) The **charge density** is a function $\mathbf{q} : \mathbb{R}^3 \longrightarrow \mathbb{R}$ (where $\mathbf{q}(x_1, x_2, x_3) = 0$ indicates electric neutrality)

(e) The **electric potential** (or **voltage**) is a function $\mathbf{V} : \mathbb{R}^3 \longrightarrow \mathbb{R}$.

(f) The **temperature distribution** in space is a function $u : \mathbb{R}^3 \longrightarrow \mathbb{R}$ (so $u(x_1, x_2, x_3)$ is the "temperature at location (x_1, x_2, x_3)") ♢

A **time-varying scalar field** is a function $u : \mathbb{R}^D \times \mathbb{R} \longrightarrow \mathbb{R}$, assigning a quantity to every point in space at each moment in time. Thus, for example, $u(\mathbf{x}; t)$ is the "temperature at location $\mathbf{x}$, at time t"

Vector Fields: A **vector field** is a function $\vec{V} : \mathbb{R}^D \longrightarrow \mathbb{R}^D$; it assigns a vector (ie. an "arrow") at every point in space.

Example 1.5:

(a) The **electric field** generated by a charge distribution (denoted $\vec{\mathbf{E}}$).

(b) The **flux** of some material flowing through space (often denoted $\vec{F}$). ◇

Thus, for example, $\vec{F}(\mathbf{x})$ is the "flux" of material at location $\mathbf{x}$.

1.2 Derivatives —Notation

If $f : \mathbb{R} \longrightarrow \mathbb{R}$, then f' is the first derivative of f; f'' is the second derivative,... $f^{(n)}$ the nth derivative, etc. If $\mathbf{x} : \mathbb{R} \longrightarrow \mathbb{R}^D$ is a path, then the **velocity** of $\mathbf{x}$ at time t is the vector

$$\dot{\mathbf{x}}(t) = \left[x_1'(t),\ x_2'(t),\ \ldots, x_D'(t)\right]$$

If $u : \mathbb{R}^D \longrightarrow \mathbb{R}$ is a scalar field, then the following notations will be used interchangeably:

$$\partial_d u \;=\; \frac{\partial u}{\partial x_d}$$

For example, if $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (ie. $u(x,y)$ is a function of two variables), then we have

$$\partial_x u \;=\; \frac{\partial u}{\partial x}; \qquad \partial_y u \;=\; \frac{\partial u}{\partial y};$$

Multiple derivatives will be indicated by iterating this procedure. For example,

$$\partial_x^3 \partial_y^2 \;=\; \frac{\partial^3}{\partial x^3} \frac{\partial^2 u}{\partial y^2}$$

Sometimes we will use **multiexponents**. If $\gamma_1, \ldots, \gamma_D$ are positive integers, and $\gamma = (\gamma_1, \ldots, \gamma_D)$, then

$$\mathbf{x}^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \ldots, x_D^{\gamma_D}$$

For example, if $\gamma = (3, 4)$, and $\mathbf{z} = (x, y)$ then $\mathbf{z}^\gamma = x^3 y^4$.

This generalizes to **multi-index** notation for derivatives. If $\gamma = (\gamma_1, \ldots, \gamma_D)$, then

$$\partial^\gamma u = \partial_1^{\gamma_1} \partial_2^{\gamma_2} \ldots \partial_D^{\gamma_D} u$$

For example, if $\gamma = (1, 2)$, then $\partial^\gamma u = \dfrac{\partial}{\partial x} \dfrac{\partial^2 u}{\partial y^2}$.

Remark: Many authors use subscripts to indicate partial derivatives. For example, they would write

$$u_x \;:=\; \partial_x u, \qquad u_{xx} \;:=\; \partial_x^2 u, \qquad u_{xy} \;:=\; \partial_x \partial_y u, \text{ etc.}$$

This notation is very compact and intuitive, but it has two major disadvantages:

1. When dealing with an N-dimensional function $u(x_1, x_2, \ldots, x_N)$ (where N is either large or indeterminate), you have only two options. You can either either use awkward 'nested subscript' expressions like

$$u_{x_3} \; := \; \partial_3 \, u, \qquad u_{x_5 x_5} \; := \; \partial_5^2 \, u, \qquad u_{x_2 x_3} \; := \; \partial_2 \partial_3 \, u, \text{ etc.,}$$

or you must adopt the 'numerical subscript' convention that

$$u_3 \; := \; \partial_3 \, u, \qquad u_{55} \; := \; \partial_5^2 \, u, \qquad u_{23} \; := \; \partial_2 \partial_3 \, u, \text{ etc.}$$

But once 'numerical' subscripts are reserved to indicate derivatives in this fashion, they can no longer be used for other purposes (e.g. indexing a sequence of functions, or indexing the coordinates of a vector-valued function). This can create further awkwardness.

2. We will often be considering functions of the form $u(x, y; t)$, where (x, y) are 'space' coordinates and t is a 'time' coordinate. In this situation, it is often convenient to fix a value of t and consider the two-dimensional scalar field $u_t(x, y) := u(x, y; t)$. Normally, when we use t as a subscript, it will be indicate a 'time-frozen' scalar field of this kind.

Thus, in this book, *we will never use subscripts to indicate partial derivatives.* Partial derivatives will always be indicated by the notation "$\partial_x u$" or "$\frac{\partial u}{\partial x}$" (almost always the first one). However, when consulting other texts, you should be aware of the 'subscript' notation for derivatives, because it is used quite frequently.

1.3 Complex Numbers

Complex numbers have the form $z = x + y\mathbf{i}$, where $\mathbf{i}^2 = -1$. We say that x is the **real part** of z, and y is the **imaginary part**; we write: $x = \mathbf{re}[z]$ and $y = \mathbf{im}[z]$.

If we imagine (x, y) as two real coordinates, then the complex numbers form a two-dimensional plane. Thus, we can also write a complex number in *polar coordinates* (see Figure 1.4) If $r > 0$ and $0 \leq \theta < 2\pi$, then we define

$$\boxed{r \, \mathtt{cis} \, \theta \; = \; r \cdot [\cos(\theta) + \mathbf{i} \sin(\theta)]}$$

Addition: If $z_1 = x_1 + y_1\mathbf{i}$, $z_2 = x_2 + y_2\mathbf{i}$, are two complex numbers, then $z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)\mathbf{i}$. (see Figure 1.5)

Multiplication: If $z_1 = x_1 + y_1\mathbf{i}$, $z_2 = x_2 + y_2\mathbf{i}$, are two complex numbers, then $z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)\,\mathbf{i}$.

Multiplication has a nice formulation in polar coordinates; If $z_1 = r_1 \, \mathtt{cis} \, \theta_1$ and $z_2 = r_2 \, \mathtt{cis} \, \theta_2$, then $z_1 \cdot z_2 \; = \; (r_1 \cdot r_2) \, \mathtt{cis} \, (\theta_1 + \theta_2)$. In other words, multiplication by the complex number $z = r \, \mathtt{cis} \, \theta$ is equivalent to *dilating* the complex plane by a factor of r, and *rotating* the plane by an angle of θ. (see Figure 1.6)

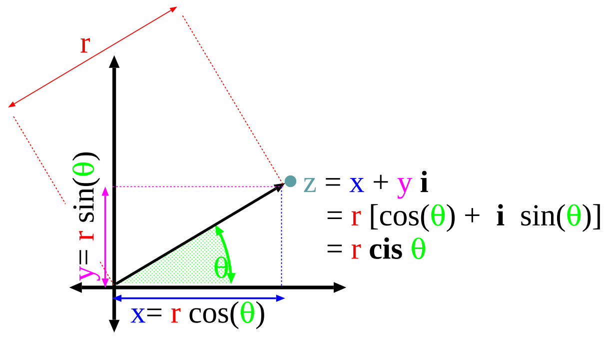


Figure 1.4: $z = x + y\mathbf{i}$; $r = \sqrt{x^2 + y^2}$, $\theta = \tan(y/x)$.

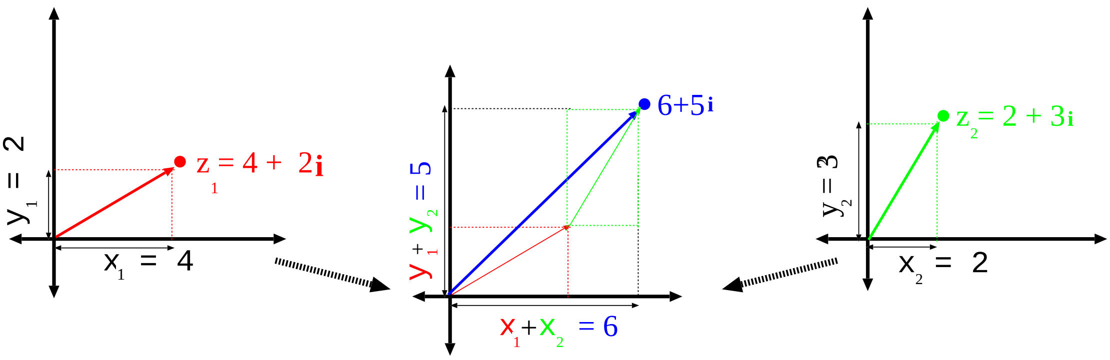


Figure 1.5: The addition of complex numbers $z_1 = x_1 + y_1\mathbf{i}$ and $z_2 = x_2 + y_2\mathbf{i}$.

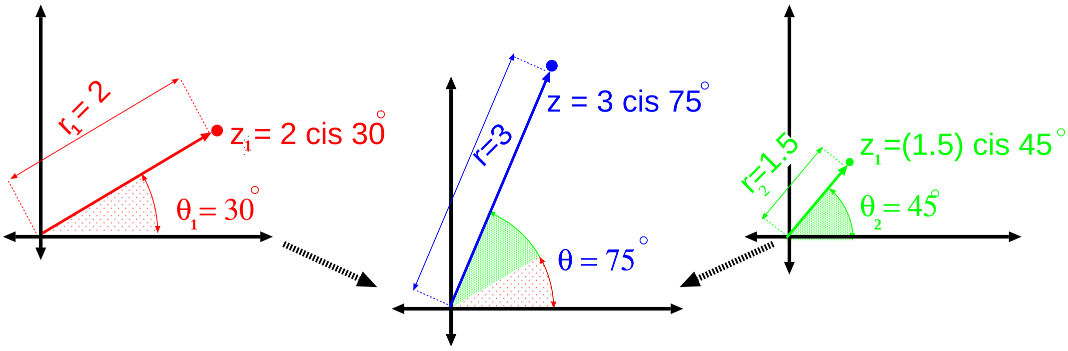


Figure 1.6: The multiplication of complex numbers $z_1 = r_1 \,\mathtt{cis}\, \theta_1$ and $z_2 = r_2 \,\mathtt{cis}\, \theta_2$.

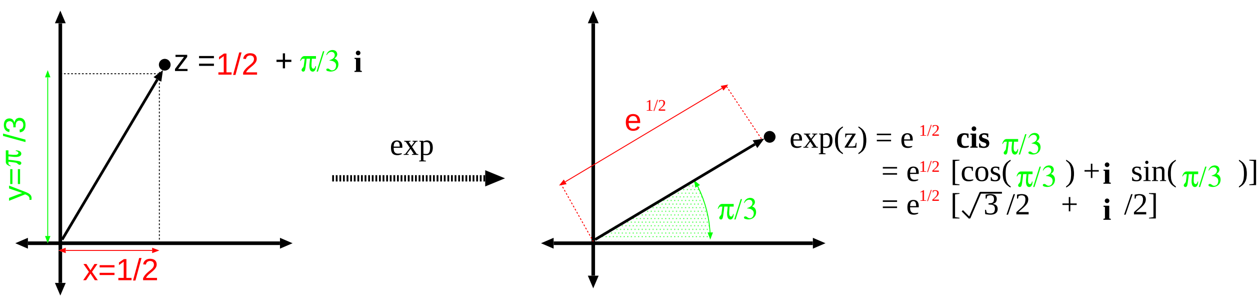


Figure 1.7: The exponential of complex number $z = x + y\mathbf{i}$.

Exponential: If $z = x + y\mathbf{i}$, then $\exp(z) \;=\; e^x \,\texttt{cis}\, y \;= e^x \cdot [\cos(y) + \mathbf{i}\sin(y)]$. (see Figure 1.7) In particular, if $x \in \mathbb{R}$, then

- $\exp(x) = e^x$ is the standard real-valued exponential function.
- $\exp(y\mathbf{i}) = \cos(y) + \sin(y)\mathbf{i}$ is a periodic function; as y moves along the real line, $\exp(y\mathbf{i})$ moves around the unit circle.

The complex exponential function shares two properties with the real exponential function:

- If $z_1, z_2 \in \mathbb{C}$, then $\exp(z_1 + z_2) = \exp(z_1) \cdot \exp(z_2)$.
- If $w \in \mathbb{C}$, and we define the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ by $f(z) = \exp(w \cdot z)$, then $f'(z) = w \cdot f(z)$.

Consequence: If $w_1, w_2, \ldots, w_D \in \mathbb{C}$, and we define $f : \mathbb{C}^D \longrightarrow \mathbb{C}$ by

$$f(z_1, \ldots, z_D) \quad = \quad \exp(w_1 z_1 + w_2 z_2 + \ldots w_D z_D),$$

then $\partial_d f(\mathbf{z}) = w_d \cdot f(\mathbf{z})$. More generally,

$$\partial_1^{n_1} \partial_2^{n_2} \ldots \partial_D^{n_D} f(\mathbf{z}) \quad = \quad w_1^{n_1} \cdot w_2^{n_2} \cdot \ldots w_D^{n_D} \cdot f(\mathbf{z}). \tag{1.1}$$

For example, if $f(x, y) = \exp(3x + 5\mathbf{i}y)$, then

$$f_{xxy}(x, y) \quad = \quad \partial_x^2 \partial_y f(x, y) \quad = \quad 45\,\mathbf{i} \cdot \exp(3x + 5\mathbf{i}y).$$

If $\mathbf{w} = (w_1, \ldots, w_D)$ and $\mathbf{z} = (z_1, \ldots, z_D)$, then we will sometimes write:

$$\exp(w_1 z_1 + w_2 z_2 + \ldots w_D z_D) \quad = \quad \exp \langle \mathbf{w}, \mathbf{z} \rangle .$$

Conjugation and Norm: If $z = x + y\mathbf{i}$, then the **complex conjugate** of z is $\overline{z} = x - y\mathbf{i}$. In polar coordinates, if $z = r\,\texttt{cis}\,\theta$, then $\overline{z} = r\,\texttt{cis}\,(-\theta)$.

The **norm** of z is $|z| = \sqrt{x^2 + y^2}$. We have the formula:

$$|z|^2 \; = \; z \cdot \overline{z}.$$

1.4 Vector Calculus

Prerequisites: §1.1, §1.2

1.4(a) Gradient

....in two dimensions:

Suppose $\mathbb{X} \subset \mathbb{R}^2$ was a two-dimensional region. To define the topography of a "landscape" on this region, it suffices[1] to specify the *height* of the land at each point. Let $u(x,y)$ be the height of the land at the point $(x,y) \in \mathbb{X}$. (Technically, we say: "$u : \mathbb{X} \longrightarrow \mathbb{R}$ is a *two-dimensional scalar field.*")

The **gradient** of the landscape measures the *slope* at each point in space. To be precise, we want the gradient to be an arrow pointing in the direction of *most rapid ascent.* The *length* of this arrow should then measure the *rate* of ascent. Mathematically, we define the **two-dimensional gradient** of u by:

$$\boxed{\nabla u(x,y) \quad = \quad \left[\frac{\partial u}{\partial x}(x,y), \quad \frac{\partial u}{\partial y}(x,y)\right]}$$

The gradient arrow points in the direction where u is increasing the most rapidly. If $u(x,y)$ was the height of a mountain at location (x,y), and you were trying to climb the mountain, then your (naive) strategy would be to always walk in the direction $\nabla u(x,y)$. Notice that, for any $(x,y) \in \mathbb{X}$, the gradient $\nabla u(x,y)$ is a two-dimensional vector —that is, $\nabla u(x,y) \in \mathbb{R}^2$. (Technically, we say "$\nabla u : \mathbb{X} \longrightarrow \mathbb{R}^2$ is a *two-dimensional vector field*".)

....in many dimensions:

This idea generalizes to any dimension. If $u : \mathbb{R}^D \longrightarrow \mathbb{R}$ is a scalar field, then the **gradient** of u is the associated vector field $\nabla u : \mathbb{R}^D \longrightarrow \mathbb{R}^D$, where, for any $\mathbf{x} \in \mathbb{R}^D$,

$$\boxed{\nabla u(\mathbf{x}) \quad = \quad \left[\partial_1 u,\ \partial_2 u, \ldots,\ \partial_D u\right](\mathbf{x})}$$

1.4(b) Divergence

....in one dimension:

Imagine a current of water flowing along the real line $\mathbb{R}$. For each point $x \in \mathbb{R}$, let $V(x)$ describe the rate at which water is flowing past this point. Now, in places where the water *slows down*, we expect the derivative $V'(x)$ to be negative. We also expect the water to *accumulate* at such locations (because water is entering the region more quickly than it leaves). In places where the water *speeds up*, we expect the derivative $V'(x)$ to be positive, and we expect the water to be *depleted* at such locations (because water is leaving the region more quickly than it arrives). Thus, if we define the **divergence** of the flow to be the *rate at which water is being depleted*, then mathematically speaking,

$$\boxed{\mathbf{div}\, V(x) \quad = \quad V'(x)}$$

[1] Assuming no overhangs!

....in two dimensions:

Let $\mathbb{X} \subset \mathbb{R}^2$ be some planar region, and consider a fluid flowing through $\mathbb{X}$. For each point $(x, y) \in \mathbb{X}$, let $\vec{V}(x, y)$ be a two-dimensional vector describing the current at that point[2].

Think of this two-dimensional current as a superposition of a *horizontal current* and a *vertical current*. For each of the two currents, we can reason as in the one-dimensional case. If the horizontal current is accelerating, we expect it to deplete the fluid at this location. If it is decelarating, we expect it to deposit fluid at this location. The divergence of the two-dimensional current is thus just the sum of the divergences of its one-dimensional components:

$$\boxed{\mathbf{div}\,\vec{V}(x, y) \quad = \quad \partial_x V_1(x, y) \quad + \quad \partial_y V_2(x, y)}$$

Notice that, although $\vec{V}(x, y)$ was a *vector*, the divergence $\mathbf{div}\,\vec{V}(x, y)$ is a *scalar*[3].

....in many dimensions:

We can generalize this idea to any number of dimensions. If $\vec{V} : \mathbb{R}^D \longrightarrow \mathbb{R}^D$ is a vector field, then the **divergence** of $\vec{V}$ is the associated scalar field $\mathbf{div}\,\vec{V} : \mathbb{R}^D \longrightarrow \mathbb{R}$, where, for any $\mathbf{x} \in \mathbb{R}^D$,

$$\boxed{\mathbf{div}\,\vec{V}(\mathbf{x}) \ = \ \partial_1 V_1(\mathbf{x}) + \partial_2 V_2(\mathbf{x}) + \ldots + \partial_D V_D(\mathbf{x})}$$

The divergence measures the rate at which $\vec{V}$ is "diverging" or "converging" near $\mathbf{x}$. For example

- If $\vec{F}$ is the flux of some material, then $\mathbf{div}\,\vec{F}(\mathbf{x})$ is the rate at which the material is "expanding" at $\mathbf{x}$.
- If $\vec{\mathbf{E}}$ is the electric field, then $\mathbf{div}\,\vec{\mathbf{E}}(\mathbf{x})$ is the amount of electric field being "generated" at $\mathbf{x}$ —that is, $\mathbf{div}\,\vec{\mathbf{E}}(\mathbf{x}) = \mathbf{q}(\mathbf{x})$ is the **charge density** at $\mathbf{x}$.

1.5 Even and Odd Functions

Prerequisites: §1.1

A function $f : [-L, L] \longrightarrow \mathbb{R}$ is **even** if $f(-x) = f(x)$ for all $x \in [0, L]$. For example, the following functions are even:

- $f(x) = 1$.
- $f(x) = |x|$.
- $f(x) = x^2$.
- $f(x) = x^k$ for any even $k \in \mathbb{N}$.

[2]Technically, we say "$\vec{V} : \mathbb{X} \longrightarrow \mathbb{R}^2$ is a *two-dimensional vector field*".
[3]Technically, we say "$\mathbf{div}\,\vec{V} : \mathbb{X} \longrightarrow \mathbb{R}^2$ is a *two-dimensional scalar field*"

- $f(x) = \cos(x)$.

A function $f : [-L, L] \longrightarrow \mathbb{R}$ is **odd** if $f(-x) = -f(x)$ for all $x \in [0, L]$. For example, the following functions are odd:

- $f(x) = x$.
- $f(x) = x^3$.
- $f(x) = x^k$ for any odd $k \in \mathbb{N}$.
- $f(x) = \sin(x)$.

Every function can be 'split' into an 'even part' and an 'odd part'.

Proposition 1.6: *For any $f : [-L, L] \longrightarrow \mathbb{R}$, there is a unique even function $\check{f}$ and a unique odd function $\acute{f}$ so that $f = \check{f} + \acute{f}$. To be specific:*

$$\check{f}(x) = \frac{f(x) + f(-x)}{2} \qquad \text{and} \qquad \acute{f}(x) = \frac{f(x) - f(-x)}{2}$$

Proof: **Exercise 1.1** Hint: Verify that $\check{f}$ is even, $\acute{f}$ is odd, and that $f = \check{f} + \acute{f}$. □

The equation $f = \check{f} + \acute{f}$ is called the **even-odd decomposition** of f.
Exercise 1.2 1. If f is *even*, show that $f = \check{f}$, and $\acute{f} = 0$.
2. If f is *odd*, show that $\check{f} = 0$, and $f = \acute{f}$.

If $f : [0, L] \longrightarrow \mathbb{R}$, then we can "extend" f to a function on $[-L, L]$ in two ways:

- The **even** extension of f is defined: $f_{\text{even}}(x) = f(|x|)$ for all $x \in [-L, L]$.
- The **odd** extension of f is defined: $f_{\text{odd}}(x) = \begin{cases} f(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -f(-x) & \text{if } x < 0 \end{cases}$

Exercise 1.3 Verify:
1. f_{even} is even, and f_{odd} is odd.
2. For all $x \in [0, L]$, $f_{\text{even}}(x) = f(x) = f_{\text{odd}}(x)$.

1.6 Coordinate Systems and Domains

Prerequisites: §1.1

Boundary Value Problems are usually posed on some "domain" —some region of space. To solve the problem, it helps to have a convenient way of mathematically representing these domains, which can sometimes be simplified by adopting a suitable coordinate system.

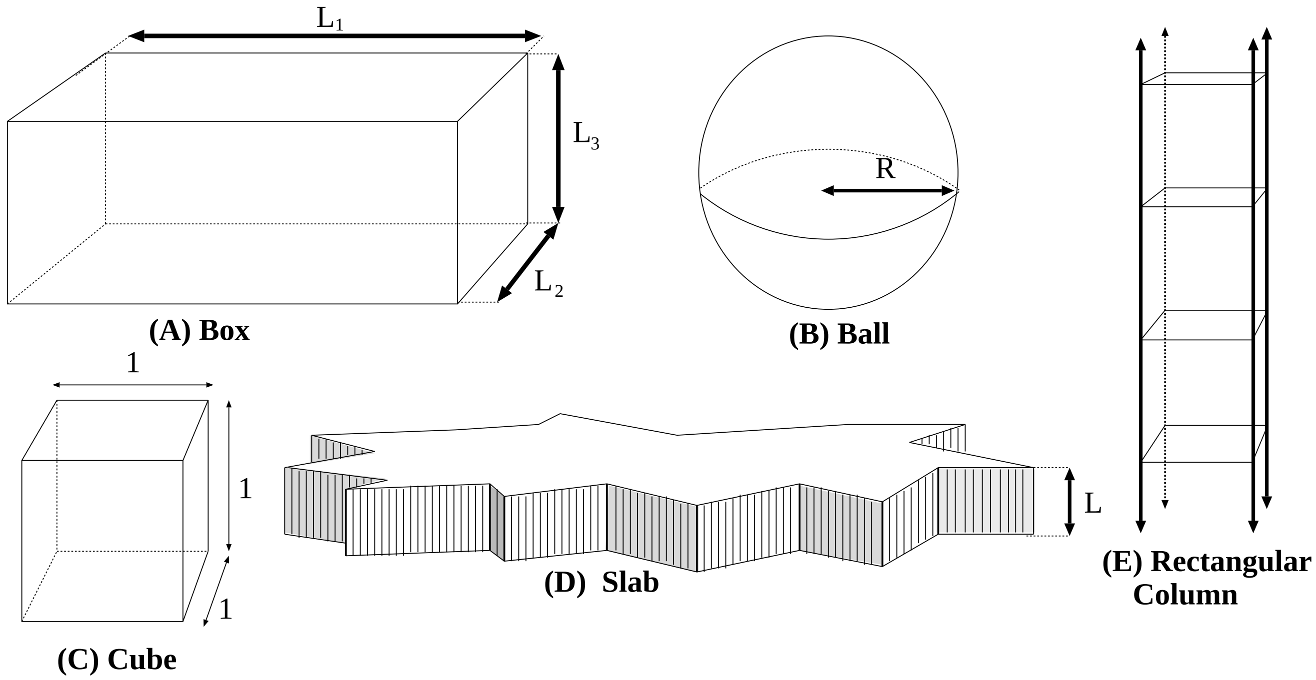


Figure 1.8: Some domains in $\mathbb{R}^3$.

1.6(a) Rectangular Coordinates

Rectangular coordinates in $\mathbb{R}^3$ are normally denoted (x, y, z). Three common domains in rectangular coordinates:

- The **slab** $\mathbb{X} = \left\{(x, y, z) \in \mathbb{R}^3 \; ; \; 0 \leq z \leq L\right\}$, where L is the thickness of the slab (see Figure 1.8D).
- The **unit cube:** $\mathbb{X} = \{(x, y, z) \in \mathbb{R}^3; \; 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad \text{and } 0 \leq z \leq 1\}$ (see Figure 1.8C).
- The **box:** $\mathbb{X} = \{(x, y, z) \in \mathbb{R}^3; \; 0 \leq x \leq L_1, \quad 0 \leq y \leq L_2, \quad \text{and } 0 \leq z \leq L_3\}$, where L_1, L_2, and L_3 are the sidelengths (see Figure 1.8A).
- The **rectangular column:** $\mathbb{X} = \left\{(x, y, z) \in \mathbb{R}^3 \; ; \; 0 \leq x \leq L_1 \text{ and } 0 \leq y \leq L_2\right\}$ (see Figure 1.8E).

1.6(b) Polar Coordinates on $\mathbb{R}^2$

Polar coordinates (r, θ) on $\mathbb{R}^2$ are defined by the transformation:

$$x = r \cdot \cos(\theta) \quad \text{and} \quad y = r \cdot \sin(\theta).$$

with reverse transformation:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan\left(\frac{y}{x}\right).$$

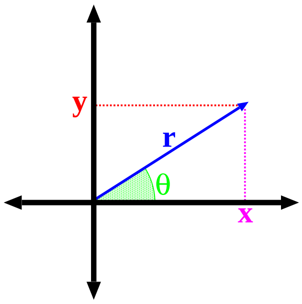


Figure 1.9: Polar coordinates

Here, the coordinate r ranges over $[0, \infty)$, while the variable θ ranges over $[-\pi, \pi)$. Three common domains in polar coordinates are:

- $\mathbb{D} = \{(r, \theta) \; ; \; r \leq R\}$ is the **disk** of radius R (see Figure 1.10A).
- $\mathbb{D}^{\complement} = \{(r, \theta) \; ; \; R \leq r\}$ is the **codisk** of inner radius R.
- $\mathbb{A} = \{(r, \theta) \; ; \; R_{\min} \leq r \leq R_{\max}\}$ is the **annulus**, of inner radius $R_{\min}$ and outer radius $R_{\max}$ (see Figure 1.10B).

1.6(c) Cylindrical Coordinates on $\mathbb{R}^3$

Cylindrical coordinates (r, θ, z) on $\mathbb{R}^3$, are defined by the transformation:

$$x = r \cdot \cos(\theta), \quad y = r \cdot \sin(\theta) \quad \text{and} \quad z = z$$

with reverse transformation:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{x}{y}\right) \quad \text{and} \quad z = z.$$

Five common domains in cylindrical coordinates are:

- $\mathbb{X} = \{(r, \theta, z) \; ; \; r \leq R\}$ is the **(infinite) cylinder** of radius R (see Figure 1.10E).
- $\mathbb{X} = \{(r, \theta, z) \; ; \; R_{\min} \leq r \leq R_{\max}\}$ is the **(infinite) pipe** of inner radius $R_{\min}$ and outer radius $R_{\max}$ (see Figure 1.10D).
- $\mathbb{X} = \{(r, \theta, z) \; ; \; r > R\}$ is the **wellshaft** of radius R.
- $\mathbb{X} = \{(r, \theta, z) \; ; \; r \leq R \text{ and } 0 \leq z \leq L\}$ is the **finite cylinder** of radius R and length L (see Figure 1.10C).
- In cylindrical coordinates on $\mathbb{R}^3$, we can write the **slab** as $\{(r, \theta, z) \; ; \; 0 \leq z \leq L\}$.

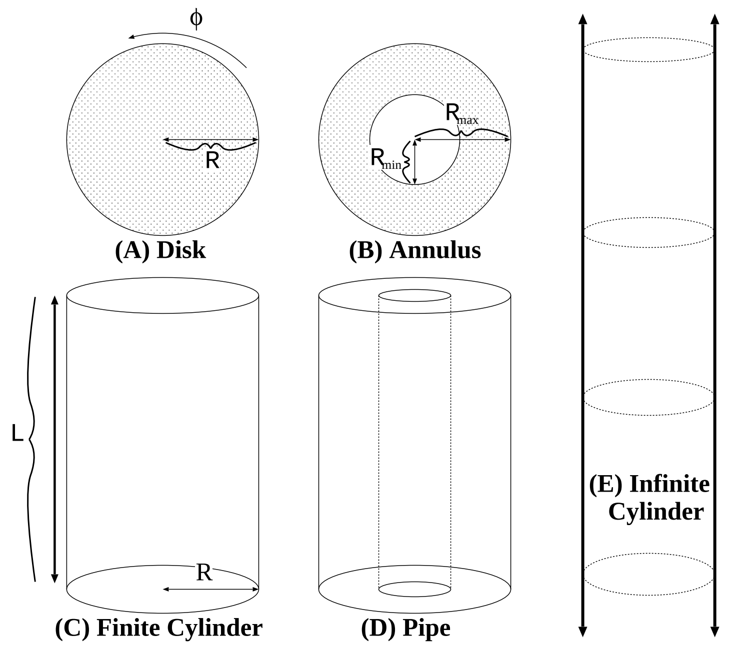


Figure 1.10: Some domains in polar and cylindrical coordinates.

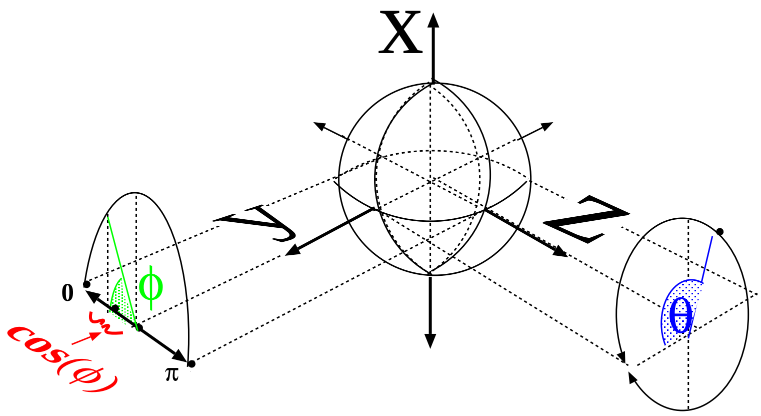


Figure 1.11: Spherical coordinates

1.6(d) Spherical Coordinates on $\mathbb{R}^3$

Spherical coordinates (r, θ, ϕ) on $\mathbb{R}^3$ are defined by the transformation:

$$\begin{aligned} x &= r \cdot \sin(\phi) \cdot \cos(\theta), \qquad y = r \cdot \sin(\phi) \cdot \sin(\theta) \\ \text{and} \quad z &= r \cdot \cos(\phi). \end{aligned}$$

with reverse transformation:

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \arctan\left(\frac{x}{y}\right) \\ \text{and} \quad \phi &= \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right). \end{aligned}$$

In spherical coordinates, the set $\mathbb{B} = \{(r, \theta, \phi) \; ; \; r \leq R\}$ is the **ball** of radius R (see Figure 1.8B).

1.7 Differentiation of Function Series

Many of our methods for solving partial differential equations will involve expressing the solution function as an infinite *series* of functions (like a Taylor series). To make sense of such solutions, we must be able to differentiate them.

Proposition 1.7: Differentiation of Series

Let $-\infty \leq a < b \leq \infty$. Suppose that, for all $n \in \mathbb{N}$, $f_n : (a, b) \longrightarrow \mathbb{R}$ is a differentiable function, and define $F : (a, b) \longrightarrow \mathbb{R}$ by

$$F(x) = \sum_{n=0}^{\infty} f_n(x), \qquad \text{for all } x \in (a, b).$$

Suppose there is a sequence $\{B_n\}_{n=1}^{\infty}$ of positive real numbers such that

(a) $\sum_{n=1}^{\infty} B_n < \infty$.

(b) *For all $x \in (a, b)$, and all $n \in \mathbb{N}$, $|f_n(x)| \leq B_n$ and $|f'_n(x)| \leq B_n$.*

Then F is differentiable, and, for all $x \in (a, b)$, $F'(x) = \sum_{n=0}^{\infty} f'_n(x)$. □

Example 1.8: Let $a = 0$ and $b = 1$. For all $n \in \mathbb{N}$, let $f_n(x) = \frac{x^n}{n!}$. Thus,

$$F(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \exp(x),$$

(because this is the Taylor series for the exponential function). Now let $B_0 = 1$ and let $B_n = \frac{1}{(n-1)!}$ for $n \geq 1$. Then

(a) $\sum_{n=1}^{\infty} B_n \;=\; \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \;<\; \infty.$

(b) For all $x \in (0,1)$, and all $n \in \mathbb{N}$, $|f_n(x)| = \frac{1}{n!}x^n < \frac{1}{n!} < \frac{1}{(n-1)!} = B_n$ and $|f_n'(x)| = \frac{n}{n!}x^{n-1} = \frac{1}{(n-1)!}x^{n-1} < \frac{1}{(n-1)!} = B_n$.

Hence the conditions of Proposition 1.7 are satisfied, so we conclude that

$$F'(x) \;=\; \sum_{n=0}^{\infty} f_n'(x) \;=\; \sum_{n=0}^{\infty} \frac{n}{n!}x^{n-1} \;=\; \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \;\underset{\text{(c)}}{=}\; \sum_{m=0}^{\infty} \frac{x^m}{m!} \;=\; \exp(x),$$

where **(c)** is the change of variables $m = n-1$. In this case, the conclusion is a well-known fact. But the same technique can be applied to more mysterious functions. ♢

Remarks: (a) The series $\sum_{n=0}^{\infty} f_n'(x)$ is sometimes called the *formal derivative* of the series $\sum_{n=0}^{\infty} f_n(x)$. It is 'formal' because it is obtained through a purely symbolic operation; it is not true in general that the 'formal' derivative is *really* the derivative of the series, or indeed, if the formal derivative series even converges. Proposition 1.7 essentially says that, under certain conditions, the 'formal' derivative equals the *true* derivative of the series.

(b) Proposition 1.7 is also true if the functions f_n involve more than one variable and/or more than one index. For example, if $f_{n,m}(x,y,z)$ is a function of three variables and two indices, and

$$F(x,y,z) \;=\; \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f_{n,m}(x,y,z), \qquad \text{for all } (x,y,z) \in (a,b)^3.$$

then under similar hypothesis, we can conclude that $\partial_y F(x,y,z) \;=\; \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \partial_y f_{n,m}(x,y,z)$, for all $(x,y,z) \in (a,b)^3$.

(c) For a proof of Proposition 1.7, see for example [Fol84], Theorem 2.27(b), p.54.

1.8 Differentiation of Integrals

Recommended: §1.7

Many of our methods for solving partial differential equations will involve expressing the solution function $F(x)$ as an *integral* of functions; ie. $F(x) \;=\; \int_{-\infty}^{\infty} f_y(x)\, dy$, where, for each $y \in \mathbb{R}$, $f_y(x)$ is a differentiable function of the variable x. This is a natural generalization of the 'solution series' spoken of in §1.7. Instead of beginning with a *discretely* paramaterized family of functions $\{f_n\}_{n=1}^{\infty}$, we begin with a *continuously* paramaterized family, $\{f_y\}_{y\in\mathbb{R}}$. Instead of combining these functions through a *summation* to get $F(x) = \sum_{n=1}^{\infty} f_n(x)$, we combine them through *integration*, to get $F(x) = \int_{-\infty}^{\infty} f_y(x)\, dy$. However, to make sense of such integrals as the solutions of differential equations, we must be able to differentiate them.

Proposition 1.9: Differentiation of Integrals

Let $-\infty \leq a < b \leq \infty$. Suppose that, for all $y \in \mathbb{R}$, $f_y : (a,b) \longrightarrow \mathbb{R}$ is a differentiable function, and define $F : (a,b) \longrightarrow \mathbb{R}$ by

$$F(x) = \int_{-\infty}^{\infty} f_y(x)\, dy, \qquad \text{for all } x \in (a,b).$$

Suppose there is a function $\beta : \mathbb{R} \longrightarrow [0,\infty)$ such that

(a) $\int_{-\infty}^{\infty} \beta(y)\, dy < \infty.$

(b) *For all $y \in \mathbb{R}$ and for all $x \in (a,b)$, $|f_y(x)| \leq \beta(y)$ and $\left|f_y'(x)\right| \leq \beta(y)$.*

Then F is differentiable, and, for all $x \in (a,b)$, $F'(x) = \int_{-\infty}^{\infty} f_n'(x)\, dy.$ □

Example 1.10: Let $a = 0$ and $b = 1$. For all $y \in \mathbb{R}$ and $x \in (0,1)$, let $f_y(x) = \frac{x^{|y|+1}}{1+y^4}$. Thus,

$$F(x) = \int_{-\infty}^{\infty} f_y(x)\, dy = \int_{-\infty}^{\infty} \frac{x^{|y|+1}}{1+y^4}\, dy.$$

Now, let $\beta(y) = \frac{1+|y|}{1+y^4}$. Then

(a) $\int_{-\infty}^{\infty} \beta(y)\, dy = \int_{-\infty}^{\infty} \frac{1+|y|}{1+y^4}\, dy < \infty$ (check this).

(b) For all $y \in \mathbb{R}$ and all $x \in (0,1)$, $|f_y(x)| = \frac{x^{|y|+1}}{1+y^4} < \frac{1}{1+y^4} < \frac{1+|y|}{1+y^4} = \beta(y)$,

and $\left|f_n'(x)\right| = \frac{(|y|+1)\cdot x^{|y|}}{1+y^4} < \frac{1+|y|}{1+y^4} = \beta(y)$.

Hence the conditions of Proposition 1.9 are satisfied, so we conclude that

$$F'(x) = \int_{-\infty}^{\infty} f_n'(x)\, dy = \int_{-\infty}^{\infty} \frac{(|y|+1)\cdot x^{|y|}}{1+y^4}\, dy.$$ ♢

Remarks: (a) Proposition 1.9 is also true if the functions f_y involve more than one variable. For example, if $f_{v,w}(x, y, z)$ is a function of five variables, and

$$F(x, y, z) \quad = \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{u,v}(x, y, z)\ du\ dv \qquad \text{for all } (x, y, z) \in (a, b)^3.$$

then under similar hypothesis, we can conclude that $\partial_y^2\, F(x, y, z) \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \partial_y^2\, f_{u,v}(x, y, z)\ du\ dv$, for all $(x, y, z) \in (a, b)^3$.

(b) For a proof of Proposition 1.9, see for example [Fol84], Theorem 2.27(b), p.54.

Notes: ....

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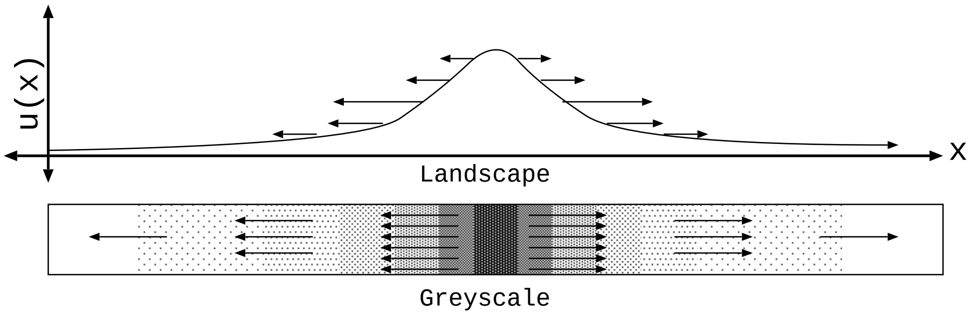


Figure 2.1: Fourier's Law of Heat Flow in one dimension

2 Heat and Diffusion

2.1 Fourier's Law

Prerequisites: §1.1 **Recommended:** §1.4

2.1(a) ...in one dimension

Figure 2.1 depicts a material diffusing through a one-dimensional domain $\mathbb{X}$ (for example, $\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = [0, L]$). Let $u(x, t)$ be the density of the material at the point $x \in \mathbb{X}$ at time $t > 0$. Intuitively, we expect the material to flow from regions of *greater* to *lesser* concentration. In other words, we expect the *flow* of the material at any point in space to be proportional to the *slope* of the curve $u(x, t)$ at that point. Thus, if $F(x, t)$ is the flow at the point x at time t, then we expect:

$$F(x,t) \quad = \quad -\kappa \cdot \partial_x u(x,t)$$

where $\kappa > 0$ is a constant measuring the rate of diffusion. This is an example of **Fourier's Law.**

2.1(b) ...in many dimensions

Prerequisites: §1.4

Figure 2.2 depicts a material diffusing through a two-dimensional domain $\mathbb{X} \subset \mathbb{R}^2$ (eg. heat spreading through a region, ink diffusing in a bucket of water, etc.). We could just as easily suppose that $\mathbb{X} \subset \mathbb{R}^D$ is a D-dimensional domain. If $\mathbf{x} \in \mathbb{X}$ is a point in space, and $t > 0$ is a moment in time, let $u(\mathbf{x}, t)$ denote the concentration at $\mathbf{x}$ at time t. (This determines a function $u : \mathbb{X} \times \mathbb{R} \longrightarrow \mathbb{R}$, called a *time-varying scalar field*.)

Now let $\vec{F}(\mathbf{x}, t)$ be a D-dimensional vector describing the *flow* of the material at the point $\mathbf{x} \in \mathbb{X}$. (This determines a *time-varying vector field* $\vec{F} : \mathbb{R}^D \times \mathbb{R} \longrightarrow \mathbb{R}^D$.)

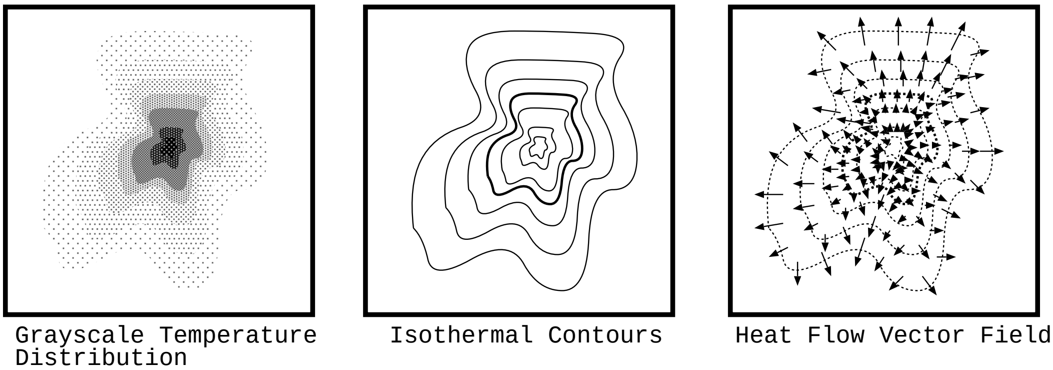


Figure 2.2: Fourier's Law of Heat Flow in two dimensions

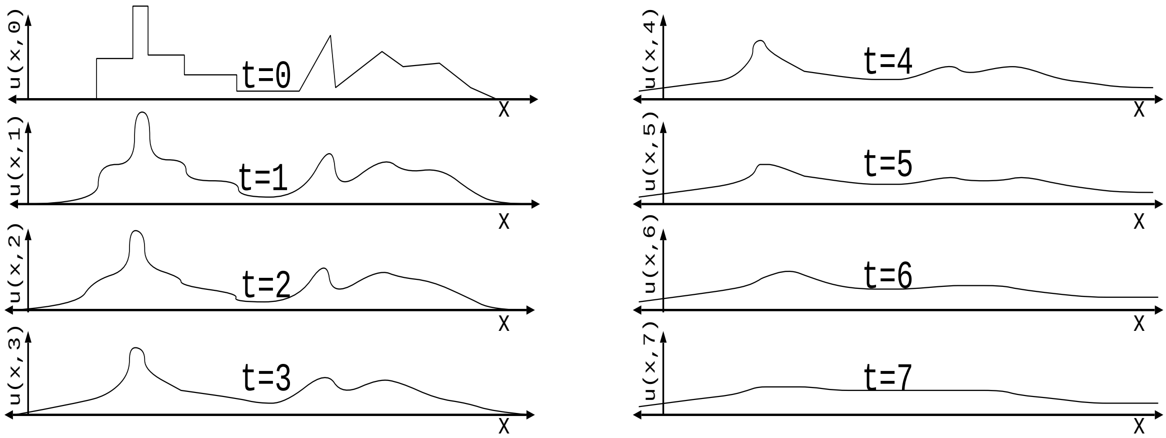


Figure 2.3: The Heat Equation as "erosion".

Again, we expect the material to flow from regions of high concentration to low concentration. In other words, material should flow *down the concentration gradient*. This is expressed by **Fourier's Law of Heat Flow** , which says:

$$\vec{F} \quad = \quad -\kappa \cdot \nabla u$$

where $\kappa > 0$ is is a constant measuring the rate of diffusion.

One can imagine u as describing a distribution of highly antisocial people; each person is always fleeing everyone around them and moving in the direction with the fewest people. The constant κ measures the average walking speed of these misanthropes.

2.2 The Heat Equation

Recommended: §2.1

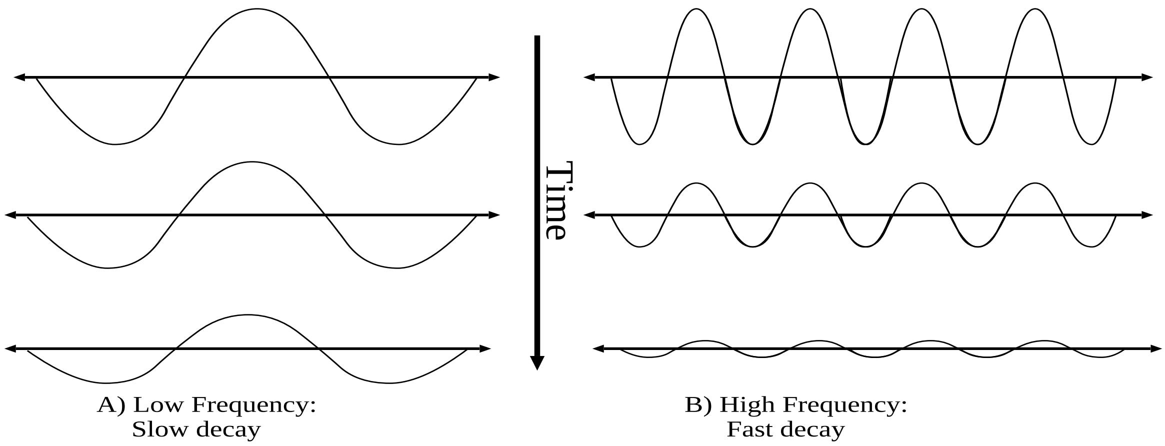


Figure 2.4: Under the Heat equation, the exponential decay of a periodic function is proportional to the square of its frequency.

2.2(a) ...in one dimension

Prerequisites: §2.1(a)

Consider a material diffusing through a one-dimensional domain $\mathbb{X}$ (for example, $\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = [0, L]$). Let $u(x,t)$ be the density of the material at the point $x \in \mathbb{X}$ at time $t > 0$, and $F(x,t)$ the flux. Consider the derivative $\partial_x F(x,t)$. If $\partial_x F(x,t) > 0$, this means that the flow is *accelerating* at this point in space, so the material there is spreading farther apart. Hence, we expect the concentration at this point to *decrease*. Conversely, if $\partial_x F(x,t) < 0$, then the flow is *decelerating* at this point in space, so the material there is crowding closer together, and we expect the concentration to *increase*. To be succinct: the concentration of material will *increase* in regions where F converges, and *decrease* in regions where F diverges. The equation describing this is:

$$\partial_t u(x,t) \quad = \quad -\partial_x F(x,t)$$

If we combine this with Fourier's Law, however, we get:

$$\partial_t u(x,t) \quad = \quad \kappa \cdot \partial_x \partial_x u(x,t)$$

which yields the **one-dimensional Heat Equation:**

$$\boxed{\partial_t u(x,t) \quad = \quad \kappa \cdot \partial_x^2 u(x,t)}$$

Heuristically speaking, if we imagine $u(x,t)$ as the height of some one-dimensional "landscape", then the Heat Equation causes this landscape to be "eroded", as if it were subjected to thousands of years of wind and rain (see Figure 2.3).

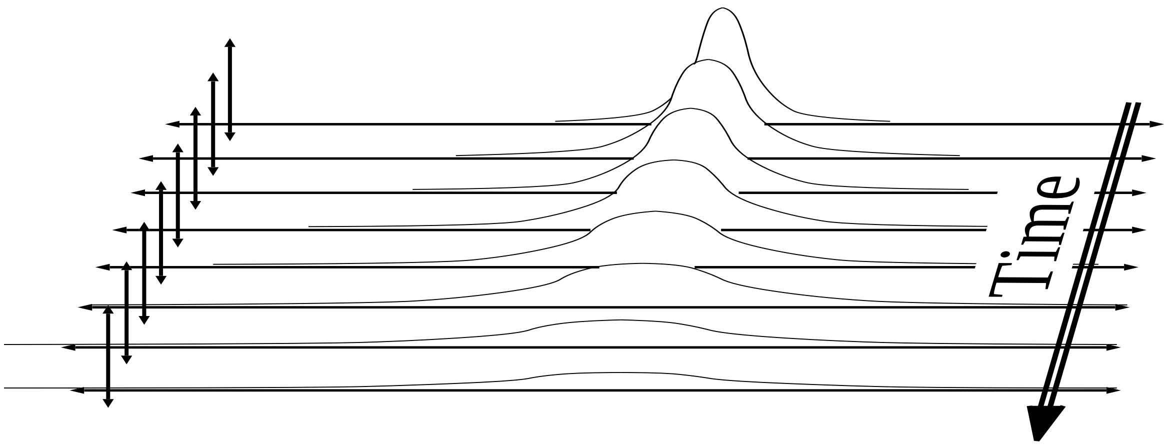


Figure 2.5: The Gauss-Weierstrass kernel under the Heat equation.

Example 2.1: For simplicity we suppose $\kappa = 1$.

(a) Let $u(x,t) = e^{-9t} \cdot \sin(3x)$. Thus, u describes a spatially sinusoidal function (with spatial frequency 3) whose magnitude decays exponentially over time.

(b) **The dissipating wave:** More generally, let $u(x,t) = e^{-\omega^2 \cdot t} \cdot \sin(\omega \cdot x)$. Then u is a solution to the one-dimensional Heat Equation, and looks like a standing wave whose amplitude decays exponentially over time (see Figure 2.4). Notice that the decay rate of the function u is proportional to the square of its frequency.

(c) **The (one-dimensional) Gauss-Weierstrass Kernel:** Let $\mathcal{G}(x;t) = \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-x^2}{4t}\right)$. Then $\mathcal{G}$ is a solution to the one-dimensional Heat Equation, and looks like a "bell curve", which starts out tall and narrow, and over time becomes broader and flatter (Figure 2.5).

<u>**Exercise 2.1**</u> Verify that the functions in Examples 2.1(a,b,c) all satisfy the Heat Equation. ♢

All three functions in Examples 2.1 starts out very tall, narrow, and pointy, and gradually become shorter, broader, and flatter. This is generally what the Heat equation does; it tends to flatten things out. If u describes a physical landscape, then the heat equation describes "erosion".

2.2(b) ...in many dimensions

Prerequisites: §2.1(b)

More generally, if $u : \mathbb{R}^D \times \mathbb{R} \longrightarrow \mathbb{R}$ is the time-varying density of some material, and $\vec{F} : \mathbb{R}^D \times \mathbb{R} \longrightarrow \mathbb{R}$ is the flux of this material, then we would expect the material to *increase* in regions where $\vec{F}$ converges, and to *decrease* in regions where $\vec{F}$ diverges. In other words, we have:

$$\partial_t u \;=\; -\mathbf{div}\,\vec{F}$$

If u is the density of some diffusing material (or heat), then $\vec{F}$ is determined by **Fourier's Law**, so we get the **Heat Equation**

$$\partial_t u \;=\; \kappa \cdot \mathbf{div}\,\nabla u \;=\; \kappa \triangle u$$

Here, $\triangle$ is the **Laplacian** operator[1], defined:

$$\boxed{\triangle u \;=\; \partial_1^2 u + \partial_2^2 u + \ldots \partial_D^2 u}$$

<u>**Exercise 2.2**</u>

- If $D = 1$, verify that $\mathbf{div}\,\nabla u(x) = u''(x) = \triangle u(x)$,
- If $D = 2$, verify that $\mathbf{div}\,\nabla u(x) = \partial_x^2 u(x) + \partial_y^2 u(x) = \triangle u(x)$.
- Verify that $\mathbf{div}\,\nabla u(x) = \triangle u(x)$ for any value of D.

By changing to the appropriate units, we can assume $\kappa = 1$, so the **Heat equation** becomes:

$$\boxed{\partial_t u \;=\; \triangle u}.$$

For example,

- If $\mathbb{X} \subset \mathbb{R}$, and $x \in \mathbb{X}$, then $\triangle u(x;t) \;=\; \partial_x^2 u(x;t)$.
- If $\mathbb{X} \subset \mathbb{R}^2$, and $(x,y) \in \mathbb{X}$, then $\triangle u(x,y;t) \;=\; \partial_x^2 u(x,y;t) + \partial_y^2 u(x,y;t)$.

Thus, as we've already seen, the one-dimensional Heat Equation is

$$\partial_t u \;=\; \partial_x^2 u$$

and the the **two dimensional Heat Equation** is:

$$\boxed{\partial_t u(x,y;t) \;=\; \partial_x^2 u(x,y;t) \;+\; \partial_y^2 u(x,y;t)}$$

Example 2.2:

(a) Let $u(x,y;t) \;=\; e^{-25\,t} \cdot \sin(3x)\sin(4y)$. Then u is a solution to the two-dimensional Heat Equation, and looks like a two-dimensional 'grid' of sinusoidal hills and valleys with horizontal spacing $1/3$ and vertical spacing $1/4$ As shown in Figure 2.6, these hills rapidly subside into a gently undulating meadow, and then gradually sink into a perfectly flat landscape.

[1] Sometimes the Laplacian is written as "∇^2".

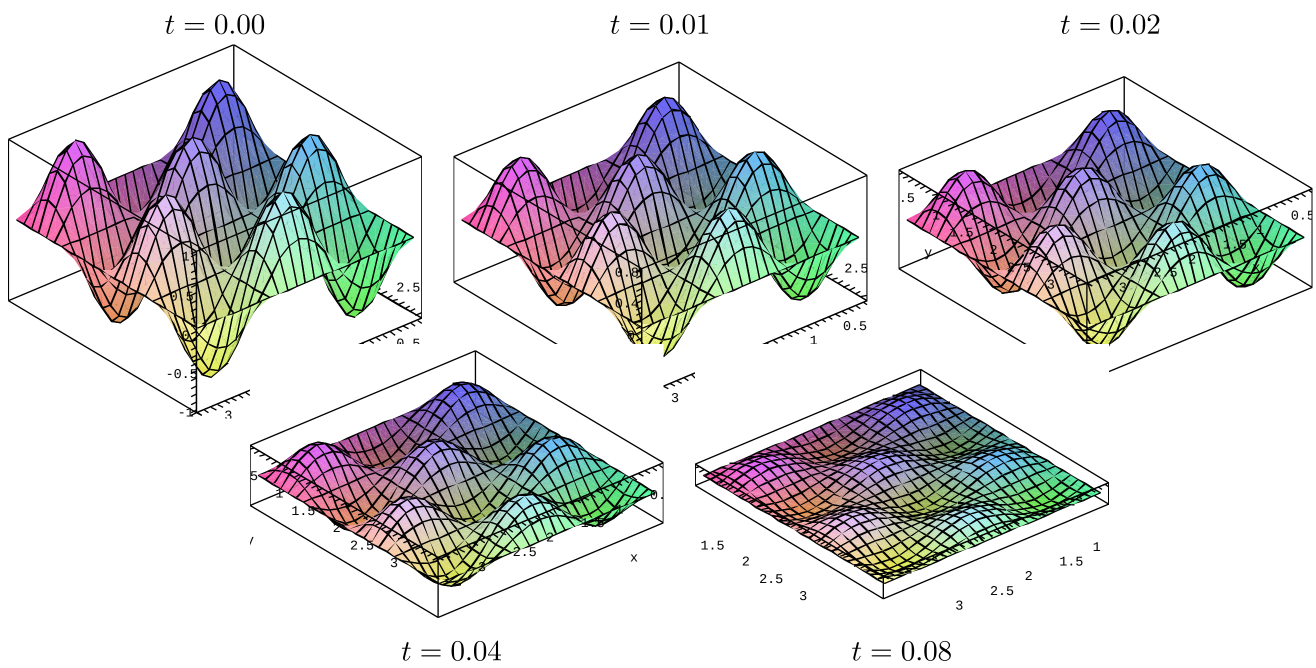


Figure 2.6: Five snapshots of the function $u(x,y;t) = e^{-25\,t} \cdot \sin(3x)\sin(4y)$ from Example 2.2.

(b) **The (two-dimensional) Gauss-Weierstrass Kernel:** Let $\mathcal{G}(x,y;\ t) = \frac{1}{4\pi t}\exp\left(\frac{-x^2-y^2}{4t}\right)$. Then $\mathcal{G}$ is a solution to the two-dimensional Heat Equation, and looks like a mountain, which begins steep and pointy, and gradually "erodes" into a broad, flat, hill.

(c) **The D-dimensional Gauss-Weierstrass Kernel** is the function $\mathcal{G} : \mathbb{R}^D \times (0,\infty) \longrightarrow \mathbb{R}$ defined

$$\boxed{\mathcal{G}(\mathbf{x};t) \;=\; \frac{1}{(4\pi t)^{D/2}}\exp\left(\frac{-\left\|\mathbf{x}\right\|^2}{4t}\right)}$$

Technically speaking, $\mathcal{G}(\mathbf{x};t)$ is a D-dimensional *symmetric normal probability distribution* with variance $\sigma = \sqrt{2t}$.

Exercise 2.3 Verify that the functions in Examples 2.2(a,b,c) both satisfy the Heat Equation.
◊

2.3 Laplace's Equation

Prerequisites: §2.2

If the Heat Equation describes the erosion/diffusion of some system, then an **equilibrium** or **steady-state** of the Heat Equation is a scalar field $h : \mathbb{R}^D \longrightarrow \mathbb{R}$ satisfying **Laplace's**

Pierre-Simon Laplace
Born: 23 March 1749 in Beaumont-en-Auge, Normandy
Died: 5 March 1827 in Paris

Equation:

$$\boxed{\triangle h \equiv 0.}$$

A scalar field satisfying the Laplace equation is called a **harmonic function**.

Example 2.3:

(a) If $D = 1$, then $\triangle h(x) = \partial_x^2 h(x) = h''(x)$; thus, the **one-dimensional Laplace equation** is just

$$\boxed{h''(x) = 0}$$

Suppose $h(x) = 3x + 4$. Then $h'(x) = 3$, and $h''(x) = 0$, so h is harmonic. More generally: the one-dimensional harmonic functions are just the *linear* functions of the form: $h(x) = ax + b$ for some constants $a, b \in \mathbb{R}$.

(b) If $D = 2$, then $\triangle h(x, y) = \partial_x^2 h(x, y) + \partial_y^2 h(x, y)$, so the **two-dimensional Laplace equation** reads:

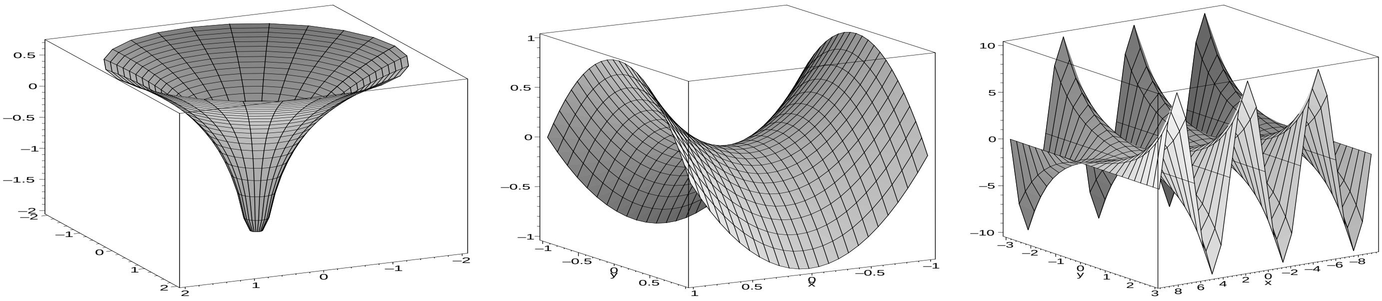
$$\boxed{\partial_x^2 h + \partial_y^2 h \quad = \quad 0,}$$

or, equivalently, $\partial_x^2 h = -\partial_y^2 h$. For example:

- Figure 2.7(B) shows the harmonic function $h(x, y) = x^2 - y^2$.
- Figure 2.7(C) shows the harmonic function $h(x, y) = \sin(x) \cdot \sinh(y)$.

<u>**Exercise 2.4**</u> Verify that these two functions are harmonic. ♢

The surfaces in Figure 2.7 have a "saddle" shape, and this is typical of harmonic functions; in a sense, a harmonic function is one which is "saddle-shaped" at every point in space. In particular, notice that $h(x, y)$ has no maxima or minima anywhere; this is a universal property



(A) **(B)** **(C)**

Figure 2.7: Three harmonic functions: **(A)** $h(x, y) = \log(x^2 + y^2)$. **(B)** $h(x, y) = x^2 - y^2$. **(C)** $h(x, y) = \sin(x) \cdot \sinh(y)$. In all cases, note the telltale "saddle" shape.

of harmonic functions (see Corollary 2.15 on page 33). The next example seems to contradict this assertion, but in fact it doesn't...

Example 2.4: Figure 2.7(A) shows the harmonic function $h(x, y) = \log(x^2 + y^2)$ for all $(x, y) \neq (0, 0)$. This function is well-defined everywhere except at $(0, 0)$; hence, contrary to appearances, $(0, 0)$ is *not* an extremal point. [Verifying that h is harmonic is problem # 3 on page 31]. ◊

When $D \geq 3$, harmonic functions no longer define nice saddle-shaped *surfaces*, but they still have similar mathematical properties.

Example 2.5:

(a) If $D = 3$, then $\triangle h(x, y, z) = \partial_x^2 h(x, y, z) + \partial_y^2 h(x, y, z) + \partial_z^2 h(x, y, z)$.

Thus, the **three-dimensional Laplace equation** reads:

$$\boxed{\partial_x^2 h + \partial_y^2 h + \partial_z^2 h \quad = \quad 0,}$$

For example, let $h(x, y, z) = \dfrac{1}{\|x, y, z\|} = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ for all $(x, y, z) \neq (0, 0, 0)$. Then h is harmonic everywhere except at $(0, 0, 0)$. [Verifying this is problem # 4 on page 31.]

(b) For any $D \geq 3$, the **D-dimensional Laplace equation** reads:

$$\boxed{\partial_1^2 h + \ldots + \partial_D^2 h \quad = \quad 0.}$$

For example, let $h(\mathbf{x}) = \dfrac{1}{\|\mathbf{x}\|^{D-2}} = \dfrac{1}{\left(x_1^2 + \cdots + x_D^2\right)^{\frac{D-2}{2}}}$ for all $\mathbf{x} \neq \mathbf{0}$. Then h is harmonic everywhere everywhere in $\mathbb{R}^D \setminus \{\mathbf{0}\}$ (**Exercise 2.5**).

(**Remark:** If we metaphorically interpret "$x^{\frac{-0}{2}}$" to mean "$-\log(x)$", then we can interpret Example 2.4 as a special case of Example (2b) for $D = 2$.) ◊

Harmonic functions have the convenient property that we can multiply together two lower-dimensional harmonic functions to get a higher dimensional one. For example:

- $h(x, y) = x \cdot y$ is a two-dimensional harmonic function (**Exercise 2.6**).
- $h(x, y, z) = x \cdot (y^2 - z^2)$ is a three-dimensional harmonic function (**Exercise 2.7**).

In general, we have the following:

Proposition 2.6: *Suppose $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is harmonic and $v : \mathbb{R}^m \longrightarrow \mathbb{R}$ is harmonic, and define $w : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}$ by $w(\mathbf{x}, \mathbf{y}) = u(\mathbf{x}) \cdot v(\mathbf{y})$ for $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$. Then w is also harmonic*

Proof: **Exercise 2.8** Hint: First prove that w obeys a kind of Liebniz rule: $\triangle w(\mathbf{x}, \mathbf{y}) = v(\mathbf{y}) \cdot \triangle u(\mathbf{x}) + u(\mathbf{x}) \cdot \triangle v(\mathbf{y})$. □

The function $w(\mathbf{x}, \mathbf{y}) = u(\mathbf{x}) \cdot v(\mathbf{y})$ is called a *separated solution*, and this theorem illustrates a technique called *separation of variables* (see Chapter 15 on page 278).

2.4 The Poisson Equation

Prerequisites: §2.3

Imagine $p(\mathbf{x})$ is the concentration of a chemical at the point $\mathbf{x}$ in space. Suppose this chemical is being *generated* (or *depleted*) at different rates at different regions in space. Thus, in the absence of diffusion, we would have the **generation equation**

$$\partial_t p(\mathbf{x}, t) = q(\mathbf{x}),$$

where $q(\mathbf{x})$ is the rate at which the chemical is being created/destroyed at $\mathbf{x}$ (we assume that q is constant in time).

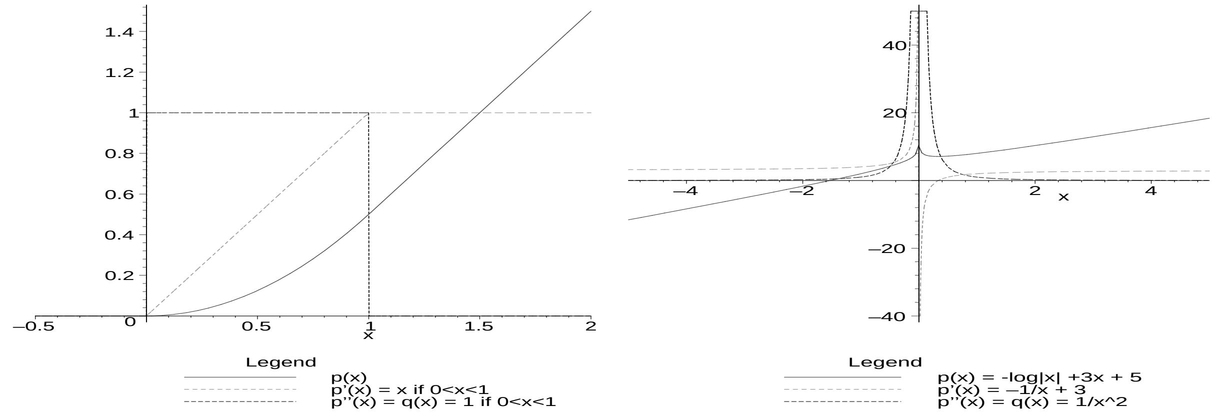
If we now included the effects of diffusion, we get the **generation-diffusion equation:**

$$\partial_t p = \kappa \triangle p + q.$$

A *steady state* of this equation is a scalar field p satisfying **Poisson's Equation:**

$$\boxed{\triangle p = Q.}$$

where $Q(x) = \dfrac{-q(x)}{\kappa}$.



(A) **(B)**

Figure 2.8: Two one-dimensional potentials

Example 2.7: One-Dimensional Poisson Equation

If $D = 1$, then $\triangle p(x) = \partial_x^2 p(x) = h''(x)$; thus, the **one-dimensional Poisson equation** is just

$$\boxed{h''(x) = Q(x)}$$

We can solve this equation by twice-integrating the function $Q(x)$. If $p(x) = \int\int Q(x)$ is some double-antiderivative of G, then p clearly satisfies the Poisson equation. For example:

(a) Suppose $Q(x) = \begin{cases} 1 & \text{if } 0 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$. Then define

$$p(x) \quad = \quad \int_0^x \int_0^y q(z)\ dz\ dy \quad = \quad \begin{cases} 0 & \text{if } x < 0; \\ x^2/2 & \text{if } 0 < x < 1; \\ x - \frac{1}{2} & \text{if } 1 < x. \end{cases} \qquad \text{(Figure 2.8A)}$$

(b) If $Q(x) = 1/x^2$ (for $x \neq 0$), then $p(x) \;=\; \int\int Q(x) \;=\; -\log|x| + ax + b$ (for $x \neq 0$), where $a, b \in \mathbb{R}$ are arbitrary constants. (see Figure 2.8B)

Exercise 2.9 Verify that the functions $p(x)$ in Examples (a) and (b) are both solutions to their respective Poisson equations. ♢

Example 2.8: Electrical/Gravitational Fields

Poisson's equation also arises in classical field theory[2]. Suppose, for any point $\mathbf{x} = (x_1, x_2, x_3)$ in three-dimensional space, that $q(\mathbf{x})$ is charge density at $\mathbf{x}$, and that $p(\mathbf{x})$ is the the electric potential field at $\mathbf{x}$. Then we have:

$$\kappa \triangle p(\mathbf{x}) \;=\; q(\mathbf{x}) \qquad (\kappa \text{ some constant}) \tag{2.1}$$

[2]For a quick yet lucid introduction to electrostatics, see [Ste95, Chap.3].

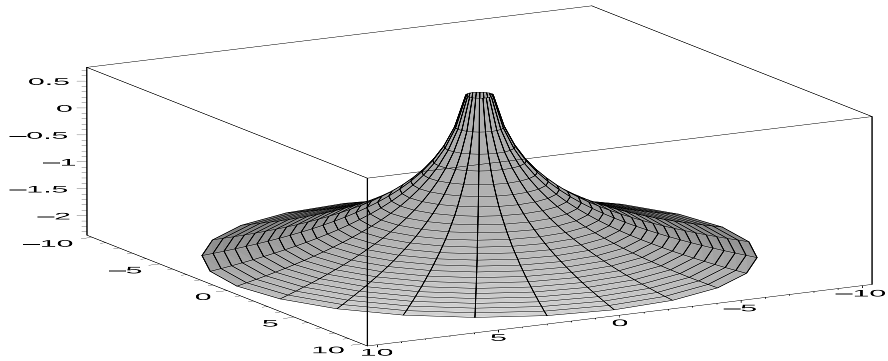


Figure 2.9: The two-dimensional potential field generated by a concentration of charge at the origin.

If $q(\mathbf{x})$ was the *mass* density at $\mathbf{x}$, and $p(\mathbf{x})$ was the *gravitational* potential energy, then we would get the same equation. (See Figure 2.9 for an example of such a potential in two dimensions).

Because of this, solutions to Poisson's Equation are sometimes called **potentials**. ♢

Example 2.9: The Coloumb Potential

Let $D = 3$, and let $p(x,y,z) = \frac{1}{\|x,y,z\|} = \frac{1}{\sqrt{x^2+y^2+z^2}}$. In Example (2a), we asserted that $p(x,y,z)$ was harmonic everywhere except at $(0,0,0)$, where it is not well-defined. For physical reasons, it is 'reasonable' to write the equation:

$$\triangle\, p(0,0,0) \quad = \quad \delta_0, \tag{2.2}$$

where δ_0 is the 'Dirac delta function' (representing an infinite concentration of charge at zero)[3]. Then $p(x,y,z)$ describes the electric potential generated by a *point charge.*

Exercise 2.10 Check that $\nabla\, p(x,y,z) = \frac{-(x,y,z)}{\|x,y,z\|^3}$. This is the electric field generated by a point charge, as given *Coloumb's Law* from classical electrostatics. ♢

Notice that the electric/gravitational potential field is *not uniquely defined* by equation (2.1). If $p(\mathbf{x})$ solves the Poisson equation (2.1), then so does $\widetilde{p}(\mathbf{x}) = p(\mathbf{x}) + a$ for any constant $a \in \mathbb{R}$. Thus, we say that the potential field is well-defined *up to addition of a constant*; this is similar to the way in which the antiderivative $\int Q(x)$ of a function is only well-defined up to some constant.[4] This is an example of a more general phenomenon:

[3]Equation (2.2) seems mathematically nonsensical, but it *can* be made mathematically meaningful, using *distribution theory.* However, this is far beyond the scope of these notes, so for our purposes, we will interpret eqn. (2.2) as purely metaphorical.

[4]For the purposes of the physical theory, this constant *does not matter*, because the field p is physically interpreted only by computing the *potential difference* between two points, and the constant a will always cancel out in this computation. Thus, the two potential fields $p(\mathbf{x})$ and $\widetilde{p}(\mathbf{x}) = p(\mathbf{x}) + a$ will generate identical physical predictions. Physicists refer to this phenomenon as *gauge invariance.*

Proposition 2.10: *Let $\mathbb{X} \subset \mathbb{R}^D$ be some domain, and let $p(\mathbf{x})$ and $h(\mathbf{x})$ be two functions defined for $\mathbf{x} \in \mathbb{X}$. Let $\widetilde{p}(\mathbf{x}) = p(\mathbf{x}) + h(\mathbf{x})$. Suppose that h is* **harmonic** *—ie. $\triangle h(\mathbf{x}) = 0$. If p satisfies the* **Poisson Equation** *"$\triangle p(\mathbf{x}) = q(\mathbf{x})$", then $\widetilde{p}$ also satisfies this Poisson equation.*

Proof: **Exercise 2.11** Hint: Notice that $\triangle\widetilde{p}(\mathbf{x}) = \triangle p(\mathbf{x}) + \triangle h(\mathbf{x})$. □

For example, if $Q(x) = 1/x^2$, as in Example 2.7(b), then $p(x) = -\log(x)$ is a solution to the Poisson equation "$p''(x) = 1/x^2$". If $h(x)$ is a one-dimensional harmonic function, then $h(x) = ax + b$ for some constants a and b (see Example 2.3 on page 26). Thus $\widetilde{p}(x) = -\log(x) + ax + b$, and we've already seen that these are also valid solutions to this Poisson equation.

Exercise 2.12 (a) Let $\mu, \nu \in \mathbb{R}$ be constants, and let $f(x, y) = e^{\mu x} \cdot e^{\nu y}$. Suppose f is harmonic; what can you conclude about the relationship between μ and ν? (Justify your assertion).

(b) Suppose $f(x, y) = X(x) \cdot Y(y)$, where $X : \mathbb{R} \longrightarrow \mathbb{R}$ and $Y : \mathbb{R} \longrightarrow \mathbb{R}$ are two smooth functions. Suppose $f(x, y)$ is harmonic

1. Prove that $\dfrac{X''(x)}{X(x)} = \dfrac{-Y''(y)}{Y(y)}$ for all $x, y \in \mathbb{R}$.
2. Conclude that the function $\dfrac{X''(x)}{X(x)}$ must equal a constant c independent of x. Hence $X(x)$ satisfies the ordinary differential equation $X''(x) = c \cdot X(x)$.

   Likewise, the function $\dfrac{Y''(y)}{Y(y)}$ must equal $-c$, independent of y. Hence $Y(y)$ satisfies the ordinary differential equation $Y''(y) = -c \cdot Y(y)$.
3. Using this information, deduce the general form for the functions $X(x)$ and $Y(y)$, and use this to obtain a general form for $f(x, y)$.

This argument is an example of *separation of variables*; see Chapter 15 on page 278.

2.5 Practice Problems

1. Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}$ be a differentiable scalar field. Show that $\mathbf{div}\, \nabla f(x_1, x_2, x_3, x_4) = \triangle f(x_1, x_2, x_3, x_4)$.
2. Let $f(x, y; t) = \exp(-34t) \cdot \sin(3x + 5y)$. Show that $f(x, y; t)$ satisfies the two-dimensional Heat Equation: $\partial_t f(x, y; t) = \triangle f(x, y; t)$.
3. Let $u(x, y) = \log(x^2 + y^2)$. Show that $u(x, y)$ satisfies the (two-dimensional) Laplace Equation, everywhere except at $(x, y) = (0, 0)$.

   Remark: If $(x, y) \in \mathbb{R}^2$, recall that $\|x, y\| = \sqrt{x^2 + y^2}$. Thus, $\log(x^2 + y^2) = 2\log\|x, y\|$. This function is sometimes called the **logarithmic potential**.
4. If $(x, y, z) \in \mathbb{R}^3$, recall that $\|x, y, z\| = \sqrt{x^2 + y^2 + z^2}$. Define

$$u(x, y, z) = \frac{1}{\|x, y, z\|} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

Show that u satisfies the (three-dimensional) Laplace equation, everywhere except at $(x, y, z) = (0, 0, 0)$.

Remark: Observe that $\nabla u(x, y, z) = \frac{-(x, y, z)}{\|x, y, z\|^3}$. What force field does this remind you of? Hint: $u(x, y, z)$ is sometimes called the **Coulomb potential**.

5. Let $u(x, y; t) = \frac{1}{4\pi t} \exp\left(\frac{-\|x, y\|^2}{4t}\right) = \frac{1}{4\pi t} \exp\left(\frac{-x^2 - y^2}{4t}\right)$ be the (two-dimensional) **Gauss-Weierstrass Kernel**. Show that u satisfies the (two-dimensional) Heat equation, $\partial_t u = \triangle u$.

6. Let α and β be real numbers, and let $h(x, y) = \sinh(\alpha x) \cdot \sin(\beta y)$.
   (a) Compute $\triangle h(x, y)$.
   (b) Suppose h is **harmonic**. Write an equation describing the relationship between α and β.

2.6 Properties of Harmonic Functions

Prerequisites: §3.1, §16.3 **Recommended:** §2.3

Recall that a function $h : \mathbb{R}^D \longrightarrow \mathbb{R}$ is **harmonic** if $\triangle u \equiv 0$. Harmonic functions have nice geometric properties, which can be loosely summarized as 'smooth and gently curving'.

Proposition 2.11: Differentiability of Harmonic functions

If $h : \mathbb{R}^D \longrightarrow \mathbb{R}$ is a harmonic function, then h is infinitely differentiable.

Proof: **Exercise 2.13** Hint: Let $G_t(\mathbf{x})$ be the Gauss-Weierstrass kernel. Recall that the harmonic function h is a stationary solution to the Heat Equation, so use Theorem 16.15 on page 315 to argue that $G_t * h(\mathbf{x}) = h(\mathbf{x})$ for all $t > 0$. Verify that G_t is infinitely differentiable. Now apply **Part 6** of Proposition 16.19 on page 320.

For a different proof, see Theorem 6 on p. 28 of [Eva91, §2.2]. □

Actually, we can go even further than this

Proposition 2.12: Harmonic functions are analytic

Let $\mathbf{X} \subset \mathbb{R}^D$ be an open set. If $h : \mathbf{X} \longrightarrow \mathbb{R}$ is a harmonic function, then h is **analytic**. *That is, for every $\mathbf{x} \in \mathbf{X}$, there is a Taylor series expansion for h around $\mathbf{x}$ with a nonzero radius of convergence.*

Proof: See Theorem 10 on p. 31 of [Eva91, §2.2]. □

Theorem 2.13: Mean Value Theorem

Let $f : \mathbb{R}^D \longrightarrow \mathbb{R}$ be a scalar field. Then f is harmonic if and only if:

$$\text{For any } \mathbf{x} \in \mathbb{R}^D, \text{ and any } R > 0, \quad f(\mathbf{x}) \;=\; \int_{\mathbb{S}(\mathbf{x};R)} f(\mathbf{s})\; d\mathbf{s}. \tag{2.3}$$

Here, $\mathbb{S}(\mathbf{x};R) = \left\{\mathbf{s} \in \mathbb{R}^D \;;\; \|\mathbf{s}-\mathbf{x}\| = R\right\}$ is the **sphere** *of radius R around $\mathbf{x}$.*

Proof: <u>**Exercise 2.14**</u>

1. First use the the **Spherical Means** formula for the Laplacian (Theorem 3.1) to show that f is harmonic *if* statement (2.3) is true.

2. Now, define $\phi : [0,\infty) \longrightarrow \mathbb{R}$ by: $\phi(R) = \int_{\mathbb{S}(\mathbf{x};R)} f(\mathbf{s})\; d\mathbf{s}$.

Show that $\phi'(R) \;=\; K \int_{\mathbb{S}(\mathbf{x};R)} \partial_\perp f(\mathbf{s})\; d\mathbf{s}$ for some constant $K > 0$.

3. Apply **Gauss's Divergence Theorem:** if $\mathbf{V} : \mathbb{R}^D \longrightarrow \mathbb{R}^D$ is any vector field and ν is the **outward normal** vector field on the sphere, then

$$\int_{\mathbb{S}(\mathbf{x};R)} \langle \mathbf{V}(\mathbf{s}), \nu\rangle \;\; d\mathbf{s} \;=\; \int_{\mathbb{B}(\mathbf{x};R)} \mathbf{div}\,\mathbf{V}(\mathbf{b})\; d\mathbf{b}$$

where $\mathbb{B}\left(\mathbf{x};R\right) = \left\{\mathbf{b} \in \mathbb{R}^D \;;\; \|\mathbf{b}-\mathbf{x}\| \leq R\right\}$ is the **ball** of radius R around $\mathbf{x}$. Use this to show that:

$$\phi'(R) = \int_{\mathbb{B}(\mathbf{x};R)} \triangle f(\mathbf{b})\; d\mathbf{b}.$$

4. Deduce that, if f is harmonic, then ϕ must be constant.

5. Use the **Spherical Means** formula for the Laplacian to show that this constant must be zero. Conclude that, if f is harmonic, then statement (2.3) must be true. □

A function $F : \mathbb{R}^D \longrightarrow \mathbb{R}$ is **spherically symmetric** if $F(\mathbf{x})$ depends only in the norm $\|\mathbf{x}\|$ (ie. $F(\mathbf{x}) = f\left(\|\mathbf{x}\|\right)$ for some function $f : [0,\infty) \longrightarrow \mathbb{R}$). For example, the Gauss-Weierstrass kernel G_t is spherically symmetric.

Corollary 2.14: *If h is harmonic and F is spherically symmetric, then $h * F = h$*

Proof: <u>**Exercise 2.15**</u> □

Corollary 2.15: Maximum Modulus Principle for Harmonic Functions

Let $\mathbb{X} \subset \mathbb{R}^D$ be a domain, and let $u : \mathbb{X} \longrightarrow \mathbb{R}$ be a nonconstant harmonic function. Then u has no maximal or minimal points anywhere in the interior of u.

Proof: <u>**Exercise 2.16**</u> □

Remark: Of course, a harmonic function *can* (and usually will) obtain a maximum somewhere on the *boundary* of the domain $\mathbb{X}$.

2.7 (∗) Transport and Diffusion

Prerequisites: §2.2, §7.1

If $u : \mathbb{R}^D \longrightarrow \mathbb{R}$ is a "mountain", then recall that $\nabla u(\mathbf{x})$ points in the direction of *most rapid ascent* at $\mathbf{x}$. If $\vec{v} \in \mathbb{R}^D$ is a vector, then $\langle \vec{v}, \nabla u(\mathbf{x})\rangle$ measures how rapidly you would be ascending if you walked in direction $\vec{v}$.

Suppose $u : \mathbb{R}^D \longrightarrow \mathbb{R}$ describes a pile of leafs, and there is a steady wind blowing in the direction $\vec{v} \in \mathbb{R}^D$. We would expect the pile to slowly move in the direction $\vec{v}$. Suppose you were an observer fixed at location $\mathbf{x}$. The pile is moving past you in direction $\vec{v}$, which is the same as you walking along the pile in direction $-\vec{v}$; thus, you would expect the height of the pile at your location to increase/decrease at rate $\langle -\vec{v}, \nabla u(\mathbf{x})\rangle$. The pile thus satisfies the **Transport Equation:**

$$\partial_t u \;=\; -\langle \vec{v}, \nabla u\rangle$$

Now, suppose that the wind does not blow in a *constant* direction, but instead has some complex spatial pattern. The wind velocity is therefore determined by a *vector field* $\vec{V} : \mathbb{R}^D \longrightarrow \mathbb{R}^D$. As the wind picks up leafs and carries them around, the **flux** of leafs at a point $\mathbf{x} \in \mathbb{X}$ is then the vector $\vec{F}(\mathbf{x}) = u(\mathbf{x}) \cdot \vec{V}(\mathbf{x})$. But the rate at which leafs are piling up at each location is the *divergence* of the flux. This results in **Liouville's Equation:**

$$\partial_t u \;=\; -\mathbf{div}\,\vec{F} \;=\; -\mathbf{div}\,(u \cdot \vec{V}) \;=\; -\left\langle \vec{V}, \nabla u \right\rangle \;-\; u \cdot \mathbf{div}\,\vec{V}.$$

(<u>**Exercise 2.17**</u> Verify that $\mathbf{div}\,(u\cdot\vec{V}) \;=\; -\left\langle \vec{V}, \nabla u\right\rangle - u\cdot\mathbf{div}\,\vec{V}$. *Hint:* This is sort of a multivariate version of the Liebniz product rule.)

Liouville's equation describes the rate at which u-material accumulates when it is being pushed around by the $\vec{V}$-vector field. Another example: $\vec{V}(\mathbf{x})$ describes the flow of water at $\mathbf{x}$, and $u(\mathbf{x})$ is the buildup of some sediment at $\mathbf{x}$.

Now suppose that, in addition to the deterministic force $\vec{V}$ acting on the leafs, there is also a "random" component. In other words, while being blown around by the wind, the leafs are also subject to some random diffusion. To describe this, we combine *Liouville's Equation* with the *Heat Equation*, to obtain the **Fokker-Plank** equation:

$$\partial_t u \;=\; \kappa \triangle u \;-\; \left\langle \vec{V}, \nabla u \right\rangle \;-\; u \cdot \mathbf{div}\,\vec{V}.$$

2.8 (∗) Reaction and Diffusion

Prerequisites: §2.2

Suppose A,B and C are three chemicals, satisfying the chemical reaction:

$$2A + B \;\Longrightarrow\; C$$

As this reaction proceeds, the A and B species are consumed, and C is produced. Thus, if a, b, c are the concentrations of the three chemicals, we have:

$$\partial_t c \;=\; R(t) \;=\; -\partial_t b \;=\; -\frac{1}{2}\partial_t a,$$

where $R(t)$ is the rate of the reaction at time t. The rate $R(t)$ is determined by the concentrations of A and B, and by a rate constant ρ. Each chemical reaction requires 2 molecules of A and one of B; thus, the reaction rate is given by

$$R(t) \;=\; \rho \cdot a(t)^2 \cdot b(t)$$

Hence, we get three ordinary differential equations, called the **reaction kinetic equations** of the system:

$$\left.\begin{array}{rcl} \partial_t a(t) &=& -2\rho \cdot a(t)^2 \cdot b(t) \\ \partial_t b(t) &=& -\rho \cdot a(t)^2 \cdot b(t) \\ \partial_t c(t) &=& \rho \cdot a(t)^2 \cdot b(t) \end{array}\right\} \qquad (2.4)$$

Now, suppose that the chemicals A, B and C are in solution, but are not uniformly mixed. At any location $\mathbf{x} \in \mathbb{X}$ and time $t > 0$, let $a(\mathbf{x}, t)$ be the concentration of chemical A at location $\mathbf{x}$ at time t; likewise, let $b(\mathbf{x}, t)$ be the concentration of B and $c(\mathbf{x}, t)$ be the concentration of C. (This determines three *time-varying scalar fields*, $a, b, c : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}$.) As the chemicals react, their concentrations at each point in space may change. Thus, the functions a, b, c will obey the equations (2.4) at each point in space. That is, for every $\mathbf{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$, we have

$$\partial_t a(\mathbf{x}; t) \;\approx\; -2\rho \cdot a(\mathbf{x}; t)^2 \cdot b(\mathbf{x}; t)$$

etc. However, the dissolved chemicals are also subject to *diffusion* forces. In other words, each of the functions a, b and c will also be obeying the Heat Equation. Thus, we get the system:

$$\begin{array}{rcl} \partial_t a &=& \kappa_a \cdot \triangle a(\mathbf{x}; t) \;-\; 2\rho \cdot a(\mathbf{x}; t)^2 \cdot b(\mathbf{x}; t) \\ \partial_t b &=& \kappa_b \cdot \triangle b(\mathbf{x}; t) \;-\; \rho \cdot a(\mathbf{x}; t)^2 \cdot b(\mathbf{x}; t) \\ \partial_t c &=& \kappa_c \cdot \triangle c(\mathbf{x}; t) \;+\; \rho \cdot a(\mathbf{x}; t)^2 \cdot b(\mathbf{x}; t) \end{array}$$

where $\kappa_a, \kappa_b, \kappa_c > 0$ are three different diffusivity constants.

This is an example of a **reaction-diffusion system**. In general, in a reaction-diffusion system involving N distinct chemicals, the concentrations of the different species is described by a **concentration vector field** $\mathbf{u} : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}^N$, and the chemical reaction is described by a **rate function** $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$. For example, in the previous example, $\mathbf{u}(\mathbf{x}, t) = \Big(a(\mathbf{x}, t),\; b(\mathbf{x}, t),\; c(\mathbf{x}, t)\Big)$, and

$$F(a, b, c) \;=\; \left[-2\rho a^2 b, \quad -\rho a^2 b, \quad \rho a^2 b\right].$$

The **reaction-diffusion equations** for the system then take the form

$$\partial_t u_n \;=\; \kappa_n \triangle u_n \;+\; F_n(\mathbf{u}),$$

for $n = 1, ..., N$

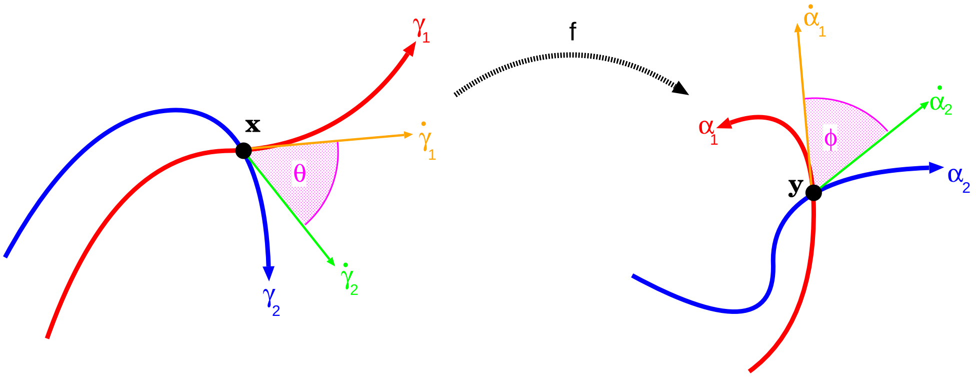


Figure 2.10: A conformal map preserves the angle of intersection between two paths.

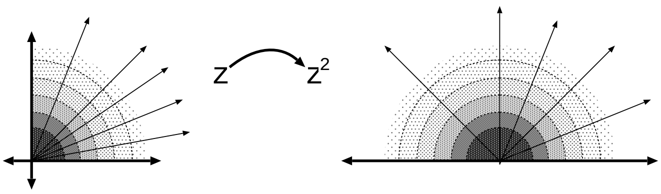


Figure 2.11: The map $f(z) = z^2$ conformally identifies the quarter plane and the half-plane.

2.9 (∗) Conformal Maps

Prerequisites: §2.2, §6.5, §1.3

A linear map $f : \mathbb{R}^D \longrightarrow \mathbb{R}^D$ is called **conformal** if it preserves the angles between vectors. Thus, for example, *rotations*, *reflections*, and *dilations* are all conformal maps.

Let $\mathbb{U}, \mathbb{V} \subset \mathbb{R}^D$ be open subsets of $\mathbb{R}^D$. A differentiable map $f : \mathbb{U} \longrightarrow \mathbb{V}$ is called **conformal** if its derivative $\mathsf{D} f(\mathbf{x})$ is a conformal linear map, for every $\mathbf{x} \in \mathbb{U}$.

One way to interpret this is depicted in Figure 2.10). Suppose two smooth paths γ_1 and γ_2 cross at $\mathbf{x}$, and their velocity vectors $\dot{\gamma}_1$ and $\dot{\gamma}_2$ form an angle θ at $\mathbf{x}$. Let $\alpha_1 = f \circ \gamma_1$ and $\alpha_2 = f \circ \gamma_2$, and let $\mathbf{y} = f(\mathbf{x})$. Then α_1 and α_2 are smooth paths, and cross at $\mathbf{y}$, forming an angle ϕ. The map f is conformal if, for every $\mathbf{x}$, γ_1, and γ_2, the angles θ and ϕ are equal.

Example 2.16: *Complex Analytic Maps*

Identify the set of complex numbers $\mathbb{C}$ with the plane $\mathbb{R}^2$ in the obvious way. If $\mathbb{U}, \mathbb{V} \subset \mathbb{C}$ are open subsets of the plane, then any complex-analytic map $f : \mathbb{U} \longrightarrow \mathbb{V}$ is conformal.

Exercise 2.18 Prove this. *Hint:* Think of the derivative f' as a linear map on $\mathbb{R}^2$, and use the Cauchy-Riemann differential equations to show it is conformal. ♢

In particular, we can often identify different domains in the complex plane via a conformal analytic map. For example:

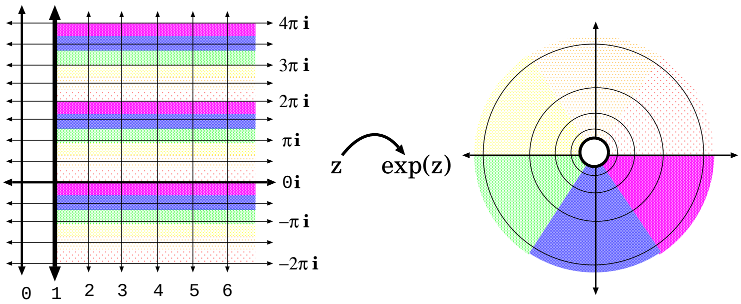


Figure 2.12: The map $f(z) = \exp(z)$ conformally projects a half-plane onto the complement of the unit disk.

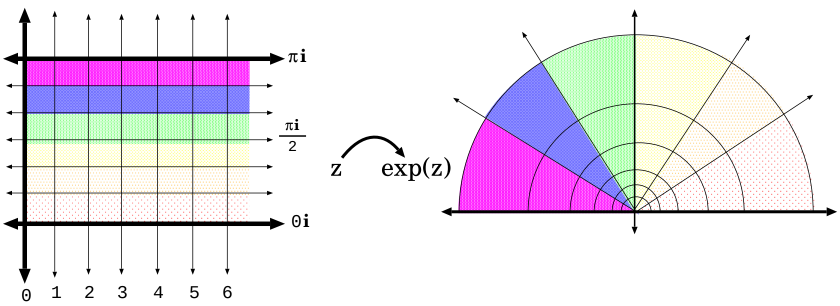


Figure 2.13: The map $f(z) = \exp(z)$ conformally identifies a half-infinite rectangle with the upper half-plane.

- In Figure 2.11, $\mathbb{U} = \{x + y\mathbf{i}\ ;\ x, y > 0\}$ is the quarter plane, and $\mathbb{V} = \{x + y\mathbf{i}\ ;\ y > 0\}$ is the half-plane, and $f(z) = z^2$. Then $f : \mathbb{U} \longrightarrow \mathbb{V}$ is a **conformal isomorphism**, meaning that it is conformal, invertible, and has a conformal inverse.
- In Figure 2.12), $\mathbb{U} = \{x + y\mathbf{i}\ ;\ x > 1\}$, and $\mathbb{V} = \left\{x + y\mathbf{i}\ ;\ x^2 + y^2 > 1\right\}$ is the complement of the unit disk, and $f(z) = \exp(z)$. Then $f : \mathbb{U} \longrightarrow \mathbb{V}$ is a conformal **covering map**. This means that f is *locally* one-to-one: for any point $u \in \mathbb{U}$, with $v = f(u) \in \mathbb{V}$, there is a neighbourhood $\mathcal{V} \subset \mathbb{V}$ of v and a neighbourhood $\mathcal{U} \subset \mathbb{U}$ of u so that $f_| : \mathcal{U} \longrightarrow \mathcal{V}$ *is* one-to-one. Note that f is *not* globally one-to-one because it is periodic in the imaginary coordinate.
- In Figure 2.13, $\mathbb{U} = \{x + y\mathbf{i}\ ;\ x > 0, 0 < y < \pi\}$ is a half-infinite rectangle, and $\mathbb{V} = \{x + y\mathbf{i}\ ;\ x > 1\}$ is the upper half plane, and $f(z) = \exp(z)$. Then $f : \mathbb{U} \longrightarrow \mathbb{V}$ is a conformal isomorphism.
- In Figure 2.14, $\mathbb{U} = \{x + y\mathbf{i}\ ;\ x > 1, 0 < y < \pi\}$ is a half-infinite rectangle, and $\mathbb{V} = \left\{x + y\mathbf{i}\ ;\ x > 1,\ x^2 + y^2 > 1\right\}$ is the "amphitheatre", and $f(z) = \exp(z)$. Then $f : \mathbb{U} \longrightarrow \mathbb{V}$ is a conformal isomorphism.

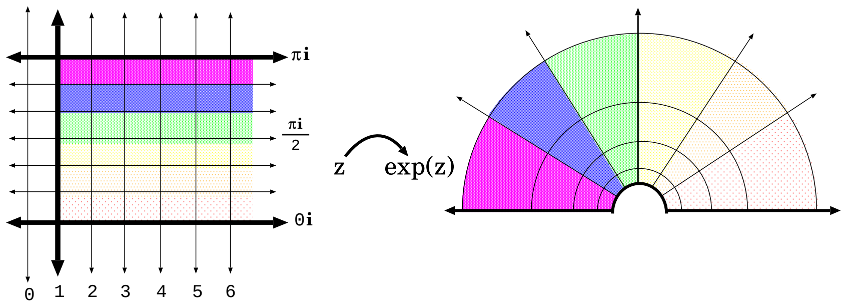


Figure 2.14: The map $f(z) = \exp(z)$ conformally identifies a half-infinite rectangle with the "amphitheatre"

Exercise 2.19 Verify each of the previous examples.

Exercise 2.20 Show that *any* conformal map is locally invertible. Show that the (local) inverse of a conformal map is also conformal.

Conformal maps are important in the theory of harmonic functions because of the following result:

Proposition 2.17: *Suppose that $f : \mathbb{U} \longrightarrow \mathbb{V}$ is a conformal map. Let $h : \mathbb{V} \longrightarrow \mathbb{R}$ be some smooth function, and define $H = h \circ f : \mathbb{U} \longrightarrow \mathbb{R}$.*

1. *h is harmonic if and only if H is harmonic.*
2. *h satisfies homogeneous* **Dirichlet** *boundary conditions*[5] *if and only if H does.*
3. *h satisfies homogeneous* **Neumann** *boundary conditions*[6] *if and only if H does.*
4. *Let $b : \partial\mathbb{V} \longrightarrow \mathbb{R}$ be some function on the boundary of $\mathbb{V}$. Then $B = b \circ f : \partial\mathbb{U} \longrightarrow \mathbb{R}$ is a function on the boundary of $\mathbb{U}$. Then h satisfies the nonhomogeneous Dirichlet boundary condition "$h(\mathbf{x}) = b(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathbb{V}$" if and only if H satisfies the nonhomogeneous Dirichlet boundary condition "$H(\mathbf{x}) = H(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathbb{U}$"*

Proof: **Exercise 2.21** Hint: Use the chain rule. Remember that rotation doesn't affect the value of the Laplacian, and dilation multiplies it by a scalar. □

We can apply this as follows: given a boundary value problem on some "nasty" domain $\mathbb{U}$, try to find a "nice" domain $\mathbb{V}$, and a conformal map $f : \mathbb{U} \longrightarrow \mathbb{V}$. Now, solve the boundary value problem in $\mathbb{V}$, to get a solution function h. Finally, "pull back" the this solution to get a solution $H = h \circ f$ to the original BVP on $\mathbb{V}$.

[5]See § 6.5(a) on page 93.
[6]See § 6.5(b) on page 95.

This may sound like an unlikely strategy. After all, how often are we going to find a "nice" domain $\mathbb{V}$ which we can conformally identify with our nasty domain? In general, not often. However, in two dimensions, we can search for conformal maps arising from complex analytic mappings, as described above. There is a deep result which basically says that it is almost always possible to find the conformal map we seek....

Theorem 2.18: Riemann Mapping Theorem

Let $\mathbb{U}, \mathbb{V} \subset \mathbb{C}$ be two open, simply connected[7] *regions of the complex plane. Then there is always a complex-analytic bijection $f : \mathbb{U} \longrightarrow \mathbb{V}$.* □

Corollary 2.19: *Let $\mathbb{D} \subset \mathbb{R}^2$ be the unit disk. If $\mathbb{U} \subset \mathbb{R}^2$ is open and simply connected, then there is a conformal isomorphism $f : \mathbb{D} \longrightarrow \mathbb{U}$.*

Proof: **Exercise 2.22** Derive this from the Riemann Mapping Theorem. □

Further Reading

An analogy of the Laplacian can be defined on any Riemannian manifold, where it is sometimes called the **Laplace-Beltrami operator**. The study of harmonic functions on manifolds yields important geometric insights [War83, Cha93].

The reaction diffusion systems from §2.8 play an important role in modern mathematical biology [Mur93].

The Heat Equation also arises frequently in the theory of Brownian motion and other Gaussian stochastic processes on $\mathbb{R}^D$ [Str93].

The discussion in §2.9 is just the beginning of the beautiful theory of the 2-dimensional Laplace equation and the conformal properties of complex-analytic maps. An excellent introduction can be found in Tristan Needham's beautiful introduction to complex analysis [Nee97]. Other good introductions are §3.4 and Chapter 4 of [Fis99], or Chapter VIII of [Lan85].

Notes: ....

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[7]Given any two points $x, y \in \mathbb{U}$, if we take a string and pin one end at x and the other end at y, then the string determines a *path* from x to y. The term **simply connected** means this: if α and β are two such paths connecting x to y, it is possible to *continuously deform* α into β, meaning that we can push the "α string" into the "β string" without pulling out the pins at the endpoints.

Heuristically speaking, a subset $\mathbb{U} \subset \mathbb{R}^2$ is simply connected if it has no "holes" in it. For example, the **disk** is simply connected, and so is the **square**. However, the **annulus** is *not* simply connected. Nor is the **punctured plane** $\mathbb{R}^2 \setminus \{(0,0)\}$.

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3 Waves and Signals

3.1 The Laplacian and Spherical Means

Prerequisites: §1.1, §1.2 **Recommended:** §2.2

Let $u : \mathbb{R}^D \longrightarrow \mathbb{R}$ be a function of D variables. Recall that the **Laplacian** of u is defined:

$$\boxed{\triangle u \;=\; \partial_1^2\, u + \partial_2^2\, u + \ldots \partial_D^2\, u}$$

In this section, we will show that $\triangle u(\mathbf{x})$ measures the discrepancy between $u(\mathbf{x})$ and the 'average' of u in a small neighbourhood around $\mathbf{x}$.

Let $\mathbb{S}(\epsilon)$ be the "sphere" of radius ϵ around 0. For example:

- If $D = 1$, then $\mathbb{S}(\epsilon)$ is just a set with two points: $\mathbb{S}(\epsilon) = \{-\epsilon, +\epsilon\}$.
- If $D = 2$, then $\mathbb{S}(\epsilon)$ is the *circle* of radius ϵ: $\mathbb{S}(\epsilon) = \left\{(x, y) \in \mathbb{R}^2 \;;\; x^2 + y^2 = \epsilon^2\right\}$
- If $D = 3$, then $\mathbb{S}(\epsilon)$ is the 3-dimensional spherical shell of radius ϵ:

$$\mathbb{S}(\epsilon) = \left\{(x, y, z) \in \mathbb{R}^3 \;;\; x^2 + y^2 + z^2 = \epsilon^2\right\}.$$

- More generally, for any dimension D, $\mathbb{S}(\epsilon) = \left\{(x_1, x_2, \ldots, x_D) \in \mathbb{R}^D \;;\; x_1^2 + x_2^2 + \ldots + x_D^2 = \epsilon^2\right\}$.

Let A_ϵ be the "surface area" of the sphere. For example:

- If $D = 1$, then $\mathbb{S}(\epsilon) = \{-\epsilon, +\epsilon\}$ is a finite set, with two points, so we say $A_\epsilon = 2$.
- If $D = 2$, then $\mathbb{S}(\epsilon)$ is the circle of radius ϵ; the *perimeter* of this circle is $2\pi\epsilon$, so we say $A_\epsilon = 2\pi\epsilon$.
- If $D = 3$, then $\mathbb{S}(\epsilon)$ is the sphere of radius ϵ, which has *surface area* $4\pi\epsilon^2$.

Let $\mathbf{M}_\epsilon\, f(0) := \dfrac{1}{A_\epsilon}\displaystyle\int_{\mathbb{S}(\epsilon)} f(\mathbf{s})\; d\mathbf{s}$; then $\mathbf{M}_\epsilon\, f(0)$ is the *average value* of $f(\mathbf{s})$ over all $\mathbf{s}$ on the surface of the ϵ-radius sphere around 0, which is called the **spherical mean** of f at 0. The interpretation of the integral sign "$\int$" depends on the dimension D of the space. For example,"$\int$" represents a *surface integral* if $D = 3$, a *line integral* if $D = 2$, and simple two-point sum if $D = 1$. Thus:

- If $D = 1$, then $\mathbb{S}(\epsilon) = \{-\epsilon, +\epsilon\}$, so that $\displaystyle\int_{\mathbb{S}(\epsilon)} f(\mathbf{s})\; d\mathbf{s} \;=\; f(\epsilon) + f(-\epsilon)$; thus,

$$\mathbf{M}_\epsilon\, f \;=\; \frac{f(\epsilon) + f(-\epsilon)}{2}.$$

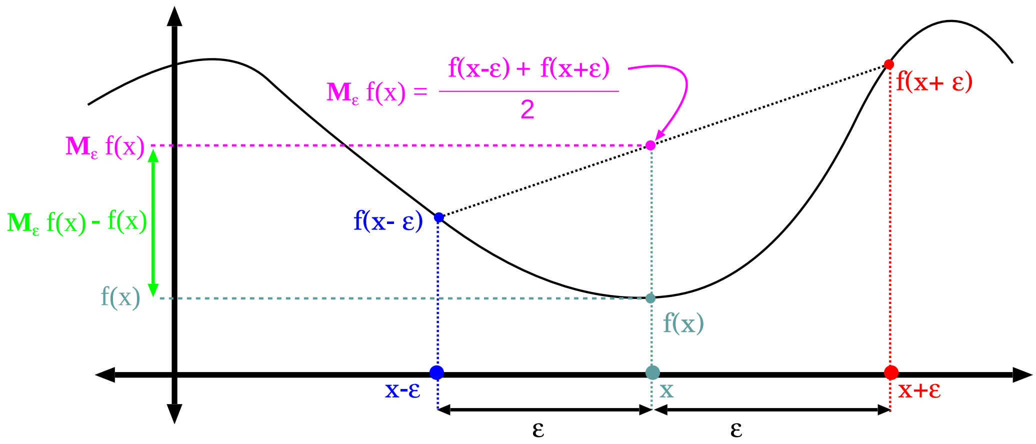

Figure 3.1: Local averages: $f(x)$ vs. $\mathbf{M}_\epsilon\, f(x) := \frac{f(x-\epsilon)+f(x+\epsilon)}{2}$.

- If $D = 2$, then any point on the circle has the form $\Big(\epsilon\cos(\theta),\ \epsilon\sin(\theta)\Big)$ for some angle $\theta \in [0, 2\pi)$. Thus, $\displaystyle\int_{\mathbb{S}(\epsilon)} f(\mathbf{s})\ d\mathbf{s} \;=\; \int_0^{2\pi} f\Big(\epsilon\cos(\theta),\ \epsilon\sin(\theta)\Big)\ \epsilon\, d\theta$, so that

$$\mathbf{M}_\epsilon\, f \quad=\quad \frac{1}{2\pi\epsilon}\int_0^{2\pi} f\Big(\epsilon\cos(\theta),\ \epsilon\sin(\theta)\Big)\ \epsilon\, d\theta \quad=\quad \frac{1}{2\pi}\int_0^{2\pi} f\Big(\epsilon\cos(\theta),\ \epsilon\sin(\theta)\Big)\ d\theta,$$

Likewise, for any $\mathbf{x} \in \mathbb{R}^D$, we define $\mathbf{M}_\epsilon\, f(\mathbf{x}) := \dfrac{1}{A_\epsilon}\displaystyle\int_{\mathbb{S}(\epsilon)} f(\mathbf{x}+\mathbf{s})\ d\mathbf{s}$ to be the average value of f over an ϵ-radius sphere around $\mathbf{x}$. Suppose $f : \mathbb{R}^D \longrightarrow \mathbb{R}$ is a smooth scalar field, and $\mathbf{x} \in \mathbb{R}^D$. One interpretation of the Laplacian is this: $\triangle f(\mathbf{x})$ measures the disparity between $f(\mathbf{x})$ and the *average value* of f in the immediate vicinity of $\mathbf{x}$. This is the meaning of the next theorem:

Theorem 3.1:

(a) *If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth scalar field, then* (as shown in Figure 3.1), *for any $x \in \mathbb{R}$,*

$$\triangle f(x) \quad=\quad \lim_{\epsilon\to 0}\frac{C}{\epsilon^2}\Big[\mathbf{M}_\epsilon\, f(\mathbf{x}) \;-\; f(\mathbf{x})\Big] \quad=\quad \lim_{\epsilon\to 0}\frac{2}{\epsilon^2}\left[\frac{f(x-\epsilon)+f(x+\epsilon)}{2} \;-\; f(x)\right].$$

(b) *If $f : \mathbb{R}^D \longrightarrow \mathbb{R}$ is a smooth scalar field, then for any $\mathbf{x} \in \mathbb{R}^D$,*

$$\triangle f(\mathbf{x}) \quad=\quad \lim_{\epsilon\to 0}\frac{C}{\epsilon^2}\Big[\mathbf{M}_\epsilon\, f(\mathbf{x}) \;-\; f(\mathbf{x})\Big] \quad=\quad \lim_{\epsilon\to 0}\frac{C}{\epsilon^2}\left[\frac{1}{A_\epsilon}\int_{\mathbb{S}(\epsilon)} f(\mathbf{x}+\mathbf{s})\ d\mathbf{s} \;-\; f(\mathbf{x})\right]$$

(Here C is a constant determined by the dimension D).

Proof: **(a)** Using *Taylor's theorem* (from first-year calculus), we have:

$$f(x+\epsilon) \quad = \quad f(x) + \ \epsilon f'(x) + \ \frac{\epsilon^2}{2} f''(x) + \ \mathcal{O}(\epsilon^3)$$
$$\text{and} \quad f(x-\epsilon) \quad = \quad f(x) - \ \epsilon f'(x) + \ \frac{\epsilon^2}{2} f''(x) + \ \mathcal{O}(\epsilon^3).$$

Here, $f'(x) = \partial_x f(x)$ and $f''(x) = \partial_x^2 f(x)$. The expression "$\mathcal{O}(\epsilon)$" means "some function (we don't care which one) such that $\lim_{\epsilon \to 0} \mathcal{O}(\epsilon) = 0$". Likewise, "$\mathcal{O}(\epsilon^3)$" means "some function (we don't care which one) such that $\lim_{\epsilon \to 0} \frac{\mathcal{O}(\epsilon)}{\epsilon^2} = 0$." Summing these two equations, we get

$$f(x+\epsilon) + f(x-\epsilon) \quad = \quad 2f(x) \ + \ \epsilon^2 \cdot f''(x) \ + \ \mathcal{O}(\epsilon^3).$$

Thus,

$$\frac{f(x-\epsilon) - 2f(x) + f(x+\epsilon)}{\epsilon^2} \quad = \quad f''(x) + \mathcal{O}(\epsilon).$$

[because $\mathcal{O}(\epsilon^3)/\epsilon^2 = \mathcal{O}(\epsilon)$.] Now take the limit as $\epsilon \to 0$, to get

$$\lim_{\epsilon \to 0} \frac{f(x-\epsilon) - 2f(x) + f(x+\epsilon)}{\epsilon^2} \quad = \quad \lim_{\epsilon \to 0} f''(x) + \mathcal{O}(\epsilon) \quad = \quad f''(x) \quad = \quad \triangle f(x),$$

as desired.

(b) Define the **Hessian derivative matrix** of f at $\mathbf{x}$:

$$\mathsf{D}^2 f(\mathbf{x}) = \begin{bmatrix} \partial_1^2 f & \partial_1\partial_2 f & \ldots & \partial_1\partial_D f \\ \partial_2\partial_1 f & \partial_2^2 f & \ldots & \partial_2\partial_D f \\ \vdots & \vdots & \ddots & \vdots \\ \partial_D\partial_1 f & \partial_D\partial_2 f & \ldots & \partial_D^2 f \end{bmatrix}$$

Then, for any $\mathbf{s} \in \mathbb{S}(\epsilon)$, the *Multivariate Taylor's theorem* (from vector calculus) says:

$$f(\mathbf{x}+\mathbf{s}) \quad = \quad f(\mathbf{x}) + \langle \mathbf{s}, \nabla f(\mathbf{x}) \rangle \quad + \quad \frac{1}{2} \left\langle \mathbf{s}, \ \mathsf{D}^2 f(\mathbf{x}) \cdot \mathbf{s} \right\rangle \quad + \quad \mathcal{O}(\epsilon^3).$$

Now, if $\mathbf{s} = (s_1, s_2, \ldots, s_D)$, then $\left\langle \mathbf{s}, \ \mathsf{D}^2 f(\mathbf{x}) \cdot \mathbf{s} \right\rangle \ = \ \sum_{c,d=1}^{D} s_c \cdot s_d \cdot \partial_c\partial_d f(\mathbf{x})$. Thus,

$$\begin{aligned} &\int_{\mathbb{S}(\epsilon)} f(\mathbf{x}+\mathbf{s}) \ d\mathbf{s} \\ &= \int_{\mathbb{S}(\epsilon)} f(\mathbf{x}) \ d\mathbf{s} \quad + \quad \int_{\mathbb{S}(\epsilon)} \langle \mathbf{s}, \nabla f(\mathbf{x}) \rangle \ d\mathbf{s} \quad + \quad \frac{1}{2} \int_{\mathbb{S}(\epsilon)} \left\langle \mathbf{s}, \mathsf{D}^2 f(\mathbf{x}) \cdot \mathbf{s} \right\rangle \ + \ \int_{\mathbb{S}(\epsilon)} \mathcal{O}(\epsilon^3) \ d\mathbf{s} \\ &= A_\epsilon f(\mathbf{x}) \quad + \quad \left\langle \nabla f(\mathbf{x}), \int_{\mathbb{S}(\epsilon)} \mathbf{s} \ d\mathbf{s} \right\rangle \quad + \quad \frac{1}{2} \int_{\mathbb{S}(\epsilon)} \left(\sum_{c,d=1}^{D} s_c s_d \cdot \partial_c\partial_d f(\mathbf{x}) \right) d\mathbf{s} \ + \ \mathcal{O}(\epsilon^{D+2}) \end{aligned}$$

$$= A_\epsilon f(\mathbf{x}) \quad + \quad \underbrace{\langle \nabla f(\mathbf{x}),\ \mathbf{0}\rangle}_{(*)} \quad + \quad \frac{1}{2}\sum_{c,d=1}^{D}\left(\partial_c\,\partial_d\, f(\mathbf{x})\cdot\left(\int_{\mathbb{S}(\epsilon)} s_c s_d\ d\mathbf{s}\right)\right) \quad + \quad \mathcal{O}(\epsilon^{D+2})$$

$$= A_\epsilon f(\mathbf{x}) \quad + \quad \underbrace{\frac{1}{2}\sum_{d=1}^{D}\left(\partial_d^2\, f(\mathbf{x})\cdot\left(\int_{\mathbb{S}(\epsilon)} s_d^2\ d\mathbf{s}\right)\right)}_{(\dagger)} \quad + \quad \mathcal{O}(\epsilon^{D+2})$$

$$= A_\epsilon f(\mathbf{x}) \quad + \quad \frac{1}{2}\triangle f(\mathbf{x})\cdot\epsilon^{D+1}K \quad + \quad \mathcal{O}(\epsilon^{D+2}),$$

where $K := \int_{\mathbb{S}(\epsilon)} s_1^2\ d\mathbf{s}$. Now, $A_\epsilon = A_1\cdot\epsilon^{D-1}$. Here, $(*)$ is because $\int_{\mathbb{S}(\epsilon)} \mathbf{s}\ d\mathbf{s} = \mathbf{0}$, because the centre-of-mass of a sphere is at its centre, namely $\mathbf{0}$. $(\dagger)$ is because, if $c,d\in[1...D]$, and $c\neq d$, then $\int_{\mathbb{S}(\epsilon)} s_c s_d\ d\mathbf{s} = 0$ (**Exercise 3.1**)[Hint: Use symmetry]. Thus,

$$\frac{1}{A_\epsilon}\int_{\mathbb{S}(\epsilon)} f(\mathbf{x}+\mathbf{s})\ d\mathbf{s} \quad - \quad f(\mathbf{x}) \quad = \quad \frac{\epsilon^2 K}{2A_1}\triangle f(\mathbf{x}) + \mathcal{O}(\epsilon^3).$$

Divide both sides by ϵ^2 and multiply by $C = \dfrac{2A_1}{K}$, and take the limit as $\epsilon\to 0$. □

Exercise 3.2 Let $f:\mathbb{R}^D\longrightarrow\mathbb{R}$ be a smooth scalar field, such that $\mathbf{M}_\epsilon f(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x}\in\mathbb{R}^D$. Show that f is harmonic.

3.2 The Wave Equation

Prerequisites: §3.1

3.2(a) ...in one dimension: the string

We want to mathematically describe vibrations propagating through a stretched elastic cord. We will represent the cord with a one-dimensional domain $\mathbb{X}$; either $\mathbb{X}=[0,L]$ or $\mathbb{X}=\mathbb{R}$. We will make several simplifying assumptions:

(W1) The cord has uniform thickness and density. Thus, there is a constant *linear density* $\rho>0$, so that a cord-segment of length ℓ has mass $\rho\ell$.

(W2) The cord is *perfectly elastic*; meaning that it is infinitely flexible and does not resist bending in any way. Likewise, there is no air friction to resist the motion of the cord.

(W3) The only force acting on the cord is *tension*, which is force of magnitude T pulling the cord to the right, balanced by an equal but opposite force of magnitude $-T$ pulling the cord to the left. These two forces are in balance, so the cord exhibits no horizontal

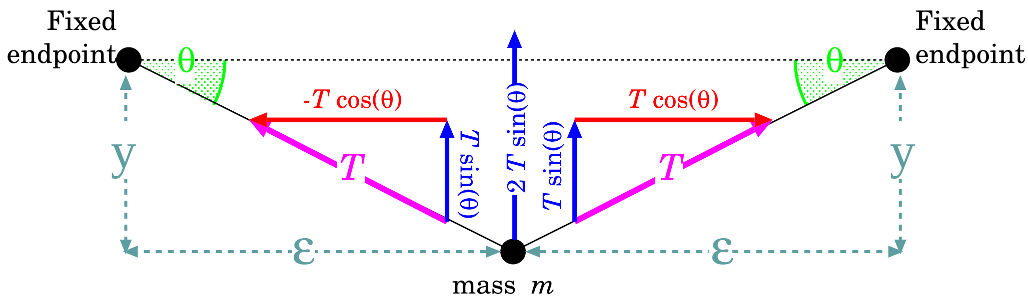

Figure 3.2: A bead on a string

motion. The tension T must be constant along the whole length of the cord. Thus, the equilibrium state for the cord is to be perfectly straight. Vibrations are deviations from this straight position.[1]

(W4) The vibrational motion of the cord is entirely *vertical*; there is no horizontal component to the vibration. Thus, we can describe the motion using a scalar-valued function $u(x,t)$, where $u(x,t)$ is the vertical displacement of the cord (from its flat equilibrium) at point x at time t. We assume that $u(x,t)$ is relatively small relative to the length of the cord, so that the cord is not significantly stretched by the vibrations[2].

For simplicity, let's first imagine a single bead of mass m suspended at the midpoint of a (massless) elastic cord of length 2ϵ, stretched between two endpoints. Suppose we displace the bead by a distance y from its equilibrium, as shown in Figure 3.2. The tension force T now pulls the bead diagonally towards each endpoint with force T. The horizontal components of the two tension forces are equal and opposite, so they cancel, so the bead experiences no net horizontal force. Suppose the cord makes an angle θ with the horizontal; then the vertical component of each tension force is $T\sin(\theta)$, so the total vertical force acting on the bead is $2T\sin(\theta)$. But $\theta = \arctan(\epsilon/y)$ by the geometry of the triangles in Figure 3.2, so $\sin(\theta) = \frac{y}{\sqrt{y^2+\epsilon^2}}$. Thus, the vertical force acting on the bead is

$$F \quad = \quad 2T\sin(\theta) \quad = \quad \frac{2Ty}{\sqrt{y^2+\epsilon^2}} \tag{3.1}$$

Now we return to our original problem of the vibrating string. Imagine that we replace the string with a 'necklace' made up of small beads of mass m separated by massless elastic

[1] We could also incorporate the force of gravity as a constant downward force. In this case, the equilibrium position for the cord is to sag downwards in a 'catenary' curve. Vibrations are then deviations from this curve. This doesn't change the mathematics of this derivation, so we will assume for simplicity that gravity is absent and the cord is straight.

[2] If $u(x,t)$ was large, then the vibrations stretch the cord, and a *restoring force* acts against this stretching, as described by *Hooke's Law*. By assuming that the vibrations are small, we are assuming we can neglect Hooke's Law.

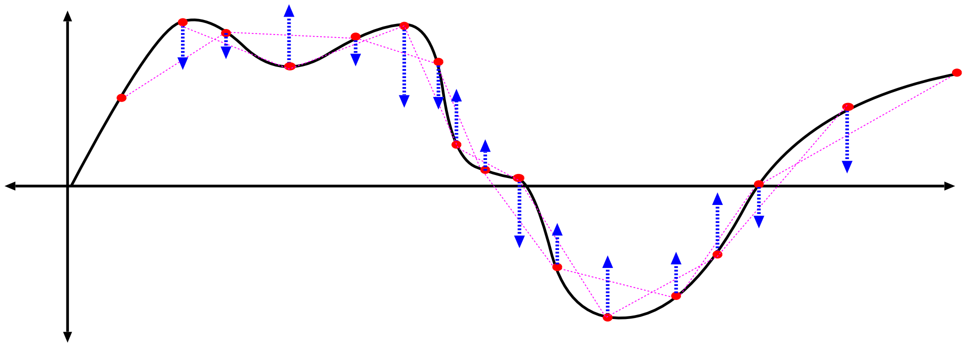

Figure 3.3: Each bead feels a negative force proportional to its deviation from the local average.

strings of length ϵ. Each of these beads, in isolation, behaves like the 'bead on a string' in Figure 3.2. However, now, the vertical displacement of each bead is not computed relative to the horizontal, but instead relative to the *average height* of the two neighbouring beads. Thus, in eqn.(3.1), we set $y := u(x) - \mathbf{M}_\epsilon\, u(x)$, where $u(x)$ is the height of bead x, and where $\mathbf{M}_\epsilon\, u := \frac{1}{2}[u(x-\epsilon)+u(x+\epsilon)]$ is the average of its neighbours. Substituting this into eqn.(3.1), we get

$$F_\epsilon(x) \quad = \quad \frac{2T[u(x) - \mathbf{M}_\epsilon\, u(x)]}{\sqrt{[u(x) - \mathbf{M}_\epsilon\, u(x)]^2 + \epsilon^2}} \tag{3.2}$$

(Here, the "ϵ" subscript in "F_ϵ" is to remind us that this is just an ϵ-bead approximation of the real string). Each bead represents a length-ϵ segment of the original string, so if the string has linear density ρ, then each bead must have mass $m_\epsilon := \rho\epsilon$. Thus, by Newton's law, the vertical acceleration of bead x must be

$$\begin{aligned} a_\epsilon(x) \quad &= \quad \frac{F_\epsilon(x)}{m_\epsilon} \quad = \quad \frac{2T[u(x) - \mathbf{M}_\epsilon\, u(x)]}{\rho\,\epsilon\,\sqrt{[u(x) - \mathbf{M}_\epsilon\, u(x)]^2 + \epsilon^2}} \\ &= \quad \frac{2T[u(x) - \mathbf{M}_\epsilon\, u(x)]}{\rho\,\epsilon^2\,\sqrt{[u(x) - \mathbf{M}_\epsilon\, u(x)]^2/\epsilon^2 + 1}} \end{aligned} \tag{3.3}$$

Now, we take the limit as $\epsilon \to 0$, to get the vertical acceleration of the string at x:

$$\begin{aligned} a(x) \quad &= \quad \lim_{\epsilon\to 0} a_\epsilon(x) \quad = \quad \frac{T}{\rho}\lim_{\epsilon\to 0}\frac{2}{\epsilon^2}\left[u(x) - \mathbf{M}_\epsilon\, u(x)\right] \cdot \lim_{\epsilon\to 0}\frac{1}{\sqrt{[u(x) - \mathbf{M}_\epsilon\, u(x)]^2/\epsilon^2 + 1}} \\ &\underset{(*)}{=} \quad \frac{T}{\rho}\,\partial_x^2\, u(x)\frac{1}{\lim_{\epsilon\to 0}\sqrt{\epsilon^2\cdot\partial_x^2\, u(x)^2 + 1}} \quad \underset{(\dagger)}{=} \quad \frac{T}{\rho}\,\partial_x^2\, u(x). \end{aligned} \tag{3.4}$$

Here, $(*)$ is because Theorem 3.1(a) on page 42 says that $\lim_{\epsilon\to 0}\frac{2}{\epsilon^2}[u(x) - \mathbf{M}_\epsilon\, u(x)] = \partial_x^2\, u(x)$. Finally, $(\dagger)$ is because, for any value of $u'' \in \mathbb{R}$, we have $\lim_{\epsilon\to 0}\sqrt{\epsilon^2 u'' + 1} = 1$. We conclude that

$$a(x) \quad = \quad \frac{T}{\rho}\,\partial_x^2\, u(x). \quad = \quad \lambda^2\,\partial_x^2\, u(x),$$

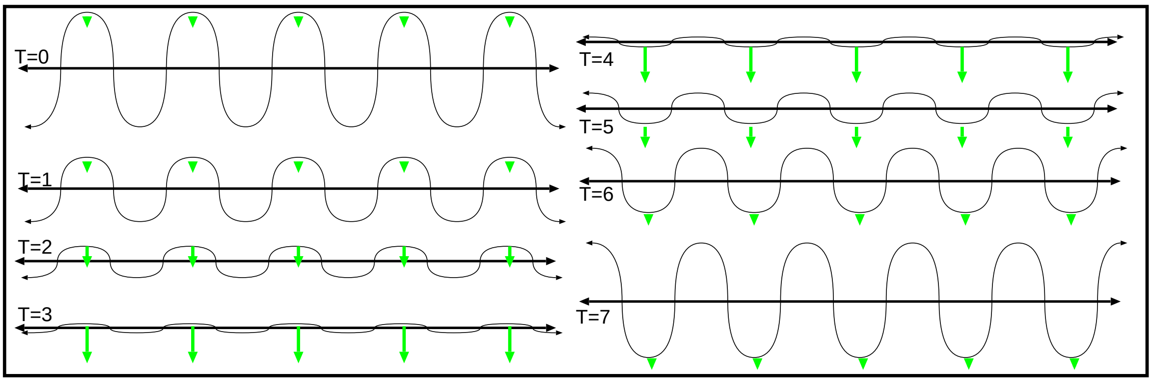

Figure 3.4: A one-dimensional standing wave.

where $\lambda := \sqrt{T/\rho}$. Now, the position (and hence, velocity and acceleration) of the cord is changing in time. Thus, a and u are functions of t as well as x. And of course, the acceleration $a(x,t)$ is nothing more than the second derivative of u with respect to t. Hence we have the **one-dimensional Wave Equation**:

$$\boxed{\partial_t^2 u(x,t) \quad = \quad \lambda^2 \cdot \partial_x^2 u(x,t).}$$

This equation describes the propagation of a transverse wave along an idealized string, or electrical pulses propagating in a wire.

Example 3.2: Standing Waves

(a) Suppose $\lambda^2 = 4$, and let $u(x;t) = \sin(3x) \cdot \cos(6t)$. Then u satisfies the Wave Equation and describes a *standing wave* with a *temporal frequency* of 6 and a *wave number* (or *spatial frequency*) of 3. (See Figure 3.4)

(b) More generally, fix $\omega > 0$; if $u(x;t) = \sin(\omega \cdot x) \cdot \cos(\lambda \cdot \omega \cdot t)$, Then u satisfies the Wave Equation and describes a *standing wave* of *temporal frequency* $\lambda \cdot \omega$ and *wave number* ω.

Exercise 3.3 Verify examples (a) and (b). ♢

Example 3.3: Travelling Waves

(a) Suppose $\lambda^2 = 4$, and let $u(x;t) = \sin(3x - 6t)$. Then u satisfies the Wave Equation and describes a *sinusoidal travelling wave* with *temporal frequency* 6 and *wave number* 3. The wave crests move rightwards along the cord with velocity 2.

(Figure 3.5A).

(b) More generally, fix $\omega \in \mathbb{R}$ and let $u(x;t) = \sin(\omega \cdot x - \lambda \cdot \omega \cdot t)$. Then u satisfies the Wave Equation and describes a sinusoidal *sinusoidal travelling wave* of *wave number* ω. The wave crests move rightwards along the cord with velocity λ.

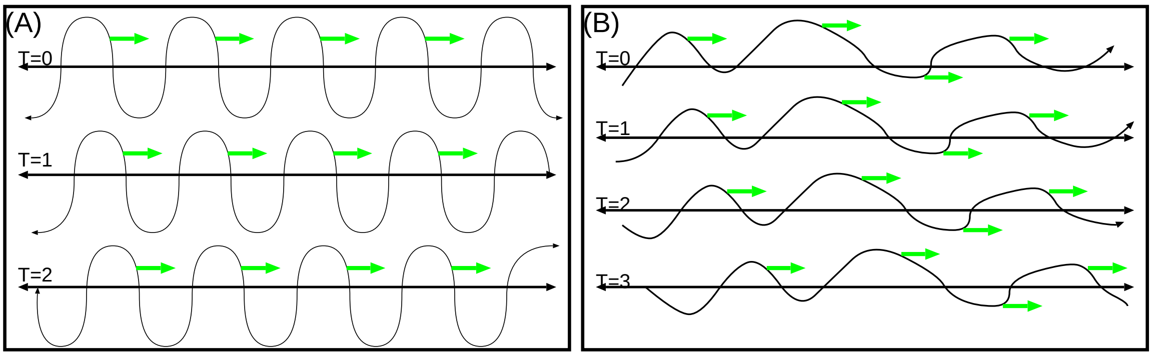

Figure 3.5: **(A)** A one-dimensional sinusoidal travelling wave. **(B)** A general one-dimensional travelling wave.

(c) More generally, suppose that f is any function of one variable, and define $u(x;t) = f(x - \lambda \cdot t)$. Then u satisfies the Wave Equation and describes a *travelling wave*, whose shape is given by f, and which moves rightwards along the cord with velocity λ (see Figure 3.5B).

Exercise 3.4 Verify examples (a),(b), and (c). ♢

Exercise 3.5 According to Example 3.3(c), you can turn any function into a travelling wave. Can you turn any function into a standing wave? Why or why not?

3.2(b) ...in two dimensions: the drum

Now, suppose $D = 2$, and imagine a two-dimensional "rubber sheet". Suppose $u(x, y; t)$ is the the vertical displacement of the rubber sheet at the point $(x, y) \in \mathbb{R}^2$ at time t. To derive the two-dimensional wave equation, we approximate this rubber sheet as a two-dimensional 'mesh' of tiny beads connected by massless, tense elastic strings of length ϵ. Each bead (x, y) feels a net vertical force $F = F_x + F_y$, where F_x is the vertical force arising from the tension in the x direction, and F_y is the vertical force from the tension n the y direction. Both of these are expressed by a formula similar to eqn.(3.2). Thus, if bead (x, y) has mass m_ϵ, then it experiences acceleration $a = F/m_\epsilon = F_x/m_\epsilon + F_y/m_\epsilon = a_x + a_y$, where $a_x := F_x/m_\epsilon$ and $a_y := F_y/m_\epsilon$, and each of these is expressed by a formula similar to eqn.(3.3). Taking the limit as $\epsilon \to 0$ as in eqn.(3.4), we deduce that

$$a(x, y) \quad = \quad \lim_{\epsilon \to 0} a_{x,\epsilon}(x, y) \; + \; \lim_{\epsilon \to 0} a_{y,\epsilon}(x, y) \quad = \quad \lambda^2 \, \partial_x^2 \, u(x, y) + \lambda^2 \, \partial_y^2 \, u(x, y),$$

where λ is a constant determined by the density and tension of the rubber membrane. Again, we recall that u and a are also functions of time, and that $a(x, y; t) = \partial_t^2 \, u(x, y; t)$. Thus, we have the **two-dimensional Wave Equation:**

$$\boxed{\partial_t^2 \, u(x, y; t) \quad = \quad \lambda^2 \cdot \partial_x^2 \, u(x, y; t) \quad + \quad \lambda^2 \cdot \partial_y^2 \, u(x, y; t)} \qquad (3.5)$$

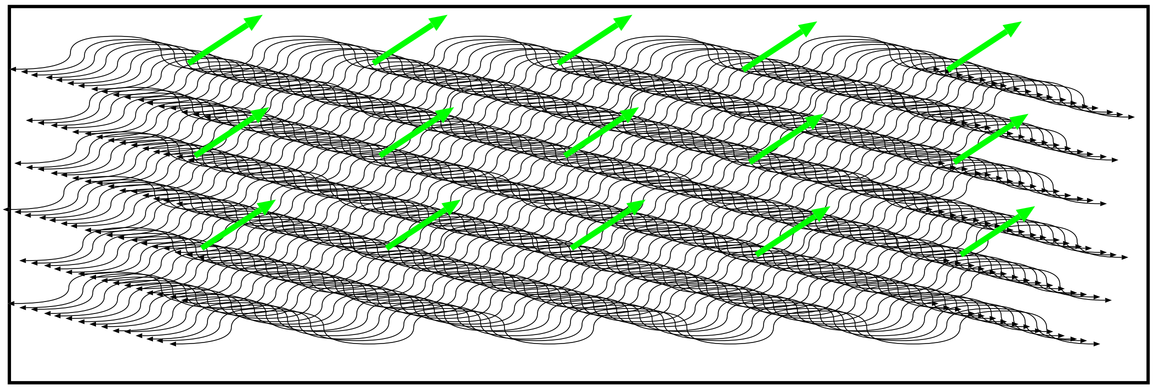

Figure 3.6: A two-dimensional travelling wave.

or, more abstractly:

$$\boxed{\partial_t^2 u \quad = \quad \lambda^2 \cdot \triangle u.}$$

This equation describes the propagation of wave energy through any medium with a linear restoring force. For example:

- Transverse waves on an idealized rubber sheet.
- Ripples on the surface of a pool of water.
- Acoustic vibrations on a drumskin.

Example 3.4: Two-dimensional Standing Waves

(a) Suppose $\lambda^2 = 9$, and let $u(x, y; t) = \sin(3x) \cdot \sin(4y) \cdot \cos(15t)$. This describes a two-dimensional standing wave with temporal frequency 15.

(b) More generally, fix $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{R}^2$ and let $\Omega = \|\omega\|_2 = \sqrt{\omega_1^2 + \omega_2^2}$.

$$u(\mathbf{x}; t) \quad = \quad \sin(\omega_1 x) \cdot \sin(\omega_2 y) \cdot \cos(\lambda \cdot \Omega t)$$

satisfies the 2-dimensional Wave Equation and describes a standing wave with temporal frequency $\lambda \cdot \Omega$.

(c) Even more generally, fix $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{R}^2$ and let $\Omega = \|\omega\|_2 = \sqrt{\omega_1^2 + \omega_2^2}$, as before.

$$\begin{aligned} \text{Let } SC_1(x) &= \text{either } \sin(x) \text{ or } \cos(x); \\ \text{let } SC_2(y) &= \text{either } \sin(y) \text{ or } \cos(y); \\ \text{and let } SC_t(t) &= \text{either } \sin(t) \text{ or } \cos(t). \end{aligned}$$

Then

$$u(\mathbf{x}; t) \quad = \quad SC_1(\omega_1 x) \cdot SC_2(\omega_2 y) \cdot SC_t(\lambda \cdot \Omega t)$$

satisfies the 2-dimensional Wave Equation and describes a standing wave with temporal frequency $\lambda \cdot \Omega$.

Exercise 3.6 Check examples (a), (b) and (c) $\diamondsuit$

Example 3.5: Two-dimensional Travelling Waves

(a) Suppose $\lambda^2 = 9$, and let $u(x, y; t) = \sin(3x + 4y + 15t)$. Then u satisfies the two-dimensional Wave Equation, and describes a sinusoidal travelling wave with **wave vector** $\boldsymbol{\omega} = (3, 4)$ and temporal frequency 15. (see Figure 3.6).

(b) More generally, fix $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{R}^2$ and let $\Omega = \|\omega\|_2 = \sqrt{\omega_1^2 + \omega_2^2}$. Then

$$u(\mathbf{x}; t) = \sin\Big(\omega_1 x + \omega_2 y + \lambda \cdot \Omega t\Big) \quad \text{and} \quad v(\mathbf{x}; t) = \cos\Big(\omega_1 x + \omega_2 y + \lambda \cdot \Omega t\Big)$$

both satisfy the two-dimensional Wave Equation, and describe sinusoidal travelling waves with **wave vector** $\boldsymbol{\omega}$ and temporal frequency $\lambda \cdot \Omega$.

Exercise 3.7 Check examples (a) and (b) $\diamondsuit$

3.2(c) ...in higher dimensions:

The same reasoning applies for $D \geq 3$. For example, the 3-dimensional wave equation describes the propagation of (small amplitude[3]) sound-waves in air or water. In general, the Wave Equation takes the form

$$\partial_t^2 u = \lambda^2 \triangle u$$

where λ is some constant (determined by the density, elasticity, pressure, etc. of the medium) which describes the speed-of-propagation of the waves.

By a suitable choice of units, we can always assume that $\lambda = 1$. Hence, from now on, we will consider the simplest form of the **Wave Equation:**

$$\boxed{\partial_t^2 u = \triangle u}$$

For example, fix $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_D) \in \mathbb{R}^D$ and let $\Omega = \|\omega\|_2 = \sqrt{\omega_1^2 + \ldots + \omega_D^2}$. Then

$$u(\mathbf{x}; t) = \sin\Big(\omega_1 x_1 + \omega_2 x_2 + \ldots + \omega_D x_D + \Omega t\Big) = \sin\Big(\langle \boldsymbol{\omega}, \mathbf{x} \rangle + \lambda \cdot \Omega \cdot t\Big)$$

satisfies the D-dimensional Wave Equation and describes a transverse wave of with **wave vector** $\boldsymbol{\omega}$ propagating across D-dimensional space. **Exercise 3.8** Check this!

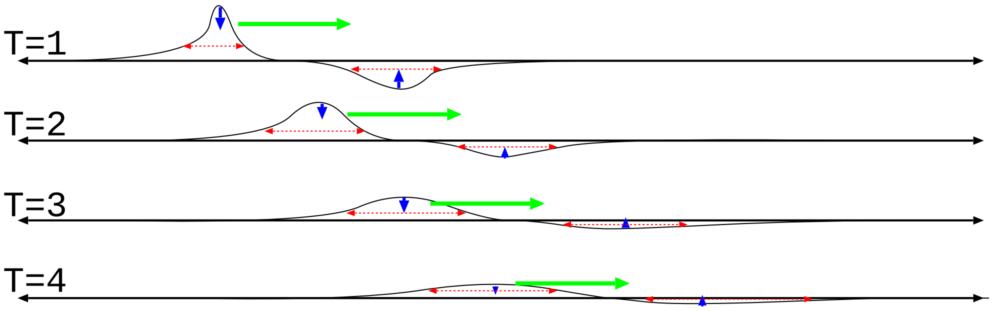

Figure 3.7: A solution to the Telegraph Equation propagates like a wave, but it also diffuses over time due to noise, and decays exponentially in magnitude due to 'leakage'.

3.3 The Telegraph Equation

Recommended: §3.2(a), §2.2(a)

Imagine a signal propagating through a medium with a linear restoring force (eg. an electrical pulse in a wire, a vibration on a string). In an ideal universe, the signal obeys the Wave Equation. However, in the real universe, *damping effects* interfere. First, energy might "leak" out of the system. For example, if a wire is imperfectly insulated, then current can leak out into surrounding space. Also, the signal may get blurred by noise or frictional effects. For example, an electric wire will pick up radio waves ("crosstalk") from other nearby wires, while losing energy to electrical resistance. A guitar string will pick up vibrations from the air, while losing energy to friction.

Thus, intuitively, we expect the signal to propagate like a wave, but to be gradually smeared out and attenuated by noise and leakage (Figure 3.7). The model for such a system is the **Telegraph Equation**:

$$\kappa_2 \partial_t^2 u + \kappa_1 \partial_t u + \kappa_0 u \quad = \quad \lambda \triangle u$$

(where $\kappa_2, \kappa_1, \kappa_0, \lambda > 0$ are constants).

Heuristically speaking, this equation is a "sum" of two equations. The first,

$$\kappa_2 \partial_t^2 u \quad = \quad \lambda_1 \triangle u$$

is a version of the Wave Equation, and describes the "ideal" signal, while the second,

$$\kappa_1 \partial_t u \quad = \quad -\kappa_0 u + \lambda_2 \triangle u$$

describes energy lost due to leakage and frictional forces.

3.4 Practice Problems

1. By explicitly computing derivatives, show that the following functions satisfy the (one-dimensional) Wave Equation $\partial_t^2 u \;=\; \partial_x^2 u$.

[3]At large amplitudes, nonlinear effects become important and invalidate the physical argument used here.

(a) $u(x,t) = \sin(7x)\cos(7t)$.

(b) $u(x,t) = \sin(3x)\cos(3t)$.

(c) $u(x,t) = \frac{1}{(x-t)^2}$ (for $x \neq t$).

(d) $u(x,t) = (x-t)^2 - 3(x-t) + 2$.

(e) $v(x,t) = (x-t)^2$.

2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be any twice-differentiable function. Define $u : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by $u(x,t) := f(x-t)$, for all $(x,t) \in \mathbb{R} \times \mathbb{R}$.

   Does u satisfies the (one-dimensional) Wave Equation $\partial_t^2 u = \triangle u$? Justify your answer.

3. Let $u(x,t)$ be as in 1(a) and let $v(x,t)$ be as in 1(e), and suppose $w(x,t) = 3u(x,t) - 2v(x,t)$. Conclude that w also satisfies the Wave Equation, without explicitly computing any derivatives of w.

4. Suppose $u(x,t)$ and $v(x,t)$ are both solutions to the Wave equation, and $w(x,t) = 5u(x,t) + 2v(x,t)$. Conclude that w also satisfies the Wave Equation.

5. Let $u(x,t) = \int_{x-t}^{x+t} \cos(y)\ dy = \sin(x+t) - \sin(x-t)$. Show that u satisfies the (one-dimensional) Wave Equation $\partial_t^2 u = \triangle u$.

6. By explicitly computing derivatives, show that the following functions satisfy the (two-dimensional) Wave Equation $\partial_t^2 u = \triangle u$.

   (a) $u(x,y;t) = \sinh(3x) \cdot \cos(5y) \cdot \cos(4t)$.

   (b) $u(x,y;t) = \sin(x)\cos(2y)\sin(\sqrt{5}t)$.

   (c) $u(x,y;t) = \sin(3x-4y)\cos(5t)$.

Notes: ....
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4 Quantum Mechanics

4.1 Basic Framework

Prerequisites: §1.3, §2.2(b)

Near the beginning of the twentieth century, physicists realized that electromagnetic waves sometimes exhibited particle-like properties, as if light was composed of discrete 'photons'. In 1923, Louis de Broglie proposed that, conversely, particles of matter might have wave-like properties. This was confirmed in 1927 by C.J. Davisson and L.H. Germer, and independently, by G.P. Thompson, who showed that an electron beam exhibited an unmistakable diffraction pattern when scattered off a metal plate, as if the beam was composed of 'electron waves'. Systems with many interacting particles exhibit even more curious phenomena. *Quantum mechanics* is a theory which explains these phenomena.

We will not attempt here to provide a physical justification for quantum mechanics. Historically, quantum theory developed through a combination of vaguely implausible physical analogies and wild guesses motivated by inexplicable empirical phenomena. By now, these analogies and guesses have been overwhelmingly vindicated by experimental evidence. The best justification for quantum mechanics is that it 'works', by which we mean that its theoretical predictions match all available empirical data with astonishing accuracy.

Unlike the Heat Equation in §2.2 and the Wave Equation in §3.2, we cannot derive quantum theory from 'first principles', because the postulates of quantum mechanics *are* the first principles. Instead, we will simply state the main assumptions of the theory, which are far from self-evident, but which we hope you will accept because of the weight of empirical evidence in their favour. Quantum theory describes any physical system via a *probability distribution* on a certain *statespace*. This probability distribution evolves over time; the evolution is driven by a potential energy function, as described by a partial differential equation called the *Schrödinger equation*. We will now examine each of these concepts in turn.

Statespace: A system of N interacting particles moving in 3 dimensional space can be completely described using the $3N$-dimensional **state space** $\mathbb{X} := \mathbb{R}^{3N}$. An element of $\mathbb{X}$ consists of list of N ordered triples:

$$\mathbf{x} \quad = \quad (x_{11}, x_{12}, x_{13};\ x_{21}, x_{22}, x_{23};\ \ldots x_{N1}, x_{N2}, x_{N3}) \quad \in \quad \mathbb{R}^{3N},$$

where (x_{11}, x_{12}, x_{13}) is ths spatial position of particle #1, (x_{21}, x_{22}, x_{23}) is ths spatial position of particle #2, and so on.

Example 4.1:

(a) *Single electron* A single electron is a one-particle system, so it would be represented using a 3-dimensional statespace $\mathbb{X} = \mathbb{R}^3$. If the electron was confined to a two-dimensional space (eg. a conducting plate), we would use $\mathbb{X} = \mathbb{R}^2$. If the electron was confined to a one-dimensional space (eg. a conducting wire), we would use $\mathbb{X} = \mathbb{R}$.

(b) *Hydrogen Atom:* The common isotope of hydrogen contains a single proton and a single electron, so it is a two-particle system, and would be represented using a 6-dimensional state space $\mathbb{X} = \mathbb{R}^6$. An element of $\mathbb{X}$ has the form $\mathbf{x} = (x_1^p, x_2^p, x_3^p;\ x_1^e, x_2^e, x_3^e)$, where (x_1^p, x_2^p, x_3^p) are the coordinates of the proton, and (x_1^e, x_2^e, x_3^e) are those of the electron. $\diamondsuit$

Readers familiar with classical mechanics may be wondering how *momentum* is represented in this statespace. Why isn't the statespace $6N$-dimensional, with 3 'position' and 3 *momentum* coordinates for each particle? The answer, as we will see later, is that the *momentum* of a quantum system is implicitly encoded in the wavefunction which describes its position (see §4.6).

Potential Energy: We define a **potential energy** (or **voltage**) function $V : \mathbb{X} \longrightarrow \mathbb{R}$, which describes which states are 'prefered' by the quantum system. Loosely speaking, the system will 'avoid' states of high potential energy, and 'seek' states of low energy. The voltage function is usually defined using reasoning familiar from 'classical' physics.

Example 4.2: Electron in ambient field

Imagine a single electron moving through an ambient electric field $\vec{\mathbf{E}}$. The statespace for this system is $\mathbb{X} = \mathbb{R}^3$, as in Example 6a. The potential function V is just the voltage of the electric field; in other words, V is any scalar function such that $-q_e \cdot \vec{\mathbf{E}} = \nabla V$, where q_e is the charge of the electron. For example:

(a) *Null field:* If $\vec{\mathbf{E}} \equiv 0$, then V will be a constant, which we can assume is zero: $V \equiv 0$.

(b) *Constant field:* If $\vec{\mathbf{E}} \equiv (E, 0, 0)$, for some constant $E \in \mathbb{R}$, then $V(x, y, z) = -q_e E x + c$, where c is an arbitrary constant, which we normally set to zero.

(c) *Coulomb field:* Suppose the electric field $\vec{\mathbf{E}}$ is generated by a (stationary) point charge Q at the origin. Let ϵ_0 be the 'permittivity of free space'. Then Coulomb's law says that the electric voltage is given by

$$V(\mathbf{x}) \quad := \quad \frac{q_e \cdot Q}{4\pi\epsilon_0 \cdot |\mathbf{x}|}, \qquad \text{for all } \mathbf{x} \in \mathbb{R}^3.$$

In SI units, $q_e \approx 1.60 \times 10^{-19}\,\mathtt{C}$, and $\epsilon_0 \approx 8.85 \times 10^{-12}\,\mathtt{C/N\,m^2}$. However, for simplicity, we will normally adopt 'atomic units' of charge and field strength, where $q_e = 1$ and $4\pi\epsilon_0 = 1$. Then the above expression becomes $V(\mathbf{x}) = Q/|\mathbf{x}|$.

(d) *Potential well:* Sometimes we confine the electron to some bounded region $\mathbb{B} \subset \mathbb{R}^3$, by setting the voltage equal to 'positive infinity' outside $\mathbb{B}$. For example, a low-energy electron in a cube made of conducting metal can move freely about the cube, but cannot leave[1] the cube. If the subset $\mathbb{B}$ represents the cube, then we define $V : \mathbb{X} \longrightarrow [0, \infty]$ by

$$V(\mathbf{x}) \quad = \quad \begin{cases} 0 & \text{if} \quad \mathbf{x} \in \mathbb{B}; \\ +\infty & \text{if} \quad \mathbf{x} \notin \mathbb{B}. \end{cases}$$

[1] 'Cannot leave' of course really means 'is very highly unlikely to leave'.

(if '$+\infty$' makes you uncomfortable, then replace it with some 'really big' number). $\diamondsuit$

Example 4.3: Hydrogen atom:

The system is an electron and a proton; the statespace of this system is $\mathbb{X} = \mathbb{R}^6$ as in Example 6b. Assuming there is no external electric field, the voltage function is defined

$$V(\mathbf{x}^p, \mathbf{x}^e) \quad := \quad \frac{q_e^2}{4\pi\epsilon_0 \cdot |\mathbf{x}^p - \mathbf{x}^e|}, \quad \text{for all } (\mathbf{x}^p, \mathbf{x}^e) \in \mathbb{R}^6.$$

where $\mathbf{x}^p$ is the position of the proton, $\mathbf{x}^e$ is the position of the electron, and q_e is the charge of the electron (which is also the charge of the proton, with reversed sign). If we adopt 'atomic' units where $q_e := 1$ and $4\pi\epsilon_0 = 1$, then this expression simplifies to

$$V(\mathbf{x}^p, \mathbf{x}^e) \quad := \quad \frac{1}{|\mathbf{x}^p - \mathbf{x}^e|}, \quad \text{for all } \ (\mathbf{x}^p, \mathbf{x}^e) \in \mathbb{R}^6, \qquad \diamondsuit$$

Probability and Wavefunctions: Our knowledge of the classical properties of a quantum system is inherently incomplete. All we have is a time-varying probability distribution $\rho : \mathbb{X} \times \mathbb{R} \longrightarrow [0, \infty)$ which describes where the particles are likely or unlikely to be at a given moment in time.

As time passes, the probability distribution ρ evolves. However, ρ itself cannot exhibit the 'wavelike' properties of a quantum system (eg. destructive interference), because ρ is a nonnegative function (and we need to add negative to positive values to get destructive interference). So, we introduce a complex-valued **wavefunction** $\omega : \mathbb{X} \times \mathbb{R} \longrightarrow \mathbb{C}$. The wavefunction ω determines ρ via the equation:

$$\rho_t(\mathbf{x}) \quad := \quad |\omega_t(\mathbf{x})|^2, \qquad \text{for all } \mathbf{x} \in \mathbb{X} \text{ and } t \in \mathbb{R}.$$

Now, ρ_t is supposed to be a probability density function, so ω_t must satisfy the condition

$$\int_{\mathbb{X}} |\omega_t(\mathbf{x})|^2 \, d\mathbf{x} \quad = \quad 1, \qquad \text{for all } t \in \mathbb{R}. \tag{4.1}$$

It is acceptable (and convenient) to relax condition (4.1), and instead simply require

$$\int_{\mathbb{X}} |\omega_t(\mathbf{x})|^2 \, d\mathbf{x} \quad = \quad W \ < \ \infty, \qquad \text{for all } t \in \mathbb{R}. \tag{4.2}$$

where W is some finite constant. In this case, we define $\rho_t(\mathbf{x}) := \frac{1}{W}|\omega_t(\mathbf{x})|^2$ for all $\mathbf{x} \in \mathbb{X}$. It follows that any physically meaningful solution to the Schrödinger equation must satisfy condition (4.2). This excludes, for example, solutions where the magnitude of the wavefunction grows exponentially in the $\mathbf{x}$ or t variables.

Condition (4.2) is usually expressed by saying that ω is *square-integrable*. Let $\mathbf{L}^2(\mathbb{X})$ denote the set of all square-integrable functions on $\mathbb{X}$. If $\omega \in \mathbf{L}^2(\mathbb{X})$, then the $\mathbf{L}^2$**-norm** of ω is defined

$$\|\omega\|_2 \quad := \quad \sqrt{\int_{\mathbb{X}} |\omega(\mathbf{x})|^2 \, d\mathbf{x}}.$$

Thus, a key postulate of quantum theory is:

Erwin Rudolf Josef Alexander Schrödinger
Born: August 12, 1887 in Erdberg, Austria
Died: January 4, 1961 in Vienna, Austria

Let $\omega : \mathbb{X} \times \mathbb{R} \longrightarrow \mathbb{C}$ be a wavefunction. To be physically meaningful, we must have $\omega_t \in \mathbf{L}^2(\mathbb{X})$ for all $t \in \mathbb{R}$. Furthermore, $\|\omega_t\|_2$ must be constant in time.

We refer the reader to § 7.2 on page 113 for more information on $\mathbf{L}^2$-norms and $\mathbf{L}^2$-spaces.

4.2 The Schrödinger Equation

Prerequisites: §4.1 **Recommended:** §5.2

The wavefunction ω evolves over time in response to the voltage field V. Let $\hbar$ be the 'rationalized' Planck constant

$$\hbar \quad := \quad \frac{h}{2\pi} \quad \approx \quad \frac{1}{2\pi} \times 6.6256 \times 10^{-34} \ \mathsf{J\,s} \quad \approx \quad 1.0545 \times 10^{-34} \ \mathsf{J\,s}.$$

Then the wavefunction's evolution is described by the **Schrödinger Equation:**

$$\mathbf{i}\hbar\, \partial_t\, \omega \quad = \quad \mathsf{H}\, \omega, \tag{4.3}$$

where H is a linear differential operator called the **Hamiltonian** operator. For wavefunctions on the 'position' statespace $\mathbb{X}$ from §4.1, with potential function $V : \mathbb{X} \longrightarrow \mathbb{R}$, the operator H is defined:

$$\mathsf{H}\, \omega_t(\mathbf{x}) \quad := \quad \frac{-\hbar^2}{2} \blacktriangle \omega_t(\mathbf{x}) \ + \ V(\mathbf{x}) \cdot \omega_t(\mathbf{x}), \qquad \text{for all } \mathbf{x} \in \mathbb{X} \text{ and } t \in \mathbb{R}. \tag{4.4}$$

Here, $\blacktriangle \omega_t$ is like the Laplacian of ω_t, except that the components for each particle are divided by the *rest mass* of that particle. Substituting eqn.(4.4) into eqn.(4.3), we get

$$\mathbf{i}\hbar\, \partial_t\, \omega \quad = \quad \frac{-\hbar^2}{2} \blacktriangle \omega \ + \ V \cdot \omega, \tag{4.5}$$

In 'atomic units', $\hbar = 1$, so the Schrödinger equation (4.5) becomes

$$\mathbf{i}\partial_t \, \omega_t(\mathbf{x}) \quad = \quad \frac{-1}{2} \, \blacktriangle \, \omega_t(\mathbf{x}) \; + \; V(\mathbf{x}) \cdot \omega_t(\mathbf{x}), \qquad \text{for all } \mathbf{x} \in \mathbb{X} \text{ and } t \in \mathbb{R}.$$

Example 4.4:

(a) *Free Electron:* Let $m_e \approx 9.11 \times 10^{-31}$ kg be the rest mass of an electron. A solitary electron in a null electric field (as in Example 4.2(a)) satisfies the *free Schrödinger equation:*

$$\mathbf{i}\hbar \, \partial_t \, \omega_t(\mathbf{x}) \quad = \quad \frac{-\hbar^2}{2\, m_e} \, \triangle \, \omega_t(\mathbf{x}). \tag{4.6}$$

(In this case $\blacktriangle = \frac{1}{m_e}\triangle$, and $V \equiv 0$ because the ambient field is null). In atomic units, we set $m_e := 1$ and $\hbar := 1$, so eqn.(4.6) becomes

$$\mathbf{i}\, \partial_t \, \omega \quad = \quad \frac{-1}{2} \, \triangle \, \omega \quad = \quad \frac{-1}{2} \left(\partial_1^2 \omega + \partial_2^2 \omega + \partial_3^2 \omega \right). \tag{4.7}$$

(b) *Electron vs. point charge:* Consider the *Coulomb* electric field, generated by a (stationary) point charge Q at the origin, as in Example 4.2(c). A solitary electron in this electric field satisfies the Schrödinger equation

$$\mathbf{i}\hbar \, \partial_t \, \omega_t(\mathbf{x}) \quad = \quad \frac{-\hbar^2}{2\, m_e} \, \triangle \, \omega_t(\mathbf{x}) \; + \; \frac{q_e \cdot Q}{4\pi\epsilon_0 \cdot |\mathbf{x}|} \, \omega_t(\mathbf{x}).$$

In atomic units, we have $m_e := 1$, $q_e := 1$, etc. Let $\widetilde{Q} = Q/q_e$ be the charge Q converted in units of electron charge. Then the previous expression simplifies to

$$\mathbf{i}\, \partial_t \, \omega_t(\mathbf{x}) \quad = \quad \frac{-1}{2} \, \triangle \, \omega_t(\mathbf{x}) \; + \; \frac{\widetilde{Q}}{|\mathbf{x}|} \, \omega_t(\mathbf{x}).$$

(c) *Hydrogen atom:* (see Example 4.3) An interacting proton-electron pair (in the absence of an ambient field) satisfies the two-particle Schrödinger equation

$$\mathbf{i}\hbar \, \partial_t \, \omega_t(\mathbf{x}^p, \mathbf{x}^e) \quad = \quad \frac{-\hbar^2}{2\, m_p} \triangle_p \, \omega_t(\mathbf{x}^p, \mathbf{x}^e) + \frac{-\hbar^2}{2\, m_e} \triangle_e \, \omega_t(\mathbf{x}^p, \mathbf{x}^e) + \frac{q_e^2 \cdot \omega_t(\mathbf{x}^p, \mathbf{x}^e)}{4\pi\epsilon_0 \cdot |\mathbf{x}^p - \mathbf{x}^e|}, \tag{4.8}$$

where $\triangle_p \, \omega := \partial_{x_1^p}^2 \omega + \partial_{x_2^p}^2 \omega + \partial_{x_3^p}^2 \omega$ is the Laplacian in the 'proton' position coordinates, and $m_p \approx 1.6727 \times 10^{-27}$ kg is the rest mass of a proton. Likewise, $\triangle_e \, \omega := \partial_{x_1^e}^2 \omega + \partial_{x_2^e}^2 \omega + \partial_{x_3^e}^2 \omega$ is the Laplacian in the 'electron' position coordinates, and m_e is the rest mass of the electron. In atomic units, we have $4\pi\epsilon_0 = 1$, $q_e = 1$, and $m_e = 1$. If $\widetilde{m}_p \approx 1836$ is the ratio of proton mass to electron mass, then $2\widetilde{m}_p \approx 3672$, and eqn.(4.8) becomes

$$\mathbf{i} \;\; \partial_t \, \omega_t(\mathbf{x}^p, \mathbf{x}^e) \quad = \quad \frac{-1}{3672} \, \triangle_p \; \omega_t(\mathbf{x}^p, \mathbf{x}^e) \; + \; \frac{-1}{2} \, \triangle_e \; \omega_t(\mathbf{x}^p, \mathbf{x}^e) \; + \; \frac{\omega_t(\mathbf{x}^p, \mathbf{x}^e)}{|\mathbf{x}^p - \mathbf{x}^e|}. \qquad \diamondsuit$$

4.3 Miscellaneous Remarks

Recommended: §4.2

Relativity: Readers familiar with special relativity will notice that the quantum mechanical framework is incompatible with the relativistic framework, for (at least) two reasons:

- *Space, Time, and simultaneity:* In quantum mechanics (as in classical physics), there is a clear distinction between space and time variables. The 'statespace' of a single quantum particle is $\mathbb{X} = \mathbb{R}^3$, and quantum 'space-time' is the 4-dimensional space $\mathbb{R}^3 \times \mathbb{R}$. The axis of 'time' is the last coordinate of this space. In relativity, there is no clear distinction between space and time. Relativistic 'space-time' is the 4-dimensional Lorenz space $\mathbb{R}^4$, in which the axis of 'time' depends upon the velocity of the observer.

  More generally, by defining the statespace of an N-particle quantum system to be $\mathbb{X} := \mathbb{R}^{3N}$, we are implicitly assuming that we can talk about the 'simultaneous state' of all particles at a particular moment in time. In special relativity, this doesn't make sense; simultaneity again depends upon the velocity of the observer.

- *Information propagation and particle velocity:* By using a potential function and associated Schrödinger equation to evolve a quantum system, we are implicitly assuming that particles instantaneously 'know' about one another's positions, no matter how far apart they are. In other words, in a quantum system, information propagates instaneously. In special relativity, however, information propagates at the speed of light (or slower). A particle cannot 'know' about anything happening outside of its light-cone.

  In particular, we will see that, in quantum systems, particles can have arbitrarily large 'velocities', and thus, can 'travel' very large distances in very short time[2]. However, in relativity, no particle can travel faster than the speed of light.

These difficulties are overcome using *relativistic quantum field theory*, but that is beyond the scope of this book. See [Ste95, Chap.7].

Spin: Some readers might be wondering about quantum 'spin' (which, confusingly, is *not* a kind of momentum). Spin is a quantum property with no classical analog, although it is described by analogy to classical angular momentum. Spin is observable by the way charged particles interact with magnetic fields. Each subatomic particle has a 'spin axis', which we can represent with an element of the unit sphere $\mathbb{S}$. Thus, if we wanted to include spin information in an N-particle quantum model, we would use the statespace $\mathbb{X} := \mathbb{R}^{3N} \times \mathbb{S}^N$. However, to keep things simple, we will not consider spin here. See [Boh79, Chap.17].

Quantum Indeterminacy: Quantum mechanics is notorious for 'indeterminacy', which refers to our inherently 'incomplete' knowledge of quantum systems, and their inherently 'random' evolution over time.

[2]I put the words 'velocity' and 'travel' in quotation marks because it is somewhat misleading to think of quantum particles as being 'located' in a particular place, and then 'traveling' to another place.

However, quantum systems are *not* random. The Schrödinger equation is completely deterministic; a particular wavefunction ω_0 at time zero completely determines the wavefunction ω_t which will be present at time t. The only 'random' thing in quantum mechanics is the transformation of a 'quantum state' (ie. a wavefunction) into a 'classical state' (ie. an observation of a 'classical' quantity, such as position or momentum).

Traditionally, physicists regarded the wavefunction as giving incomplete (ie. probabilistic) information about the 'hidden' or 'undetermined' classical state of a quantum system. An act of 'observation' (eg. an experiment, a measurement), supposedly 'collapsed' this wavefunction, forcing the 'indeterminate' quantum system to take on a 'determined' classical state. The outcome of this observation was a random variable, whose probability distribution was described by the wavefunction at the moment of collapse. This theory of the relationship between (deterministic) quantum systems and (apparently random) classical observables is called *quantum measurement theory*; see [Boh79, Part IV].

Implicit in this description is the assumption that the 'true' state of a system is *classical*, and can be described via classical quantities like position or momentum, having precise values. The quantum wavefunction was merely an annoying mirage. Implicit also in this description was the assumption that macroscopic entities (eg. human experimenters) were not themselves quantum mechanical in nature; the Schrödinger equation didn't apply to people.

However, many physicists now feel that the 'true' state of a quantum system is the *wavefunction*, not a set of classical variables like position and momentum. The classical variables which we obtain through laboratory measurements are merely incomplete 'snapshots' of this true quantum state.

To understand the so-called 'collapse' of the wavefunction, we must recall that human experimenters *themselves* are also quantum systems. Suppose a human observes a quantum system (say, an electron). We must consider the *joint* quantum system, which consists of the human-plus-electron, and which possesses a *joint* wavefunction, describing the quantum state of the human, the quantum state of the electron, and any relationship (ie. correlation) between the two. An 'experiment' is just a (carefully controlled) interaction between the human and electron. The apparent 'collapse' of the electron's wavefunction is really just a very strong *correlation* which occurs between the wavefunctions of the electron and the human, as a result of this interaction. However, the wavefunction of the entire system (human-plus-electron) never 'collapses' during the interaction; it just evolves continuously and deterministically, in accord with the Schrödinger equation.

The experiment-induced correlation of the human wavefunction to the electron wavefunction is called 'decoherence'. From the perspective of one component of the joint system (eg. the human), decoherence looks like the apparent 'collapse' of wavefunction for the other component (the electron). Sadly, a full discussion of decoherence is beyond the scope of this book.

The meaning of phase: At any point $\mathbf{x}$ in space and moment t in time, the wavefunction $\omega_t(\mathbf{x})$ can be described by its *amplitude* $A_t(\mathbf{x}) := |\omega_t(\mathbf{x})|$ and its *phase* $\phi_t(\mathbf{x}) := \omega_t(\mathbf{x})/A_t(\mathbf{x})$. We have already discussed the physical meaning of the amplitude: $|A_t(\mathbf{x})|^2$ is the *probability* that a classical measurement will produce the outcome $\mathbf{x}$ at time t. What is the meaning of phase?

The phase $\phi_t(\mathbf{x})$ is a complex number of modulus one —an element of the unit circle in the complex plane (hence $\phi_t(\mathbf{x})$ is sometimes called the *phase angle*). The 'oscillation' of the wavefunction ω over time can be imagined in terms of the 'rotation' of $\phi_t(\mathbf{x})$ around the circle. The 'wavelike' properties of quantum systems (eg. interference patterns) are because wavefunctions with different phases will partially cancel one another when they are superposed. In other words, it is because of *phase* that the Schrodinger Equation yields 'wave-like' phenomena, instead of yielding 'diffusive' phenomena like the Heat Equation.

However, like potential energy, phase is *not directly physically observable.* We can observe the phase *difference* between wavefunction α and wavefunction β (by observing cancelation between α and β), just as we can observe the potential energy *difference* between point A and point B (by measuring the energy released by a particle moving from point A to point B). However, it is not physically meaningful to speak of the 'absolute phase' of wavefunction α, just as it is not physically meaningful to speak of the 'absolute potential energy' of point A.

Indeed, inspection of the Schrödinger equation (4.5) on page 56 will reveal that the speed of phase rotation of a wavefunction ω at point $\mathbf{x}$ is determined by the magnitude of the potential function V at $\mathbf{x}$. But we can arbitrarily increase V by a constant, without changing its physical meaning. Thus, we can arbitrarily 'accelerate' the phase rotation of the wavefunction without changing the physical meaning of the solution.

4.4 Some solutions to the Schrödinger Equation

Prerequisites: §4.2

The major mathematical problems of quantum mechanics come down to finding solutions to the Schrödinger equations for various physical systems. In general it is very difficult to solve the Schrödinger equation for most 'realistic' potential functions. We will confine ourselves to a few 'toy models' to illustrate the essential ideas.

Example 4.5: Free Electron with Known Velocity (Null Field)

Consider a single electron in the absence of an ambient magnetic field. Suppose an experiment has precisely measured the 'classical' velocity of the electron, and determined it to be $\mathbf{v} = (v_1, 0, 0)$. Then the wavefunction of the electron is given[3]

$$\omega_t(\mathbf{x}) \quad = \quad \exp\left(\frac{-\mathbf{i}}{\hbar}\,\frac{m_e v_1^2}{2}\,t\right) \cdot \exp\left(\frac{\mathbf{i}}{\hbar}\,m_e v_1 \cdot x_1\right). \qquad \text{(see Figure 4.1)} \qquad (4.9)$$

This ω satisfies the free Schrödinger equation (4.6). [See practice problem # 1 on page 73 of §4.7.]

Exercise 4.1 (a) Check that the spatial wavelength λ of the function ω is given $\lambda = \dfrac{2\pi\hbar}{p_1} = \dfrac{h}{m_e v}$. This is the so-called *de Broglie wavelength* of an electron with velocity v.

[3]We will not attempt here to justify *why* this is the correct wavefunction for a particle with this velocity. It is not obvious.

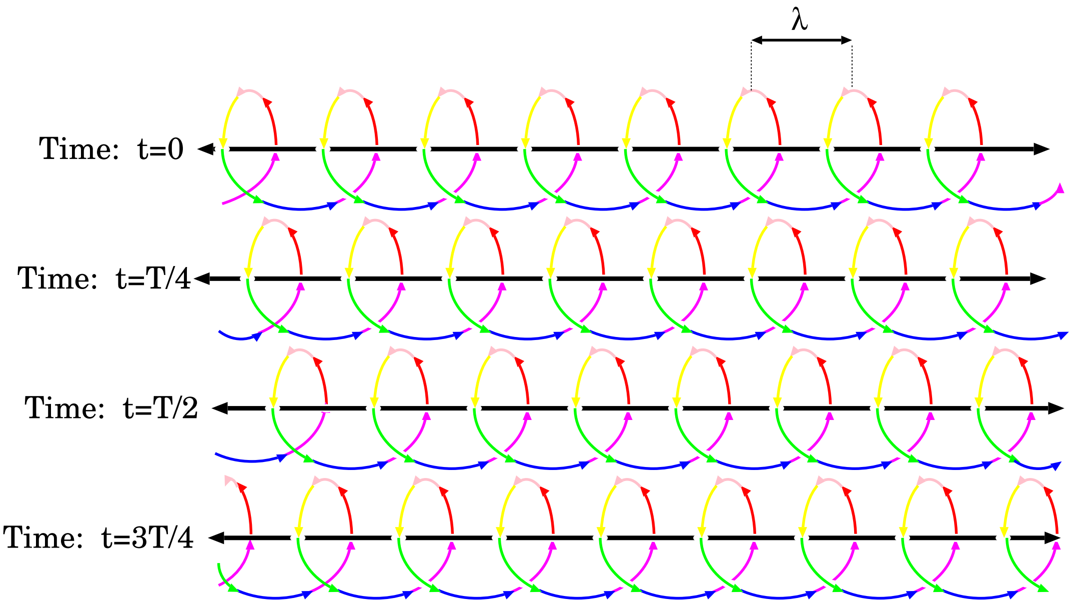


Figure 4.1: Four successive 'snapshots' of the wavefunction of a single electron in a zero potential, with a precisely known velocity. Only one spatial dimension is shown. Colour indicates complex phase.

(b) Check that the temporal period of ω is $T := \dfrac{2h}{m_e v^2}$.

(c) Conclude the *phase velocity* of ω (ie. the speed at which the wavefronts propagate through space) is equal to v.

More generally, suppose the electron has a precisely known velocity $\mathbf{v} = (v_1, v_2, v_3)$, with corresponding momentum vector $\mathbf{p} := m_e\,\mathbf{v}$. Then the wavefunction of the electron is given

$$\omega_t(\mathbf{x}) \quad = \quad \exp\left(\frac{-\mathbf{i}}{\hbar}\, E_k\, t\right) \cdot \exp\left(\frac{\mathbf{i}}{\hbar}\, \mathbf{p} \bullet \mathbf{x}\right),$$

where $E_k := \frac{1}{2} m_e |\mathbf{v}|^2$ is kinetic energy, and $\mathbf{p} \bullet \mathbf{v} \ := \ p_1 v_1 + p_2 v_2 + p_3 v_3$. If we convert to atomic units, then $E_k = \frac{1}{2}|\mathbf{v}|^2$ and $\mathbf{p} = \mathbf{v}$, and this function takes the simpler form

$$\omega_t(\mathbf{x}) \quad = \quad \exp\left(\frac{-\mathbf{i}\,|\mathbf{v}|^2\, t}{2}\right) \cdot \exp\left(\mathbf{i}\, \mathbf{v} \bullet \mathbf{x}\right).$$

This ω satisfies the free Schrödinger equation (4.7) [See practice problem # 2 on page 74 of §4.7.]

Let $\rho(\mathbf{x}; t) \ = \ |\omega_t(\mathbf{x})|^2$ be the probability distribution defined by this wavefunction. It is easy to see that $\rho(\mathbf{x}; t) \equiv 1$ for all $\mathbf{x}$ and t. Thus, this wavefunction represents a state of maximal uncertainty about the position of the electron; the electron literally 'could be anywhere'. This is manifestation of the infamous *Heisenberg Uncertainty Principle*; by assuming that the electron's *velocity* was 'precisely determined', we have forced it's *position* to be entirely undetermined.

Indeed, the astute reader will notice that, strictly speaking, ρ is *not* a probability distribution, because $\int_{\mathbb{R}^3} \rho(\mathbf{x})\, d\mathbf{x} = \infty$. In other words, ω_t is *not* square-integrable. This means that our starting assumption —an electron with a precisely known velocity —leads to a contradiction. One interpretation: a quantum particle can *never* have a precisely known classical velocity. Any physically meaningful wavefunction must contain a 'mixture' of several velocities. ♢

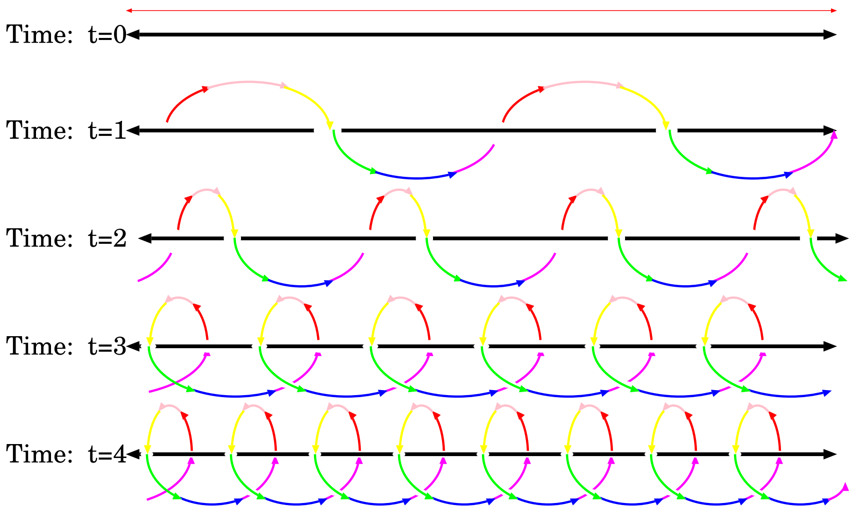


Figure 4.2: Five successive 'snapshots' of the wavefunction of an electron accelerating from initial zero velocity in a constant electric field. Only one spatial dimension is shown. Colour indicates complex phase. Notice how, as the electron accelerates, its spatial wavelength becomes shorter.

Example 4.6: Electron accelerating in constant ambient field

Consider a single electron in a constant electric field $\vec{\mathbf{E}} \equiv (-E, 0, 0)$. Recall from Example 6b on page 54 that the potential function in this case is $V(x_1, x_2, x_3) = -q_e E x_1$, where q_e is the charge of the electron. The corresponding Schrödinger equation is

$$\mathbf{i}\hbar\, \partial_t\, \omega_t(\mathbf{x}) \quad = \quad \frac{-\hbar^2}{2\, m_e} \triangle\, \omega_t(\mathbf{x}) \; - \; q_e E x_1 \cdot \omega_t(\mathbf{x}). \tag{4.10}$$

Assume that the electron is at rest at time zero (meaning that it's classical velocity is precisely known to be $(0, 0, 0)$ at time zero). Then the solution to eqn.(4.10) is given by:

$$\omega_t(\mathbf{x}) \quad = \quad \exp\left(\frac{-\mathbf{i}}{\hbar}\, \frac{q_e^2 E^2 t^3}{6 m_e}\right) \cdot \exp\left(\frac{\mathbf{i}}{\hbar}\, q_e E t x_1\right). \qquad \text{(Figure 4.2)} \tag{4.11}$$

Exercise 4.2 Verify this by substituting eqn.(4.11) into eqn.(4.10). ♢

Exercise 4.3 From classical electrodynamics, we expect the electron to experience a force $\vec{F} = (q_e E,\ 0, 0)$, causing an acceleration $\vec{a} = (q_e E/m_e,\ 0, 0)$, so that its velocity at time t is $\mathbf{v}(t) =$

$(q_e Et/m_e,\ 0,0)$. Example 4.5 says that an electron with velocity $\mathbf{v} = (v,0,0)$ has wavefunction $\exp\left(\frac{-\mathbf{i}}{\hbar}\frac{m_e v^2}{2}t\right)\cdot\exp\left(\frac{\mathbf{i}}{\hbar}m_e v_1\cdot x_1\right)$. Substituting $\mathbf{v}(t) := (q_e Et/m_e,\ 0,0)$, we get

$$\omega_t(\mathbf{x}) = \exp\left(\frac{-\mathbf{i}}{\hbar}\frac{m_e\, q_e^2 E^2 t^2}{2m_e^2}t\right)\cdot\exp\left(\frac{\mathbf{i}}{\hbar}m_e\frac{q_e Et}{m_e}x_1\right) = \exp\left(\frac{-\mathbf{i}}{\hbar}\frac{q_e^2 E^2 t^3}{2m_e}\right)\cdot\exp\left(\frac{\mathbf{i}}{\hbar}q_e Etx_1\right).$$

Notice that this disagrees with the solution (4.11), because one contains a factor of '6' and the other contains a factor of '2' in the denominator. *Explain this discrepancy.* Note that it is not good enough to simply assert that 'classical physics is wrong', because the *Correspondence Principle* says that quantum mechanics *must* agree with the predictions of classical physics in the macroscopic limit.

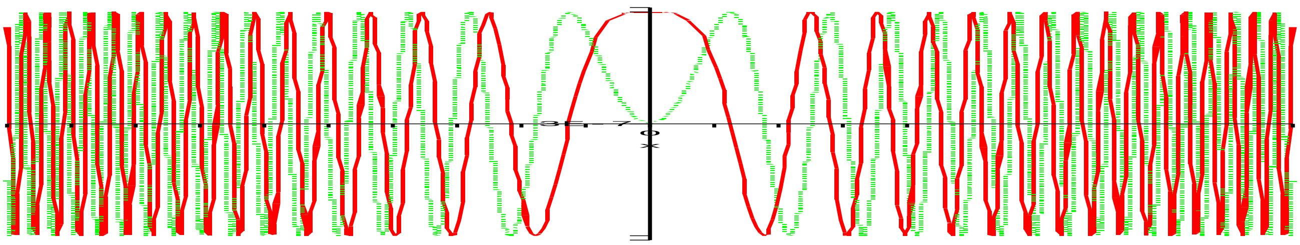


Figure 4.3: The real and imaginary parts of the 'pseudo-gaussian' solution to the free Schrödinger equation. As $x \to \infty$, the wavelength of the coils becomes smaller and smaller. As $t \to 0$, the wavelength of the coils also becomes tighter, and the amplitude grows to infinity like $1/\sqrt{t}$.

Example 4.7: Pseudo-Gaussian solution

Recall that one important solution to the one-dimensional Heat Equation is the *Gauss-Weierstrass Kernel*

$$\mathcal{G}(x;t) := \frac{1}{2\sqrt{\pi t}}\exp\left(\frac{-x^2}{4t}\right) \qquad \text{(see Example 2.1(c) on 23).}$$

We want a similar solution to the one-dimensional Schrödinger equation for a free electron:

$$\mathbf{i}\hbar\,\partial_t\,\omega_t(\mathbf{x}) = \frac{-\hbar^2}{2\,m_e}\partial_x^2\,\omega_t(\mathbf{x}). \tag{4.12}$$

Claim 1: Let $\beta \in \mathbb{C}$ be some constant, and define

$$\omega_t(x) := \frac{1}{\sqrt{t}}\exp\left(\frac{\beta\, x^2}{t}\right), \qquad \text{for all } x \in \mathbb{R} \text{ and } t > 0.$$

Then ω is a solution to eqn.(4.12) if and only if $\beta = \dfrac{m_e\,\mathbf{i}}{2\hbar}$.

Proof: **Exercise 4.4** (a) Show that $\partial_t\,\omega = -\left(\frac{1}{2t}+\frac{\beta x^2}{t^2}\right)\cdot\omega$.

(b) Show that $\partial_x^2\,\omega = 4\left(\frac{\beta}{2t}+\frac{\beta^2 x^2}{t^2}\right)\cdot\omega$.

(c) Conclude that ω satisfies eqn.(4.12) if and only if $\beta = \frac{m_e\,\mathbf{i}}{2\hbar}$. $\diamond_{\text{Claim 1}}$

Lemma 1 yields the 'pseudo-Gaussian' solution to the free Schrödinger equation (4.12):

$$\omega_t(x) \quad := \quad \frac{1}{\sqrt{t}} \exp\left(\frac{m_e \, \mathbf{i} \, x^2}{2\hbar \, t}\right), \qquad \text{for all } x \in \mathbb{R} \text{ and } t > 0. \qquad \text{(see Figure 4.3)}$$

This solution is somewhat problematical, because it is not square-integrable. Nevertheless, ω_t plays an important role as the 'fundamental solution' for (4.12). But this is somewhat complicated and beyond the scope of our discussion. ♢

Exercise 4.5 Fix $t > 0$. Show that $|\omega_t(x)| = \frac{1}{\sqrt{t}}$ for all $x \in \mathbb{R}$. Conclude that $\omega_t \notin \mathbf{L}^2(\mathbb{R})$.

Exercise 4.6 Generalize Lemma 1 to obtain a 'pseudo-Gaussian' solution to the three-dimensional free Schrödinger equation.

4.5 The Stationary Schrödinger Equation and the Eigenfunctions of the Hamiltonian

Prerequisites: §4.2 **Recommended:** §4.4, §5.2(d)

A 'stationary' state of a quantum system is one where the probability density does not change with time. This represents a physical system which is in some kind of long-term equilibrium. Note that a stationary quantum state does *not* mean that the particles are 'not moving' (whatever 'moving' means for quanta). It instead means that they are moving in some kind of regular, confined pattern (ie. an 'orbit') which remains qualitatively the same over time. For example, the orbital of an electron in a hydrogen atom should be a stationary state, because (unless the atom received absorbs or emits energy) the orbital should stay the same over time.

Mathematically speaking, a stationary wavefunction ω yields a time-invariant probability density function $\rho : \mathbb{X} \longrightarrow \mathbb{R}$ so that, for any $t \in \mathbb{R}$,

$$|\omega_t(\mathbf{x})|^2 \quad = \quad \rho(\mathbf{x}), \qquad \text{for all } \mathbf{x} \in \mathbb{X}.$$

The simplest way to achieve this is to assume that ω has the *separated* form

$$\omega_t(\mathbf{x}) \quad = \quad \phi(t) \cdot \omega_0(\mathbf{x}), \tag{4.13}$$

where $\omega_0 : \mathbb{X} \longrightarrow \mathbb{C}$ and $\phi : \mathbb{R} \longrightarrow \mathbb{C}$ satisfy the conditions

$$|\phi(t)| \; = \; 1, \quad \text{for all } t \in \mathbb{R}, \qquad \text{and} \qquad |\omega_0(\mathbf{x})| \; = \; \sqrt{\rho(\mathbf{x})}, \quad \text{for all } t \in \mathbb{R}. \tag{4.14}$$

Lemma 4.8: *Suppose* $\omega_t(\mathbf{x}) = \phi(t) \cdot \omega_0(\mathbf{x})$ *is a separated solution to the Schrödinger equation, as in* eqn.(4.13) *and* eqn.(4.14). *Then there is some constant* $E \in \mathbb{R}$ *so that*

- $\phi(t) = \exp(-\mathbf{i}Et/\hbar)$, *for all* $t \in \mathbb{R}$.

- $\mathsf{H}\,\omega_0 \;=\; E\cdot\omega_0$; *in other words* ω_0 *is an* <u>*eigenfunction*</u>[4] *of the Hamiltonian operator* H, *with* <u>*eigenvalue*</u> E.

- *Thus,* $\omega_t(\mathbf{x}) \;=\; e^{-\mathbf{i}Et/\hbar}\cdot\omega_0(\mathbf{x})$, *for all* $\mathbf{x}\in\mathbb{X}$ *and* $t\in\mathbb{R}$.

Proof: <u>**Exercise 4.7**</u> Hint: apply standard separation-of-variables[5] arguments. □

Physically speaking, E corresponds to the *total energy* (potential + kinetic) of the quantum system[6]. Thus, this lemma yields one of the key concepts of quantum theory:

> *Eigenvectors of the Hamiltonian correspond to stationary quantum states. The eigenvalues of these eigenvectors correspond to the energy level of these states.*

Thus, to get stationary states, we must solve the **stationary Schrödinger equation:**

$$\mathsf{H}\,\omega_0 \quad=\quad E\cdot\omega_0,$$

where $E\in\mathbb{R}$ is an unknown constant (the energy eigenvalue), and $\omega_0:\mathbb{X}\longrightarrow\mathbb{C}$ is an unknown wavefunction.

Example 4.9: The Free Electron

Recall 'free electron' of Example 4.5. If the electron has velocity v, then the function ω in eqn.(4.9) yields a solution to the stationary Schrödinger equation, with eigenvalue $E=\frac{1}{2}m_e\,v^2$. (See practice problem # 4 on page 74 of §4.7).

Observe that E corresponds to the classical *kinetic energy* of an electron with velocity v. ♢

Example 4.10: One-dimensional square potential well; finite voltage

Consider an electron confined to a one-dimensional environment (eg. a long conducting wire). Thus, $\mathbb{X}:=\mathbb{R}$, and the wavefunction $\omega_0:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{C}$ obeys the one-dimensional Schrödinger equation

$$\mathbf{i}\partial_t\,\omega_0 \quad=\quad \frac{-1}{2}\partial_x^2\,\omega_0 + V\cdot\omega_0,$$

where $V:\mathbb{R}\longrightarrow\mathbb{R}$ is the potential energy function, and we have adopted atomic units. Let $V_0>0$ be some constant, and suppose that

$$V(x) \quad=\quad \begin{cases} 0 & \text{if} \quad 0\leq x\leq L; \\ V_0 & \text{if} \quad x<0 \text{ or } L<x. \end{cases}$$

[4] See § 5.2(d) on page 81.

[5] See Chapter 15 on page 278.

[6] This is not obvious, but it's a consequence of the fact that the Hamiltonian $\mathsf{H}\,\omega$ measures the total energy of the wavefunction ω. Loosely speaking, the term $\frac{\hbar^2}{2}\blacktriangle\,\omega$ represents the 'kinetic energy' of ω, while the term $V\cdot\omega$ represents the 'potential energy'.

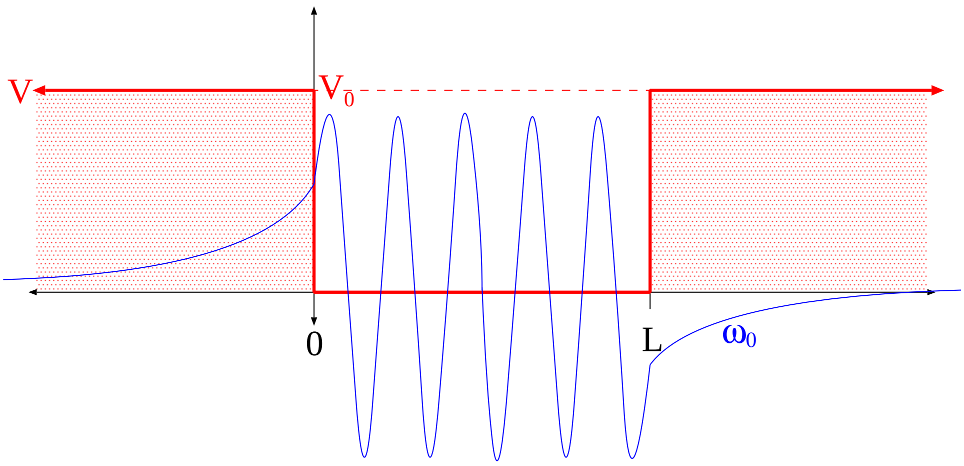


Figure 4.4: The (stationary) wavefunction of an electron in a one-dimensional 'square' potential well, with finite voltage gaps.

Physically, this means that V defines a 'potential energy well', which tries to confine the electron in the interval $[0, L]$, between two 'walls', which are voltage gaps of height V_0 (see Figure 4.4). The corresponding stationary Schrödinger equation is:

$$\frac{-1}{2}\partial_x^2 \omega_0 + V \cdot \omega_0 \quad = \quad E \cdot \omega_0, \tag{4.15}$$

where $E > 0$ is an (unknown) eigenvalue which corresponds to the energy of the electron. The function V only takes two values, so we can split eqn.(4.15) into two equations, one inside the interval $[0, L]$, and one outside it:

$$\begin{array}{rcll} \frac{-1}{2}\partial_x^2 \omega_0(x) & = & E \cdot \omega_0(x), & \text{for } x \in [0, L]; \\ \frac{-1}{2}\partial_x^2 \omega_0(x) & = & (E - V_0) \cdot \omega_0(x), & \text{for } x \notin [0, L]. \end{array} \tag{4.16}$$

Assume that $E < V_0$. This means that the electron's energy is less than the voltage gap, so the electron has insufficient energy to 'escape' the interval (at least in classical theory). The (physically meaningful) solutions to eqn.(4.16) have the form

$$\omega_0(x) \quad = \quad \left\{ \begin{array}{rl} C\exp(-\epsilon' x), & \text{if } x \in (-\infty, 0]; \\ A\sin(\epsilon x) + B\cos(\epsilon x), & \text{if } x \in [0, L]; \\ D\exp(\epsilon' x), & \text{if } L \in [L, \infty). \end{array} \right. \qquad \text{[see Fig. 4.4]} \tag{4.17}$$

Here, $\epsilon := \sqrt{2E}$ and $\epsilon' := \sqrt{2E - 2V_0}$, and $A, B, C, D \in \mathbb{C}$ are constants. The corresponding solution to the full Schrödinger equation is:

$$\omega_t(x) \quad = \quad \left\{ \begin{array}{rl} Ce^{-\mathbf{i}(E-V_0)t} \cdot \exp(-\epsilon' x), & \text{if } x \in (-\infty, 0]; \\ e^{-\mathbf{i}Et} \cdot (A\sin(\epsilon x) + B\cos(\epsilon x)), & \text{if } x \in [0, L]\ ; \\ De^{-\mathbf{i}(E-V_0)t} \cdot \exp(\epsilon' x), & \text{if } L \in [L, \infty). \end{array} \right. \qquad \text{for all } t \in \mathbb{R}.$$

This has two consequences:

(a) With nonzero probability, the electron might be found *outside* the interval $[0, L]$. In other words, it is quantumly possible for the electron to 'escape' from the potential well, something which is classically impossible[7]. This phenomenon called *quantum tunnelling* (because the electron can 'tunnel' through the wall of the well).

(b) The system has a physically meaningful solution only for certain values of E. In other words, the electron is only 'allowed' to reside at certain discrete *energy levels*; this phenomenon is called *quantization of energy.*

To see **(a)**, recall that the electron has probability distribution

$$\rho(x) \;:=\; \frac{1}{W}|\omega_0(x)|^2, \quad \text{where } W \;:=\; \int_{-\infty}^{\infty} |\omega_0(x)|^2 \, dx.$$

Thus, if $C \neq 0$, then $\rho(x) \neq 0$ for $x < 0$, while if $D \neq 0$, then $\rho(x) \neq 0$ for $x > L$. Either way, the electron has nonzero probability of 'tunnelling' out of the well.

To see **(b)**, note that we must choose A, B, C, D so that ω_0 is continuously differentiable at the boundary points $x = 0$ and $x = L$. This means we must have

$$\left.\begin{array}{rcl} B \;=\; A\sin(0) + B\cos(0) & = & C\exp(0) \;=\; C \\ \epsilon A \;=\; A\epsilon\cos(0) - B\epsilon\sin(0) & = & -\epsilon' C\exp(0) \;=\; -\epsilon' C \\ A\sin(\epsilon L) + B\cos(\epsilon L) & = & D\exp(\epsilon' L) \\ A\epsilon\cos(\epsilon L) - B\epsilon\sin(\epsilon L) & = & \epsilon' D\exp(\epsilon' L) \end{array}\right\} \tag{4.18}$$

Clearly, we can satisfy the first two equations in (4.18) by setting $B := C := \frac{-\epsilon}{\epsilon'}A$. The third and fourth equations in (4.18) then become

$$e^{-\epsilon' L}\cdot\left(\sin(\epsilon L) - \frac{\epsilon}{\epsilon'}\cos(\epsilon L)\right)\cdot A \quad = \quad D \quad = \quad \frac{\epsilon}{\epsilon'}e^{-\epsilon' L}\cdot\left(\cos(\epsilon L) + \frac{\epsilon}{\epsilon'}\sin(\epsilon L)\right) A, \tag{4.19}$$

Cancelling the factors $e^{-\epsilon' L}$ and A from both sides and substituting $\epsilon := \sqrt{2E}$ and $\epsilon' := \sqrt{2E - 2V_0}$, we see that eqn.(4.19) is satisfiable if and only if

$$\sin\left(\sqrt{2E}\cdot L\right) - \frac{\sqrt{E}\cdot\cos(\sqrt{2E}\cdot L)}{\sqrt{E - V_0}} \quad = \quad \frac{\sqrt{E}\cdot\cos\left(\sqrt{2E}\cdot L\right)}{\sqrt{E - V_0}} + \frac{E\cdot\sin(\sqrt{2E}\cdot L)}{E - V_0}. \tag{4.20}$$

Hence, eqn.(4.16) has a physically meaningful solution only for those values of E which satisfy the transcendental equation (4.20). The set of solutions to eqn.(4.20) is an infinite discrete subset of $\mathbb{R}$; each solution for eqn.(4.20) corresponds to an allowed 'energy level' for the physical system. ♢

[7] Many older texts observe that the electron 'can penetrate the classically forbidden region', which has caused mirth to generations of physics students.

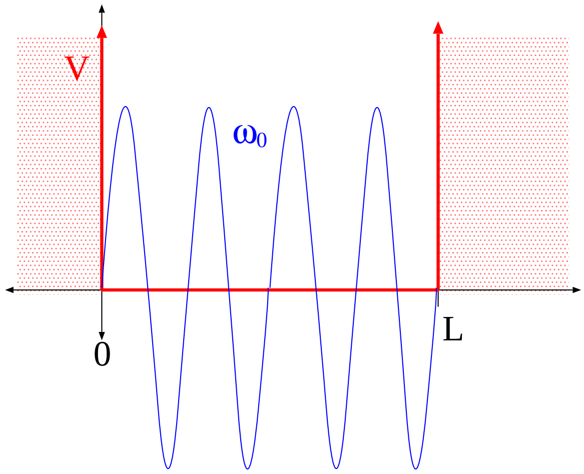


Figure 4.5: The (stationary) wavefunction of an electron in an infinite potential well.

Example 4.11: One-dimensional square potential well; infinite voltage

We can further simplify the model of Example 4.10 by setting $V_0 := +\infty$, which physically represents a 'huge' voltage gap that totally confines the electron within the interval $[0, L]$ (see Figure 4.5). In this case, $\epsilon' = -\infty$, so $\exp(-\epsilon x) = 0$ for all $x < 0$ and $\exp(\epsilon x) = 0$ for all $x > L$. Hence, if ω_0 is as in eqn.(4.17), then $\omega_0(x) \equiv 0$ for all $x \notin [0, L]$, and the constants C and D are no longer physically meaningful; we set $C = 0 = D$ for simplicity. Also, we must have $\omega_0(0) = 0 = \omega_0(L)$ to get a continuous solution; thus, we must set $B := 0$ in eqn.(4.17). Thus, the stationary solution in eqn.(4.17) becomes

$$\omega_0(x) \quad = \quad \begin{cases} 0 & \text{if } x \notin [0, L]; \\ A \cdot \sin(\sqrt{2E}\, x) & \text{if } x \in [0, L], \end{cases}$$

where A is a constant, and E satisfies the equation

$$\sin(\sqrt{2E}\, L) = 0. \qquad \text{(Figure 4.5)} \qquad (4.21)$$

Assume for simplicity that $L := \pi$. Then eqn.(4.21) is true if and only if $\sqrt{2E}$ is an integer, which means $2E \in \{0, 1, 4, 9, 16, 25, \ldots\}$, which means $E \in \{0, \frac{1}{2}, 2, \frac{9}{2}, 8, \frac{25}{2}, \ldots\}$. Here we see the phenomenon of *quantization of energy* in its simplest form. ♢

The set of eigenvalues of a linear operator is called the **spectrum** of that operator. For example, in Example 4.11, the spectrum of the Hamiltonian operator H is the set $\{0, \frac{1}{2}, 2, \frac{9}{2}, 8, \frac{25}{2}, \ldots\}$. In quantum theory, the spectrum of the Hamiltonian is the set of allowed energy levels of the system.

Example 4.12: Three-dimensional square potential well; infinite voltage

We can easily generalize Example 4.11 to three dimensions. Let $\mathbb{X} := \mathbb{R}^3$, and let $\mathbb{B} := [0,\pi]^3$ be a cube with one corner at the origin, having sidelength $L = \pi$. We use the potential function $V : \mathbb{X} \longrightarrow \mathbb{R}$ defined

$$V(\mathbf{x}) \quad = \quad \begin{cases} 0 & \text{if} \quad \mathbf{x} \in \mathbb{B}; \\ +\infty & \text{if} \quad \mathbf{x} \notin \mathbb{B}. \end{cases}$$

Physically, this represents an electron confined within a cube of perfectly conducting material with perfectly insulating boundaries[8]. Suppose the electron has energy E. The corresponding stationary Schrödinger equation is

$$\begin{array}{rcll} \frac{-1}{2} \triangle \omega_0(\mathbf{x}) & = & E \cdot \omega_0(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbb{B}; \\ \frac{-1}{2} \triangle \omega_0(\mathbf{x}) & = & -\infty \cdot \omega_0(\mathbf{x}) & \text{for } \mathbf{x} \notin \mathbb{B}; \end{array} \tag{4.22}$$

(in atomic units). By reasoning similar Example 4.11, we find that the physically meaningul solutions to eqn.(4.22) have the form

$$\omega_0(\mathbf{x}) \quad = \quad \begin{cases} \frac{1}{\pi^{3/2}} \sin(n_1 x_1) \cdot \sin(n_2 x_2) \cdot \sin(n_3 x_3) & \text{if } \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{B}; \\ 0 & \text{if } \mathbf{x} \notin \mathbb{B}. \end{cases} \tag{4.23}$$

where n_1, n_2, and n_3 are arbitrary integers (called the *quantum numbers* of the solution), and $E = \frac{1}{2}(n_1^2 + n_2^2 + n_3^2)$ is the associated energy eigenvalue.

Exercise 4.8 (a) Check that eqn.(4.23) is a solution for eqn.(4.22).

(b) Check that $\rho := |\omega|^2$ is a probability density, by confirming that

$$\int_{\mathbb{X}} |\omega_0(\mathbf{x})|^2 \, d\mathbf{x} \;=\; \frac{1}{\pi^{3/2}} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \sin(n_1 x_1)^2 \cdot \sin(n_2 x_2)^2 \cdot \sin(n_3 x_3)^2 \, dx_1 \, dx_2 \, dx_3 \;=\; 1,$$

(this is the reason for using the constant $\frac{1}{\pi^{3/2}}$).

The corresponding solution to the full Schrödinger equation is

$$\text{For all } t \in \mathbb{R}, \quad \omega_t(\mathbf{x}) \quad = \quad \begin{cases} \frac{1}{\pi^{3/2}} e^{-\mathbf{i}(n_1^2+n_2^2+n_3^2)t/2} \cdot \sin(n_1 x_1) \sin(n_2 x_2) \sin(n_3 x_3) & \text{if } \mathbf{x} \in \mathbb{B}; \\ 0 & \text{if } \mathbf{x} \notin \mathbb{B}. \end{cases}$$

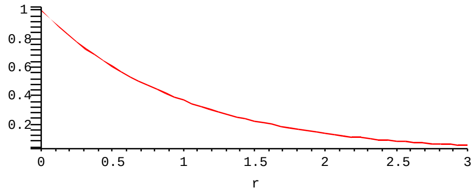
♢

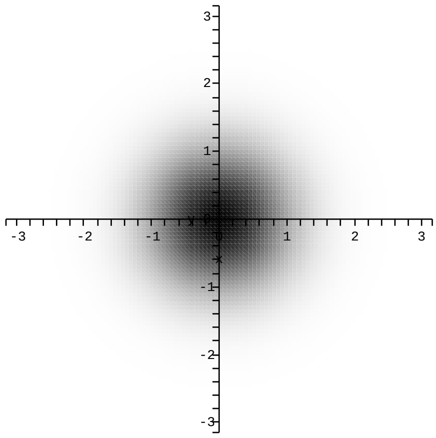
Example 4.13: Hydrogen Atom

In Example 4.3 on page 55, we described the hydrogen atom as a two-particle system, with a six-dimensional state space. However, the corresponding Schrödinger equation (Example 6c on page 57) is already too complicated for us to solve it here, so we will work with a simplified model.

Because the proton is 1836 times as massive as the electron, we can treat the proton as remaining effectively immobile while the electron moves around it. Thus, we can model the hydrogen atom as a *one*-particle system: a single electron moving in a Coulomb potential

[8] Alternately, it could be any kind of particle, confined in a cubical cavity with impenetrable boundaries.





(A) **(B)**

Figure 4.6: The groundstate wavefunction for a hydrogen atom. **(A)** Probability density as a function of distance from the nucleus. **(B)** Probability density visualized in three dimensions.

well, as described in Example 6b on page 57. The electron then satisfies the Schrödinger equation

$$\mathbf{i}\hbar\,\partial_t\,\omega_t(\mathbf{x}) \quad = \quad \frac{-\hbar^2}{2\,m_e}\,\triangle\,\omega_t(\mathbf{x}) \;+\; \frac{q_e^2}{4\pi\epsilon_0\cdot|\mathbf{x}|}\cdot\omega_t(\mathbf{x}), \quad \forall\,\mathbf{x}\in\mathbb{R}^3. \tag{4.24}$$

(Recall that m_e is the mass of the electron, q_e is the charge of both electron and proton, ϵ_0 is the 'permittivity of free space', and $\hbar$ is the rationalized Plank constant.) Assuming the electron is in a stable orbital, we can replace eqn.(4.24) with the *stationary* Schrödinger equation

$$\frac{-\hbar^2}{2\,m_e}\,\triangle\,\omega_0(\mathbf{x}) \;+\; \frac{q_e^2}{4\pi\epsilon_0\cdot|\mathbf{x}|}\cdot\omega_0(\mathbf{x}) \quad = \quad E\cdot\omega_0(\mathbf{x}), \quad \forall\,\mathbf{x}\in\mathbb{R}^3, \tag{4.25}$$

where E is the 'energy level' of the electron. One solution to this equation is

$$\omega(\mathbf{x}) \quad = \quad \frac{b^{3/2}}{\sqrt{\pi}}\exp(-b|\mathbf{x}|), \quad \text{where } b \;:=\; \frac{m\,q_e^2}{4\pi\epsilon_0\,\hbar^2}, \tag{4.26}$$

with corresponding energy eigenvalue

$$E \quad = \quad \frac{-\hbar^2}{2m}\cdot b^2 \quad = \quad \frac{-m\,q_e^4}{32\pi^2\epsilon_0^2\,\hbar^2} \tag{4.27}$$

Exercise 4.9 (a) Verify that the function ω_0 in eqn.(4.26) is a solution to eqn.(4.25), with E given by eqn.(4.27).

(b) Verify that the function ω_0 defines a probability density, by checking that $\int_{\mathbb{X}}|\omega|^2 \;=\; 1$.

There are many other, more complicated solutions to eqn.(4.25). However, eqn.(4.26) is the simplest solution, and has the *lowest* energy eigenvalue E of any solution. In other words, the solution (4.25) describes an electron in the *ground state*: the orbital of lowest potential energy, where the electron is 'closest' to the nucleus.

This solution immediately yields two experimentally testable predictions:

(a) The *ionization potential* for the hydrogen atom, which is the energy required to 'ionize' the atom, by stripping off the electron and removing it to an infinite distance from the nucleus.

(b) The *Bohr radius* of the hydrogen atom —that is, the 'most probable' distance of the electron from the nucleus.

To see **(a)**, recall that E is the sum of potential and kinetic energy for the electron. We assert (without proof) that there exist solutions to the stationary Schrödinger equation (4.25) with energy eigenvalues arbitrarily close to zero (note that E is negative). These zero-energy solutions represent orbitals where the electron has been removed to some very large distance from the nucleus, and the atom is essentially ionized. Thus, the energy difference between these 'ionized' states and ω_0 is $E - 0 = E$, and this is the energy necessary to 'ionize' the atom when the electron is in the orbital described by ω_0.

By substituting in numerical values $q_e \approx 1.60 \times 10^{-19}\,\mathtt{C}$, $\epsilon_0 \approx 8.85 \times 10^{-12}\,\mathtt{C/N\,m^2}$, $m_e \approx 9.11 \times 10^{-31}$ $\mathtt{kg}$, and $\hbar \approx 1.0545 \times 10^{-34}$ $\mathtt{J\,s}$, the reader can verify that, in fact, $E \approx -2.1796 \times 10^{-18}$ $\mathtt{J} \approx -13.605$ $\mathtt{eV}$, which is very close to -13.595 $\mathtt{eV}$, the experimentally determined ionization potential for a hydrogen atom[9]

To see **(b)**, observe that the probability density function for the distance r of the electron from the nucleus is given by

$$P(r) \quad = \quad 4\pi r^2 |\omega(r)|^2 \quad = \quad 4b^3 r^2 \exp(-2b|\mathbf{x}|). \qquad (\textbf{\underline{Exercise 4.10}})$$

The *mode* of the radial probability distribution is the maximal point of $P(r)$; if we solve the equation $P'(r) = 0$, we find that the mode occurs at

$$r \quad := \quad \frac{1}{b} \quad = \quad \frac{4\pi\epsilon_0\hbar^2}{m_e\, q_e^2} \quad \approx \quad 5.29172 \times 10^{-11}\ \mathtt{m}. \qquad \diamondsuit$$

The Balmer Lines: Recall that the **spectrum** of the Hamiltonian operator H is the set of all eigenvalues of H. Let $\mathcal{E} = \{E_0 < E_1 < E_2 < \ldots\}$ be the spectrum of the Hamiltonian of the hydrogen atom from Example 4.13, with the elements listed in increasing order. Thus, the smallest eigenvalue is $E_0 \approx -13.605$, the energy eigenvalue of the aforementioned ground state ω_0. The other, larger eigenvalues correspond to electron orbitals with higher potential energy.

When the electron 'falls' from a high energy orbital (with eigenvalue E_n, for some $n \in \mathbb{N}$) to a low energy orbital (with eigenvalue E_m, where $m < n$), it releases the energy difference, and emits a photon with energy $(E_n - E_m)$. Conversely, to 'jump' from a low E_m-energy orbital

[9]The error of 0.01 $\mathtt{eV}$ is mainly due to our simplifying assumption of an 'immobile' proton.

to a higher E_n-energy orbital, the electron must *absorb* a photon, and this photon must have exactly energy $(E_n - E_m)$.

Thus, the hydrogen atom can only emit or absorb photons of energy $|E_n - E_m|$, for some $n, m \in \mathbb{N}$. Let $\mathcal{E}' := \{|E_n - E_m| \; ; \; n, m \in \mathbb{N}\}$. We call $\mathcal{E}'$ the *energy spectrum* of the hydrogen atom.

Planck's law says that a photon with energy E has frequency $f = E/h$, where $h \approx 6.626 \times 10^{-34}$ $\mathtt{J\,s}$ is Planck's constant. Thus, if $\mathcal{F} = \{E/h \; ; \; E \in \mathcal{E}'\}$, then a hydrogen atom can only emit/absorb a photon whose frequency is in $\mathcal{F}$; we say $\mathcal{F}$ is the *frequency spectrum* of the hydrogen atom.

Here lies the explanation for the empirical observations of 19th century physicists such as Balmer, Lyman, Rydberg, and Paschen, who found that an energized hydrogen gas has a distinct *emission spectrum* of frequencies at which it emits light, and an identical *absorption spectrum* of frequencies which the gas can absorb. Indeed, every chemical element has its own distinct spectrum; astronomers use these 'spectral signatures' to measure the concentrations of chemical elements in the stars of distant galaxies. Now we see that

> *The (frequency) spectrum of an atom is determined by the (eigenvalue) spectrum of the corresponding Hamiltonian.*

4.6 The Momentum Representation

Prerequisites: §4.2, §17.4

The wavefunction ω allows us to compute the probability distribution for the classical *positions* of quantum particles. However, it seems to say nothing about the classical *momentum* of these particles. In Example 4.5 on page 60, we stated (without proof) the wavefunction of a particle with a particular known velocity. Now we make a more general assertion:

> *Suppose a particle has wavefunction $\omega : \mathbb{R}^3 \longrightarrow \mathbb{C}$. Let $\widehat{\omega} : \mathbb{R}^3 \longrightarrow \mathbb{C}$ be the Fourier transform of ω, and let $\widetilde{\rho}(\mathbf{p}) = |\widehat{\omega}|^2(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^3$. Then $\widetilde{\rho}$ is the probability distribution for the classical momentum of the particle.*

In other words, $\widehat{\omega}$ is the wavefunction for the *momentum representation* of the particle. Recall that we can reconstruct ω from $\widehat{\omega}$ via the inverse Fourier transform. Hence, the (positional) wavefunction ω implicitly encodes the (momentum) wavefunction $\widehat{\omega}$, and conversely the (momentum) wavefunction $\widehat{\omega}$ implicitly encodes the (positional) wavefunction ω. This answers the question we posed on page 54 of §4.1.

Because the momentum wavefunction contains exactly the same information as the positional wavefunction, we can reformulate the Schrödinger equation in momentum terms. Indeed, suppose the quantum system has potential energy function $V : \mathbb{X} \longrightarrow \mathbb{R}$. Let $\widehat{V}$ be the Fourier transform of V. Then the momentum wavefunction $\widehat{\omega}$ evolves according to the **momentum Schrödinger Equation:**

$$\mathbf{i}\partial_t \, \widehat{\omega}_t(\mathbf{p}) \quad = \quad \frac{\hbar^2}{2m}|\mathbf{p}|^2 \cdot \widehat{\omega}(\mathbf{p}) \; + \; \widehat{V} * \widehat{\omega}. \tag{4.28}$$

(here, if $\mathbf{p} = (p_1, p_2, p_3)$, then $|\mathbf{p}|^2 = p_1^2 + p_2^2 + p_3^2$). In particular, if the potential field is trivial, we get the *free* momentum Schrödinger equation:

$$\mathbf{i}\partial_t \widehat{\omega}_t(p_1, p_2, p_3) \quad = \quad \frac{\hbar^2}{2m}(p_1^2 + p_2^2 + p_3^2) \cdot \widehat{\omega}(p_1, p_2, p_3).$$

Exercise 4.11 Verify eqn.(4.28) by applying the Fourier transform to the (positional) Schrödinger equation eqn.(4.5) on page 56. Hint: Use Theorem 17.16 on page 343 to show that $\widehat{\triangle\omega}(\mathbf{p}) \;=\; -|\mathbf{p}|^2 \cdot \widehat{\omega}(\mathbf{p})$.

The Fourier transform relationship between position and momentum is the origin of Werner Heisenberg's famous **Uncertainty Principle**, which states:

> *In any quantum mechanical system, our certainty about the (classical) position of the particles is directly proportional to our uncertainty about their (classical) momentum, and vice versa. Thus, we can never simultaneously possess perfect knowledge of the position and momentum of a particle.*

This is just the physical interpretation of a mathematical phenomenon:

> *Let $\omega : \mathbb{R}^N \longrightarrow \mathbb{C}$ be a function with Fourier transform $\widehat{\omega} : \mathbb{R}^N \longrightarrow \mathbb{C}$. Then the more 'concentrated' the mass distribution of ω is, the more 'spread out' the mass distribution of $\widehat{\omega}$ becomes.*

It is possible to turn this vaguely worded statement into a precise theorem, but we do not have space for this here. Instead, we note that a perfect illustration of the Uncertainty Principle is the Fourier transform of a normal probability distribution. Let $\omega(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left(\dfrac{-x^2}{2\sigma^2}\right)$ be a normal probability distribution with mean 0 and variance σ^2 (recall that the *variance* measures how 'spread out' the distribution is). Then Theorem 17.17(b) on page 345 says that $\widehat{\omega}(p) \;=\; \frac{1}{2\pi} \exp\left(\frac{-\sigma^2 p^2}{2}\right)$. In other words, $\widehat{\omega}$ looks like a Gaussian distribution with variance $1/\sigma^2$. Thus, the more 'concentrated' ω is (ie. the smaller its variance σ^2 is), the more 'spread out' $\widehat{\omega}$' becomes (ie. the larger *its* variance, $1/\sigma^2$ becomes).

Further Reading: For an excellent, fast, yet precise introduction to quantum mechanics, see [McW72]. For a more comprehensive textbook, see [Boh79]. An completely different approach to quantum theory uses Feynman's *path integrals*; for a good introduction to this approach, see [Ste95], which also contains excellent introductions to classical mechanics, electromagnetism, statistical physics, and special relativity. For a rigorous mathematical approach to quantum theory, an excellent introduction is [Pru81]; another source is [BEH94].

4.7 Practice Problems

1. Let $v_1 \in \mathbb{R}$ be a constant. Consider the function $\omega : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{C}$ defined:

$$\omega_t(x_1, x_2, x_3) \;=\; \exp\left(\frac{-\mathbf{i}}{\hbar}\,\frac{m_e v_1^2}{2}\,t\right) \cdot \exp\left(\frac{\mathbf{i}}{\hbar}\, m_e v_1 \cdot x_1\right).$$

Show that ω satisfies the (free) *Schrödinger equation:* $\mathbf{i}\hbar\,\partial_t\,\omega_t(\mathbf{x}) \quad = \quad \frac{-\hbar^2}{2\,m_e}\,\triangle\,\omega_t(\mathbf{x})$.

2. Let $\mathbf{v} := (v_1, v_2, v_3)$ be a three-dimensional velocity vector, and let $|\mathbf{v}|^2 = v_1^2 + v_2^2 + v_3^2$. Consider the function $\omega : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{C}$ defined:

$$\omega_t(x_1, x_2, x_3) \quad = \quad \exp\left(-\mathbf{i}\,|\mathbf{v}|^2\,t/2\right) \cdot \exp\left(\mathbf{i}\,\mathbf{v} \bullet \mathbf{x}\right).$$

Show that ω satisfies the (free) *Schrödinger equation:* $\mathbf{i}\,\partial_t\,\omega \quad = \quad \frac{-1}{2}\,\triangle\,\omega$

3. Exercise 4.4(a-c) (p.63);

4. Consider the stationary Schrödinger equation for a null potential:

$$\mathsf{H}\,\omega_0 \quad = \quad E \cdot \omega_0, \qquad \text{where} \qquad \mathsf{H} = \frac{-\hbar^2}{2m_e}\triangle$$

Let $v \in \mathbb{R}$ be a constant. Consider the function $\omega : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{C}$ defined:

$$\omega_t(x_1, x_2, x_3) \;=\; \exp\left(\frac{-\mathbf{i}}{\hbar}\,\frac{m_e v_1^2}{2}\,t\right) \cdot \exp\left(\frac{\mathbf{i}}{\hbar}\,m_e v_1 \cdot x_1\right).$$

Show that ω is a solution to the above free Schrödinger equation, with eigenvalue $E = \frac{1}{2} m_e\,v^2$.

5. Exercise 4.8(a) (page 69).

6. Exercise 4.9(a) (page 70).

Notes: ....

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II General Theory

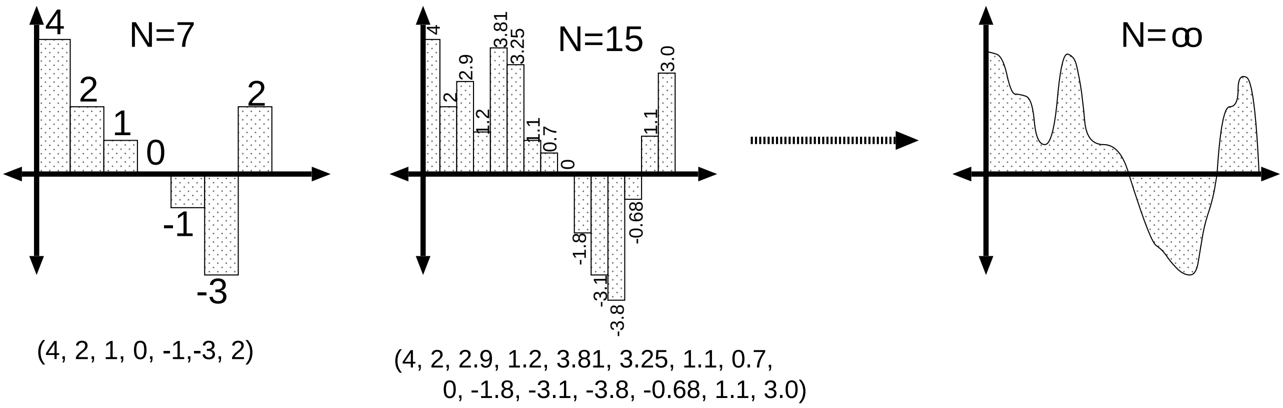


Figure 5.1: We can think of a function as an "infinite-dimensional vector"

5 Linear Partial Differential Equations

5.1 Functions and Vectors

Prerequisites: §1.1

Vectors: If $\mathbf{v} = \begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1.5 \\ 3 \\ 1 \end{bmatrix}$, then we can add these two vectors *componentwise*:

$$\mathbf{v} + \mathbf{w} \;=\; \begin{bmatrix} 2 - 1.5 \\ 7 + 3 \\ -3 + 1 \end{bmatrix} \;=\; \begin{bmatrix} 0.5 \\ 10 \\ -2 \end{bmatrix}.$$

In general, if $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, then $\mathbf{u} = \mathbf{v} + \mathbf{w}$ is defined by:

$$u_n = v_n + w_n, \quad \text{for } n = 1, 2, 3 \tag{5.1}$$

(see Figure 5.2A) Think of $\mathbf{v}$ as a function $v : \{1, 2, 3\} \longrightarrow \mathbb{R}$, where $v(1) = 2$, $v(2) = 7$, and $v(3) = -3$. If we likewise represent $\mathbf{w}$ with $w : \{1, 2, 3\} \longrightarrow \mathbb{R}$ and $\mathbf{u}$ with $u : \{1, 2, 3\} \longrightarrow \mathbb{R}$, then we can rewrite eqn.(5.1) as "$u(n) = v(n) + w(n)$ for $n = 1, 2, 3$". In a similar fashion, any N-dimensional vector $\mathbf{u} = (u_1, u_2, \ldots, u_N)$ can be thought of as a function $u : [1...N] \longrightarrow \mathbb{R}$.

Functions as Vectors: Letting N go to infinity, we can imagine any function $f : \mathbb{R} \longrightarrow \mathbb{R}$ as a sort of "infinite-dimensional vector" (see Figure 5.1). Indeed, if f and g are two functions, we can add them *pointwise*, to get a new function $h = f + g$, where

$$h(x) = f(x) + g(x), \quad \text{for all } x \in \mathbb{R} \tag{5.2}$$

(see Figure 5.2B) Notice the similarity between formulae (5.2) and (5.1), and the similarity between Figures 5.2A and 5.2B.

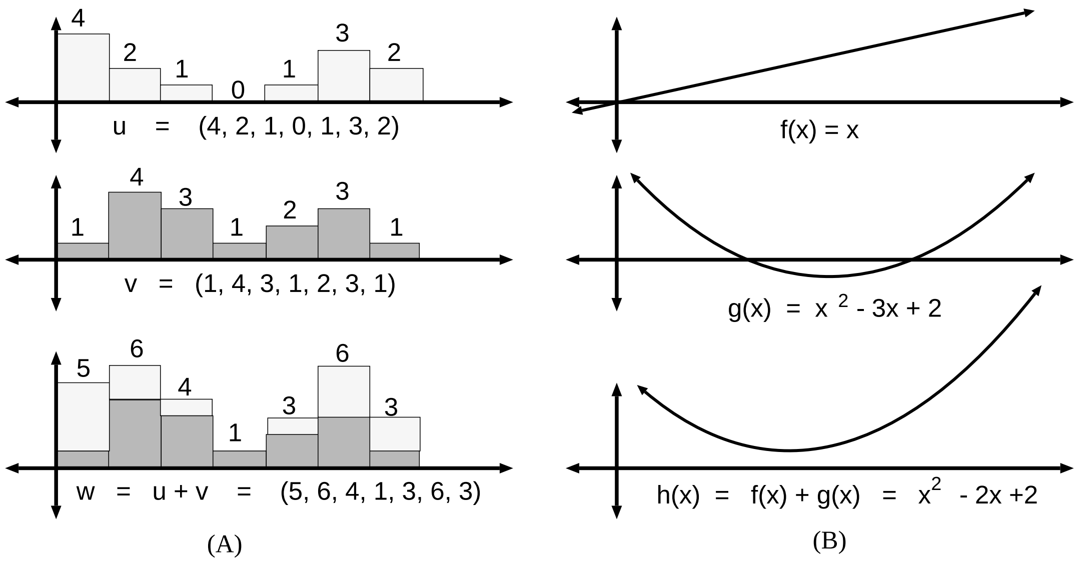


Figure 5.2: **(A)** We add vectors *componentwise*: If $\mathbf{u} = (4,2,1,0,1,3,2)$ and $\mathbf{v} = (1,4,3,1,2,3,1)$, then the equation "$\mathbf{w} = \mathbf{v} + \mathbf{w}$" means that $\mathbf{w} = (5,6,4,1,3,6,3)$. **(B)** We add two functions *pointwise*: If $f(x) = x$, and $g(x) = x^2 - 3x + 2$, then the equation "$h = f + g$" means that $h(x) = f(x) + g(x) = x^2 - 2x + 2$ for every x.

One of the most important ideas in the theory of PDEs is that *functions are infinite-dimensional vectors*. Just as with finite vectors, we can add them together, act on them with linear operators, or represent them in different *coordinate systems* on infinite-dimensional space. Also, the vector space $\mathbb{R}^D$ has a natural geometric structure; we can identify a similar geometry in infinite dimensions.

Let $\mathbb{X} \subseteq \mathbb{R}^D$ be some domain. The vector space of all continuous functions from $\mathbb{X}$ into $\mathbb{R}^m$ is denoted $\mathcal{C}(\mathbb{X};\mathbb{R}^m)$. That is:

$$\mathcal{C}(\mathbb{X};\mathbb{R}^m) \quad := \quad \{f : \mathbb{X} \longrightarrow \mathbb{R}^m \; ; \; f \text{ is continuous}\}.$$

When $\mathbb{X}$ and $\mathbb{R}^m$ are obvious from context, we may just write "$\mathcal{C}$".

Exercise 5.1 Show that $\mathcal{C}(\mathbb{X};\mathbb{R}^m)$ is a vector space.

A scalar field $f : \mathbb{X} \longrightarrow \mathbb{R}$ is **infinitely differentiable** (or **smooth**) if, for every $N > 0$ and every $i_1, i_2, \ldots, i_N \in [1...D]$, the Nth derivative $\partial_{i_1}\partial_{i_2}\cdots\partial_{i_N} f(\mathbf{x})$ exists at each $\mathbf{x} \in \mathbb{X}$. A vector field $f : \mathbb{X} \longrightarrow \mathbb{R}^m$ is **infinitely differentiable** (or **smooth**) if $f(\mathbf{x}) := (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$, where each of the scalar fields $f_1, \ldots, f_m : \mathbf{X} \longrightarrow \mathbb{R}$ is infinitely differentiable. The vector space of all smooth functions from $\mathbb{X}$ into $\mathbb{R}^m$ is denoted $\mathcal{C}^\infty(\mathbb{X};\mathbb{R}^m)$. That is:

$$\mathcal{C}^\infty(\mathbb{X};\mathbb{R}^m) \quad := \quad \{f : \mathbb{X} \longrightarrow \mathbb{R}^m \; ; \; f \text{ is infinitely differentiable}\}.$$

When $\mathbb{X}$ and $\mathbb{R}^m$ are obvious from context, we may just write "$\mathcal{C}^\infty$".

Example 5.1:

(a) $\mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R})$ is the space of all smooth *scalar fields* on the *plane* (ie. all functions $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$).

(b) $\mathcal{C}^\infty(\mathbb{R};\mathbb{R}^3)$ is the space of all smooth *curves* in *three-dimensional space.* ♢

Exercise 5.2 Show that $\mathcal{C}^\infty(\mathbb{X};\mathbb{R}^m)$ is a vector space, and thus, a linear subspace of $\mathcal{C}(\mathbb{X};\mathbb{R}^m)$.

5.2 Linear Operators

Prerequisites: §5.1

5.2(a) ...on finite dimensional vector spaces

If $\mathbf{v} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1.5 \\ 3 \end{bmatrix}$, then $\mathbf{u} = \mathbf{v}+\mathbf{w} = \begin{bmatrix} 0.5 \\ 10 \end{bmatrix}$. If $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}$, then $\mathbf{A}\cdot\mathbf{u} = \mathbf{A}\cdot\mathbf{v}+\mathbf{A}\cdot\mathbf{w}$. That is:

$$\begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}\cdot\begin{bmatrix} 0.5 \\ 10 \end{bmatrix} = \begin{bmatrix} -9.5 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \end{bmatrix}+\begin{bmatrix} -4.5 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}\cdot\begin{bmatrix} 2 \\ 7 \end{bmatrix}+\begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}\cdot\begin{bmatrix} -1.5 \\ 3 \end{bmatrix};$$

Also, if $\mathbf{x} = 3\mathbf{v} = \begin{bmatrix} 6 \\ 21 \end{bmatrix}$, then $\mathbf{Ax} = 3\mathbf{Av}$. That is:

$$\begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}\cdot\begin{bmatrix} 6 \\ 21 \end{bmatrix} = \begin{bmatrix} -15 \\ 24 \end{bmatrix} = 3\begin{bmatrix} -5 \\ 8 \end{bmatrix} = 3\begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}\cdot\begin{bmatrix} 2 \\ 7 \end{bmatrix}.$$

In other words, multiplication by the matrix $\mathbf{A}$ is a **linear operation** on vectors. In general, a function $L : \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is **linear** if:

- For all $\mathbf{v},\mathbf{w} \in \mathbb{R}^N$, $L(\mathbf{v}+\mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w})$
- For all $\mathbf{v} \in \mathbb{R}^N$ and $r \in \mathbb{R}$, $L(r\cdot\mathbf{v}) = r\cdot L(\mathbf{v})$.

Every linear function from $\mathbb{R}^N$ to $\mathbb{R}^M$ corresponds to multiplication by some $N \times M$ matrix.

Example 5.2:

(a) **Difference Operator:** Suppose $D : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ is the function:

$$D\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_3 \\ x_5 - x_4 \end{bmatrix}.$$

Then D corresponds to multiplication by the matrix $\begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix}$.

(b) **Summation operator:** Suppose $S : \mathbb{R}^4 \longrightarrow \mathbb{R}^5$ is the function:

$$S \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 \\ x_1 + x_2 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 + x_4 \end{bmatrix}$$

Then S corresponds to multiplication by the matrix $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

(c) **Multiplication operator:** Suppose $M : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ is the function

$$M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \cdot x_1 \\ 2 \cdot x_2 \\ -5 \cdot x_3 \\ \frac{3}{4} \cdot x_4 \\ \sqrt{2} \cdot x_5 \end{bmatrix}$$

Then M corresponds to multiplication by the matrix $\begin{bmatrix} 3 & & & & \\ & 2 & & & \\ & & -5 & & \\ & & & \frac{3}{4} & \\ & & & & \sqrt{2} \end{bmatrix}$. ◊

Remark: Notice that the transformation D is an inverse to the transformation S, but not vice versa.

5.2(b) ...on $\mathcal{C}^\infty$

Recommended: §2.2, §2.3, §3.2

In the same way, a transformation $\mathsf{L} : \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty$ is called **linear** if, for any two differentiable functions $f, g \in \mathcal{C}^\infty$, we have $\mathsf{L}(f + g) = \mathsf{L}(f) + \mathsf{L}(g)$, and, for any real number $r \in \mathbb{R}$, $\mathsf{L}(r \cdot f) = r \cdot \mathsf{L}(f)$.

Example 5.3:

(a) **Differentiation:** If $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are differentiable functions, and $h = f + g$, then we know that, for any $x \in \mathbb{R}$,

$$h'(x) = f'(x) + g'(x)$$

Also, if $h = r \cdot f$, then $h'(x) = r \cdot f'(x)$. Thus, if we define the operation $\mathsf{D} : \mathcal{C}^\infty(\mathbb{R};\mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R};\mathbb{R})$ by $\mathsf{D}[f] \;=\; f'$, then D is a linear transformation of $\mathcal{C}^\infty(\mathbb{R};\mathbb{R})$. For example, sin and cos are elements of $\mathcal{C}^\infty(\mathbb{R};\mathbb{R})$, and we have

$$\mathsf{D}[\sin] \;=\; \cos, \qquad \text{and} \qquad \mathsf{D}[\cos] \;=\; -\sin.$$

More generally, if $f, g : \mathbb{R}^D \longrightarrow \mathbb{R}$ and $h = f + g$, then for any $i \in [1..D]$,

$$\partial_j\, h = \partial_j\, f + \partial_j\, g.$$

Also, if $h = r \cdot f$, then $\partial_j\, h = r \cdot \partial_j\, f$. In other words, $\partial_j : \mathcal{C}^\infty(\mathbb{R}^D;\mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^D;\mathbb{R})$ is a linear operator.

(b) **Integration:** If $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are integrable functions, and $h = f + g$, then we know that, for any $x \in \mathbb{R}$,

$$\int_0^x h(y)\; dy \;=\; \int_0^x f(y)\; dy \;+\; \int_0^x g(y)\; dy$$

Also, if $h = r \cdot f$, then $\displaystyle\int_0^x h(y)\; dy \;=\; r \cdot \int_0^x f(y)\; dy$.

Thus, if we define the operation $\mathsf{S} : \mathcal{C}^\infty(\mathbb{R};\mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R};\mathbb{R})$ by

$$\mathsf{S}[f](x) \;=\; \int_0^x f(y)\; dy$$

then S is a linear transformation. For example, sin and cos are elements of $\mathcal{C}^\infty(\mathbb{R};\mathbb{R})$, and we have

$$\mathsf{S}[\sin] \;=\; 1 - \cos, \qquad \text{and} \qquad \mathsf{S}[\cos] \;=\; \sin.$$

(c) **Multiplication:** If $\gamma : \mathbb{R}^D \longrightarrow \mathbb{R}$ is a scalar field, then define the operator $\Gamma : \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty$ by: $\Gamma[f] = \gamma \cdot f$. In other words, for all $\mathbf{x} \in \mathbb{R}^D$, $\Gamma[f](\mathbf{x}) = \gamma(\mathbf{x}) \cdot f(\mathbf{x})$. Then Γ is a linear function, because, for any $f, g \in \mathcal{C}^\infty$, $\Gamma[f+g] = \gamma \cdot [f+g] = \gamma \cdot f + \gamma \cdot g = \Gamma[f] + \Gamma[g]$. ♢

Remark: Notice that the transformation D is an inverse for the transformation S; this is the Fundamental Theorem of Calculus.

Exercise 5.3 Compare the three linear transformations in Example 5.3 with those from Example 5.2. Do you notice any similarities?

Remark: Unlike linear transformations on $\mathbb{R}^N$, there is in general no way to express a linear transformation on $\mathcal{C}^\infty$ in terms of multiplication by some matrix. To convince yourself of this, try to express the three transformations from example 5.3 in terms of "matrix multiplication".

Any combination of linear operations is also a linear operation. In particular, any combination of differentiation and multiplication operations is linear. Thus, for example, the second-derivative operator $\mathsf{D}^2[f] \;=\; \partial_x^2 f$ is linear, and the Laplacian operator

$$\triangle f = \partial_1^2 f + \ldots + \partial_D^2 f$$

is also linear; in other words, $\triangle[f+g] = \triangle f + \triangle g$.

A linear transformation that is formed by adding and/or composing multiplications and differentiations is called a **linear differential operator** . Thus, for example, the Laplacian is a linear differential operator.

5.2(c) Kernels

If L is a linear function, then the **kernel** of L is the set of all vectors $\mathbf{v}$ so that $L(\mathbf{v}) = 0$.

Example 5.4:

(a) Consider the differentiation operator ∂_x on the space $\mathcal{C}^\infty(\mathbb{R};\mathbb{R})$. The kernel of ∂_x is the set of all functions $u : \mathbb{R} \longrightarrow \mathbb{R}$ so that $\partial_x u \equiv 0$ —in other words, the set of all **constant** functions.

(b) The kernel of ∂_x^2 is the set of all functions $u : \mathbb{R} \longrightarrow \mathbb{R}$ so that $\partial_x^2 u \equiv 0$ —in other words the set of all **flat** functions of the form $u(x) = ax + b$. ♢

Many partial differential equations are really equations for the kernel of some differential operator.

Example 5.5:

(a) **Laplace's equation** "$\triangle u \equiv 0$" really just says: "u is in the kernel of $\triangle$."

(b) The **Heat Equation** "$\partial_t u = \triangle u$" really just says: "$u$ is in the kernel of the operator $\mathsf{L} = \partial_t - \triangle$." ♢

5.2(d) Eigenvalues, Eigenvectors, and Eigenfunctions

If L is a linear function, then an **eigenvector** of L is a vector $\mathbf{v}$ so that

$$L(\mathbf{v}) = \lambda \cdot \mathbf{v},$$

for some constant $\lambda \in \mathbb{C}$, called the associated **eigenvalue**.

Example 5.6: If $L = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\mathbf{v} = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$, then $L(\mathbf{v}) = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right] = -\mathbf{v}$, so $\mathbf{v}$ is an eigenvector for L, with eigenvalue $\lambda = -1$. ♢

If L is a linear operator on $\mathcal{C}^\infty$, then an eigenvector of L is sometimes called an **eigenfunction**.

Example 5.7: Let $n, m \in \mathbb{N}$. Define $u(x,y) = \sin(n\cdot x)\cdot\sin(m\cdot y)$. Then it is **Exercise 5.4** to check that

$$\triangle u(x,y) \quad = \quad -(n^2+m^2)\cdot\sin(n\cdot x)\cdot\sin(m\cdot y) \quad = \quad \lambda\cdot u(x,y),$$

where $\lambda = -(n^2+m^2)$. Thus, u is an eigenfunction of the linear operator $\triangle$, with eigenvalue λ. ♢

Eigenfunctions of linear differential operators (particularly eigenfunctions of $\triangle$) play a central role in the solution of linear PDEs. This is implicit in throughout Part III (Chapters 11-13) and Chapter 18, and is made explicit in Part V.

5.3 Homogeneous vs. Nonhomogeneous

Prerequisites: §5.2

If L is a linear differential operator, then the equation "$\mathsf{L}u \equiv 0$" is called a **homogeneous linear** partial differential equation.

Example 5.8: The following are linear homogeneous PDEs

(a) **Laplace's Equation[1]:** Here, $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{R}^D;\ \mathbb{R})$, and $\mathsf{L} = \triangle$.

(b) **Heat Equation[2]:** $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{R}^D \times \mathbb{R};\ \mathbb{R})$, and $\mathsf{L} = \partial_t - \triangle$.

(c) **Wave Equation[3]:** $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{R}^D \times \mathbb{R};\ \mathbb{R})$, and $\mathsf{L} = \partial_t^2 - \triangle$.

(d) **Schrödinger Equation[4]:** $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{R}^{3N} \times \mathbb{R};\ \mathbb{C})$, and, for any $\omega \in \mathcal{C}^\infty$ and $(\mathbf{x};t) \in \mathbb{R}^{3N} \times \mathbb{R}$, $\quad \mathsf{L}\,\omega(\mathbf{x};t) \;:=\; \dfrac{-\hbar^2}{2}\,\blacktriangle\,\omega(\mathbf{x};t) \;+\; V(\mathbf{x})\cdot\omega(\mathbf{x};t) \;-\; \mathbf{i}\hbar\,\partial_t\,\omega(\mathbf{x};t)$.

(Here, $V : \mathbb{R}^{3N} \longrightarrow \mathbb{R}$ is some *potential function*, and $\blacktriangle$ is like a Laplacian operator, except that the components for each particle are divided by the rest mass of that particle.)

(e) **Fokker-Plank[5]:** $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{R}^D \times \mathbb{R};\ \mathbb{R})$, and, for any $u \in \mathcal{C}^\infty$,

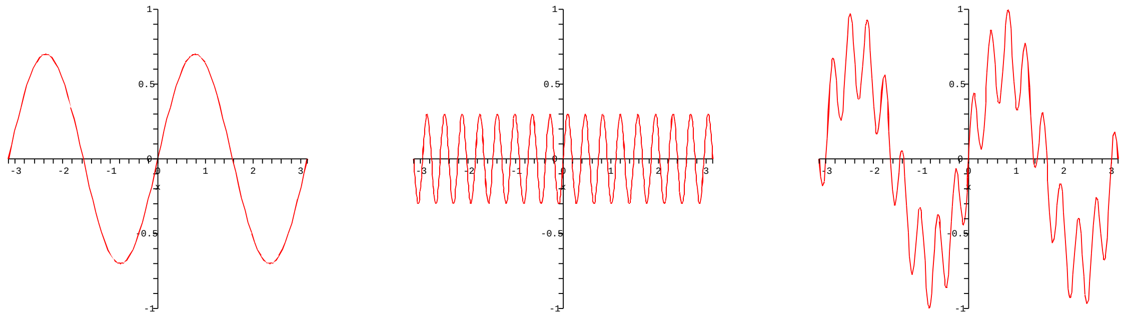
$$\mathsf{L}(u) \;=\; \partial_t\,u \;-\; \triangle u \;+\; \left\langle \vec{V}, \nabla u \right\rangle \;+\; u \cdot \mathbf{div}\,\vec{V}. \qquad \diamondsuit$$

Linear Homogeneous PDEs are nice because we can combine two solutions together to obtain a third solution....

Example 5.9:

(a) Let $u(x;t) = \frac{7}{10}\sin[2t+2x]$ and $v(x;t) = \frac{3}{10}\sin[17t+17x]$ be two travelling wave solutions to the Wave Equation. Then $w(x;t) = u(x;t) + v(x;t) = \frac{7}{10}\sin(2t+2x) + \frac{3}{10}\sin(17t+17x)$ is also a solution (see Figure 5.3). To use a musical analogy: if we think of u and v as two "pure tones", then we can think of w as a "chord".

[1] See § 2.3 on page 25.
[2] See § 2.2 on page 21.
[3] See § 3.2 on page 44.
[4] See § 4.2 on page 56.
[5] See § 2.7 on page 34.



$u(x,t) = \frac{7}{10}\sin(2t + 2x)$ $\qquad$ $v(x;t) = \frac{3}{10}\sin(17t + 17x)$ $\qquad$ $w(x,t) = u(x;t) + v(x;t)$

Figure 5.3: Example 5.9(a).

(b) Let $f(x;t) = \dfrac{1}{2\sqrt{\pi t}} \exp\left[\dfrac{-x^2}{4t}\right]$, $g(x;t) = \dfrac{1}{2\sqrt{\pi t}} \exp\left[\dfrac{-(x-3)^2}{4t}\right]$, and $h(x;t) = \dfrac{1}{2\sqrt{\pi t}} \exp\left[\dfrac{-(x-5)^2}{4t}\right]$ be one-dimensional Gauss-Weierstrass kernels, centered at 0, 3, and 5, respectively. Thus, f, g, and h are all solutions to the Heat Equation. Then, $F(x) = f(x) + 7 \cdot g(x) + h(x)$ is also a solution to the Heat Equation. If a Gauss-Weierstrass kernel models the erosion of a single "mountain", then the function F models the erosion of an little "mountain range", with peaks at 0, 3, and 5, and where the middle peak is seven times higher than the other two. $\diamondsuit$

These examples illustrate a general principle:

Theorem 5.10: Superposition Principle for Homogeneous Linear PDEs

Suppose L *is a linear differential operator, and* $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{C}^\infty$ *are solutions to the homogeneous linear PDE "*$\mathsf{L}\mathbf{u} = 0$.*" Then, for any* $c_1, c_2 \in \mathbb{R}$, $u = c_1 \cdot \mathbf{u}_1 + c_2 \cdot \mathbf{u}_2$ *is also a solution.*

Proof: <u>**Exercise 5.5**</u> $\Box$

If $q \in \mathcal{C}^\infty$ is some fixed nonzero function, then the equation "$\mathsf{L}p \equiv q$" is called a **nonhomogeneous** linear partial differential equation.

Example 5.11: The following are linear *non*homogeneous PDEs

(a) The **antidifferentiation equation** $p' = q$ is familiar from first year calculus. The *Fundamental Theorem of Calculus* effectively says that the solution to this equation is the integral $p(x) = \displaystyle\int_0^x q(y)\, dy$.

(b) The **Poisson Equation**[6], "$\triangle p = q$", is a *nonhomogeneous* linear PDE. ♢

Recall Examples 2.7 and 2.8 on page 29, where we obtained *new* solutions to a nonhomogeneous equation by taking a single solution, and adding solutions of the *homogeneous* equation to this solution. These examples illustrates a general principle:

Theorem 5.12: Subtraction Principle for nonhomogeneous linear PDEs

Suppose L *is a linear differential operator, and* $q \in \mathcal{C}^\infty$. *Let* $p_1 \in \mathcal{C}^\infty$ *be a solution to the nonhomogeneous linear PDE "*$\mathsf{L}p_1 = q$.*" If* $h \in \mathcal{C}^\infty$ *is any solution to the homogeneous equation (ie.* $\mathsf{L}h = 0$*), then* $p_2 = p_1 + h$ *is another solution to the* non*homogeneous equation. In summary:*

$$\Big(\mathsf{L}p_1 = q; \quad \mathsf{L}h = 0; \quad \text{and } p_2 = p_1 + h. \Big) \Longrightarrow \Big(\mathsf{L}p_2 = q \Big).$$

Proof: **Exercise 5.6** □

If L is *not* a linear operator, then a PDE of the form "$\mathsf{L}u \equiv 0$" or "$\mathsf{L}u \equiv g$" is called a **nonlinear** PDE. For example, a general **reaction-diffusion equation**[7]**:**

$$\partial_t \mathbf{u} = \triangle \mathbf{u} + \mathsf{L}(\mathbf{u}),$$

is *nonlinear* (because L is generally a nonlinear function of $\mathbf{u}$)

The theory of linear partial differential equations is well-developed, because solutions to linear PDEs interact in very nice ways, as shown by Theorems 5.10 and 5.12. The theory of *non*linear PDEs is much less developed, and indeed, many of the methods which *do* exist for solving nonlinear PDEs involve somehow 'approximating' them with linear ones. In this book we shall concern ourselves only with linear PDEs.

5.4 Practice Problems

1. For each of the following equations: u is an unknown function; q is always some fixed, predetermined function; and λ is always a constant.

   In each case, is the equation linear? If it is linear, is it homogeneous? Justify your answers.

   (a) Heat Equation: $\partial_t u(\mathbf{x}) = \triangle u(\mathbf{x})$.

   (b) Poisson Equation: $\triangle u(\mathbf{x}) = q(\mathbf{x})$.

   (c) Laplace Equation: $\triangle u(\mathbf{x}) = 0$.

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[6]See § 2.4 on page 28

[7]See § 2.8 on page 34

(d) Monge-Ampère Equation: $q(x,y) = \det\begin{bmatrix} \partial_x^2 u(x,y) & \partial_x\partial_y u(x,y) \\ \partial_x \partial_y u(x,y) & \partial_y^2 u(x,y) \end{bmatrix}$.

(e) Reaction-Diffusion $\partial_t u(\mathbf{x};t) = \triangle u(\mathbf{x};t) + q\Big(u(\mathbf{x};t)\Big)$.

(f) Scalar conservation Law $\partial_t u(x;t) = -\partial_x (q \circ u)(x;t)$.

(g) Helmholtz Equation: $\triangle u(\mathbf{x}) = \lambda \cdot u(\mathbf{x})$.

(h) Airy's Equation: $\partial_t u(x;t) = -\partial_x^3 u(x;t)$.

(i) Beam Equation: $\partial_t u(x;t) = -\partial_x^4 u(x;t)$.

(j) Schrödinger Equation: $\partial_t u(\mathbf{x};t) = \mathbf{i} \triangle u(\mathbf{x};t) + q(\mathbf{x};t) \cdot u(\mathbf{x};t)$.

(k) Burger's Equation: $\partial_t u(x;t) = -u(x;t) \cdot \partial_x u(x;t)$.

(l) Eikonal Equation: $|\partial_x u(x)| = 1$.

2. Which of the following are eigenfunctions for the 2-dimensional Laplacian $\triangle = \partial_x^2 + \partial_y^2$? In each case, if u is an eigenfunction, what is the eigenvalue?

(a) $u(x,y) = \sin(x)\sin(y)$ (Figure 6.8(A) on page 109)

(b) $u(x,y) = \sin(x) + \sin(y)$ (Figure 6.8(B) on page 109)

(c) $u(x,y) = \cos(2x) + \cos(y)$ (Figure 6.8(C) on page 109)

(d) $u(x,y) = \sin(3x) \cdot \cos(4y)$.

(e) $u(x,y) = \sin(3x) + \cos(4y)$.

(f) $u(x,y) = \sin(3x) + \cos(3y)$.

(g) $u(x,y) = \sin(3x) \cdot \cosh(4y)$.

(h) $u(x,y) = \sinh(3x) \cdot \cosh(4y)$.

(i) $u(x,y) = \sinh(3x) + \cosh(4y)$.

(j) $u(x,y) = \sinh(3x) + \cosh(3y)$.

(k) $u(x,y) = \sin(3x + 4y)$.

(l) $u(x,y) = \sinh(3x + 4y)$.

(m) $u(x,y) = \sin^3(x) \cdot \cos^4(y)$.

(n) $u(x,y) = e^{3x} \cdot e^{4y}$.

(o) $u(x,y) = e^{3x} + e^{4y}$.

(p) $u(x,y) = e^{3x} + e^{3y}$.

Notes: ....

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6 Classification of PDEs and Problem Types

6.1 Evolution vs. Nonevolution Equations

Recommended: §2.2, §2.3, §3.2, §5.2

An **evolution equation** is a PDE with a distinguished "time" coordinate, t. In other words, it describes functions of the form $u(\mathbf{x};t)$, and the equation has the form:

$$\mathsf{D}_t\, u \;=\; \mathsf{D}_{\mathbf{x}}\, u$$

where D_t is some differential operator involving only derivatives in the t variable (eg. ∂_t, ∂_t^2, etc.), while $\mathsf{D}_{\mathbf{x}}$ is some differential operator involving only derivatives in the $\mathbf{x}$ variables (eg. ∂_x, ∂_y^2, $\triangle$, etc.)

Example 6.1: The following are evolution equations:

(a) The Heat Equation "$\partial_t\, u = \triangle u$" of §2.2.

(b) The Wave Equation "$\partial_t^2\, u = \triangle u$" of §3.2.

(c) The Telegraph Equation "$\kappa_2 \partial_t^2\, u \;+\; \kappa_1 \partial_t\, u \;=\; -\kappa_0 u \;+\; \triangle u$" of §3.3.

(d) The Schrödinger equation "$\partial_t\, \omega \;=\; \frac{1}{\mathbf{i}\hbar} \mathsf{H}\, \omega$" of §4.2 (here H is a Hamiltonian operator).

(e) Liouville's Equation, the Fokker-Plank equation, and Reaction-Diffusion Equations. ♢

Nonexample 6.2: The following are *not* evolution equations:

(a) The Laplace Equation "$\triangle u = 0$" of §2.3.

(b) The Poisson Equation "$\triangle u = q$" of §2.4.

(c) The Helmholtz Equation "$\triangle u = \lambda u$" (where $E \in \mathbb{C}$ is a constant eigenvalue).

(d) The Stationary Schrödinger equation $\mathsf{H}\ \omega_0 \;=\; E \cdot \omega_0$ (where $E \in \mathbb{C}$ is a constant eigenvalue). ♢

In mathematical models of physical phenomena, most PDEs are evolution equations. Nonevolutionary PDEs generally arise as **stationary state** equations for evolution PDEs (eg. Laplace's equation) or as **resonance states** (eg. Sturm-Liouville, Helmholtz).

Order: The **order** of the differential operator $\partial_x^2 \partial_y^3$ is $2 + 3 = 5$. More generally, the **order** of the differential operator $\partial_1^{k_1} \partial_2^{k_2} \ldots \partial_D^{k_D}$ is the sum $k_1 + \ldots + k_D$. The **order** of a general differential operator is the highest order of any of its terms. For example, the Laplacian is second order. The **order** of a PDE is the highest order of the differential operator that appears in it. Thus, the Transport Equation, Liouville's Equation, and the (nondiffusive) Reaction Equation is *first order*, but all the other equations we have looked at (the Heat Equation, the Wave Equation, etc.) are of *second order*.

6.2 Classification of Second Order Linear PDEs (*)

Prerequisites: §6.1 **Recommended:** §2.2, §2.3, §2.7, §3.2

6.2(a) ...in two dimensions, with constant coefficients

Recall that $\mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R})$ is the space of all differentiable scalar fields on the two-dimensional plane. In general, a second-order linear differential operator L on $\mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R})$ with constant coefficients looks like:

$$\mathsf{L}u \quad = \quad a\cdot\partial_x^2 u \;+\; b\cdot\partial_x\partial_y u \;+\; c\cdot\partial_y^2 u \;+\; d\cdot\partial_x u \;+\; e\cdot\partial_y u \;+\; f\cdot u \tag{6.1}$$

where a, b, c, d, e, f are constants. Define:

$$\alpha = f, \qquad \boldsymbol{\beta} \;=\; \begin{bmatrix} d \\ e \end{bmatrix} \qquad \text{and} \qquad \Gamma \;=\; \begin{bmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{bmatrix} \;=\; \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}.$$

Then we can rewrite (6.1) as:

$$\mathsf{L}u \;=\; \sum_{c,d=1}^{2} \gamma_{c,d}\cdot\partial_c\partial_d u \;+\; \sum_{d=1}^{2}\beta_d\cdot\partial_d u \;+\; \alpha\cdot u,$$

Any 2×2 symmetric matrix Γ defines a **quadratic form** $G:\mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$G(x,y) \;=\; [x\ y]\cdot\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}\cdot\begin{bmatrix} x \\ y \end{bmatrix} \;=\; \gamma_{11}\cdot x^2 \;+\; \left(\gamma_{12}+\gamma_{21}\right)\cdot xy \;+\; \gamma_{22}\cdot y^2.$$

We say Γ is **positive definite** if, for all $x, y \in \mathbb{R}$, we have:

- $G(x,y) \geq 0$;
- $G(x,y) = 0$ if and only if $x = 0 = y$.

Geometrically, this means that the graph of G defines an *elliptic paraboloid* in $\mathbb{R}^2\times\mathbb{R}$, which curves upwards in every direction. Equivalently, Γ is positive definite if there is a constant $K > 0$ so that

$$G(x,y) \geq K\cdot(x^2+y^2)$$

for every $(x,y)\in\mathbb{R}^2$. We say Γ is **negative definite** if $-\Gamma$ is positive definite.

The differential operator L from equation (6.1) is called **elliptic** if the matrix Γ is either positive definite or negative definite.

Example: If $\mathsf{L} = \triangle$, then $\Gamma = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ is just the identity matrix. while $\boldsymbol{\beta} = 0$ and $\alpha = 0$. The identity matrix is clearly positive definite; thus, $\triangle$ is an elliptic differential operator.____

Suppose that L is an elliptic differential operator. Then:

- An **elliptic** PDE is one of the form: $\mathsf{L}u = 0$ (or $\mathsf{L}u = g$). For example, the **Laplace equation** is elliptic.
- A **parabolic** PDE is one of the form: $\partial_t = \mathsf{L}u$. For example, the two-dimensional **Heat Equation** is parabolic.
- A **hyperbolic** PDE is one of the form: $\partial_t^2 = \mathsf{L}u$. For example, the two-dimensional **Wave Equation** is hyperbolic.

Exercise 6.1 Show that Γ is positive definite if and only if $0 < \det(\Gamma) = ac - \frac{1}{4}b^2$. In other words, L is elliptic if and only if $4ac - b^2 > 0$ (this is the condition on page 9 of Pinsky).

6.2(b) ...in general

Recall that $\mathcal{C}^\infty(\mathbb{R}^D;\mathbb{R})$ is the space of all differentiable scalar fields on D-dimensional space The general second-order linear differential operator on $\mathcal{C}^\infty(\mathbb{R}^D;\mathbb{R})$ has the form

$$\mathsf{L}u \quad = \quad \sum_{c,d=1}^{D} \gamma_{c,d} \cdot \partial_c \partial_d u \quad + \quad \sum_{d=1}^{D} \beta_d \cdot \partial_d u \quad + \quad \alpha \cdot u, \tag{6.2}$$

where $\alpha : \mathbb{R}^D \times \mathbb{R} \longrightarrow \mathbb{R}$ is some time-varying scalar field, $(\beta_1, \ldots, \beta_D) = \boldsymbol{\beta} : \mathbb{R}^D \times \mathbb{R} \longrightarrow \mathbb{R}^D$ is a time-varying vector field, and $\gamma_{c,d} : \mathbb{R}^D \times \mathbb{R} \longrightarrow \mathbb{R}$ are functions so that, for any $\mathbf{x} \in \mathbb{R}^D$ and $t \in \mathbb{R}$, the matrix

$$\Gamma(\mathbf{x};t) = \begin{bmatrix} \gamma_{11}(\mathbf{x};t) & \ldots & \gamma_{1D}(\mathbf{x};t) \\ \vdots & \ddots & \vdots \\ \gamma_{D1}(\mathbf{x};t) & \ldots & \gamma_{DD}(\mathbf{x};t) \end{bmatrix}$$

is **symmetric** (ie. $\gamma_{cd} = \gamma_{dc}$).

Example 6.3:

(a) If $\mathsf{L} = \triangle$, then $\boldsymbol{\beta} \equiv 0$, $\alpha = 0$, and $\Gamma \equiv \mathbf{Id} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$.

(b) The Fokker-Plank Equation has the form $\partial_t u = \mathsf{L}u$, where $\alpha = -\mathbf{div}\,\vec{V}(\mathbf{x})$, $\boldsymbol{\beta}(\mathbf{x}) = -\nabla\vec{V}(\mathbf{x})$, and $\Gamma \equiv \mathbf{Id}$. ♢

If the functions $\gamma_{c,d}$, β_d and α are independent of $\mathbf{x}$, then we say L is **spacially homogeneous**. If they are also independent of t, we say that L has **constant coefficients**.

Any symmetric matrix Γ defines a **quadratic form** $G : \mathbb{R}^D \longrightarrow \mathbb{R}$ by

$$G(\mathbf{x}) = [x_1 ... x_D] \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1D} \\ \vdots & \ddots & \vdots \\ \gamma_{D1} & \cdots & \gamma_{DD} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_D \end{bmatrix} = \sum_{c,d=1}^{D} \gamma_{c,d} \cdot x_c \cdot x_d$$

Γ is called **positive definite** if, for all $\mathbf{x} \in \mathbb{R}^D$, we have:

- $G(\mathbf{x}) \geq 0$;
- $G(\mathbf{x}) = 0$ if and only if $\mathbf{x} = 0$.

Equivalently, Γ is positive definite if there is a constant $K > 0$ so that $G(\mathbf{x}) \geq K \cdot \|\mathbf{x}\|^2$ for every $\mathbf{x} \in \mathbb{R}^D$. On the other hand, Γ is **negative definite** if $-\Gamma$ is positive definite.

The differential operator L from equation (6.2) is **elliptic** if the matrix $\Gamma(\mathbf{x})$ is either positive definite or negative definite for every $\mathbf{x} \in \mathbb{R}^D$.

For example, the Laplacian and the Fokker-Plank operator are both elliptic.

Suppose that L is an elliptic differential operator. Then:

- An **elliptic** PDE is one of the form: $\mathsf{L}u = 0$ (or $\mathsf{L}u = g$).
- A **parabolic** PDE is one of the form: $\partial_t = \mathsf{L}u$.
- A **hyperbolic** PDE is one of the form: $\partial_t^2 = \mathsf{L}u$.

Example 6.4:

(a) Laplace's Equation and Poisson's Equation are *elliptic* PDEs.

(b) The Heat Equation and the Fokker-Plank Equation are *parabolic*.

(c) The Wave Equation is *hyperbolic*. ♢

Parabolic equations are "generalized Heat Equations", describing *diffusion through an inhomogeneous*[1]*, anisotropic*[2] *medium with drift.* The terms in $\Gamma(\mathbf{x};t)$ describe the inhomogeneity and anisotropy of the diffusion[3], while the vector field $\boldsymbol{\beta}$ describes the drift.

Hyperbolic equations are "generalized Wave Equations", describing *wave propagation* through an inhomogeneous, anisotropic medium with drift —for example, sound waves propagating through an air mass with variable temperature and pressure and wind blowing.

[1]**Homogeneous** means, "Looks the same everywhere in space", whereas **inhomogeneous** is the opposite.
[2]**Isotropic** means "looks the same in every direction"; **anisotropic** means the opposite.
[3]If the medium was homogeneous, then Γ would be constant. If the medium was isotropic, then $\Gamma = \mathbf{Id}$.

6.3 Practice Problems

For each of the following equations: u is an unknown function; q is always some fixed, predetermined function; and λ is always a constant. In each case, determine the *order* of the equation, and decide: is this an *evolution equation*? Why or why not?

1. Heat Equation: $\partial_t u(\mathbf{x}) \;=\; \triangle u(\mathbf{x})$.
2. Poisson Equation: $\triangle u(\mathbf{x}) \;=\; q(\mathbf{x})$.
3. Laplace Equation: $\triangle u(\mathbf{x}) \;=\; 0$.
4. Monge-Ampère Equation: $q(x,y) \;=\; \det \begin{bmatrix} \partial_x^2 u(x,y) & \partial_x \partial_y u(x,y) \\ \partial_x \partial_y u(x,y) & \partial_y^2 u(x,y) \end{bmatrix}$.
5. Reaction-Diffusion $\partial_t u(\mathbf{x};t) \;=\; \triangle u(\mathbf{x};t) \;+\; q\Big(u(\mathbf{x};t)\Big)$.
6. Scalar conservation Law $\partial_t u(x;t) \;=\; -\partial_x (q \circ u)(x;t)$.
7. Helmholtz Equation: $\triangle u(\mathbf{x}) \;=\; \lambda \cdot u(\mathbf{x})$.
8. Airy's Equation: $\partial_t u(x;t) \;=\; -\partial_x^3 u(x;t)$.
9. Beam Equation: $\partial_t u(x;t) \;=\; -\partial_x^4 u(x;t)$.
10. Schrödinger Equation: $\partial_t u(\mathbf{x};t) \;=\; \mathbf{i} \triangle u(\mathbf{x};t) \;+\; q(\mathbf{x};t) \cdot u(\mathbf{x};t)$.
11. Burger's Equation: $\partial_t u(x;t) \;=\; -u(x;t) \cdot \partial_x u(x;t)$.
12. Eikonal Equation: $|\partial_x u(x)| \;=\; 1$.

Notes: ....

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6.4 Initial Value Problems

Prerequisites: §6.1

Let $\mathbb{X} \subset \mathbb{R}^D$ be some domain, and let L be a differential operator on $\mathcal{C}^\infty(\mathbb{X};\mathbb{R})$. Consider evolution equation

$$\partial_t u = \mathsf{L}\, u \tag{6.3}$$

for unknown $u : \mathbb{X} \times \mathbb{R} \longrightarrow \mathbb{R}$. An **initial value problem (IVP)** or **Cauchy problem** for equation (6.3) is the following problem: Given some function $f_0 : \mathbb{X} \longrightarrow \mathbb{R}$ (the **initial conditions**), find a function u which satisfies (6.3) and also satisfies:

$$\text{For all } \mathbf{x} \in \mathbb{X}, \ \ u(\mathbf{x}, 0) = f_0(\mathbf{x}).$$

For example, suppose the domain $\mathbb{X}$ is an iron pan and its contents resting on a hot electric stove burner. You turn off the stove (so there is no further heat entering the system) and then throw some vegetables into the pan. Thus, (6.3) is the Heat Equation, and f_0 describes the initial distribution of heat: cold vegetables in a hot pan on a hotter stove. The initial value problem basically askes, "How fast do the vegetables cook? How fast does the pan cool?"

Next, consider the second order-evolution equation

$$\partial_t^2 u = \mathsf{L}\, u \tag{6.4}$$

An **initial value problem** for (6.4) is as follows: Fix a function $f_0 : \mathbb{X} \longrightarrow \mathbb{R}$ (the **initial conditions**), and/or another function $f_1 : \mathbb{X} \longrightarrow \mathbb{R}$ (the **initial velocity**) and then search for a function u satisfying (6.4) and also satisfying:

$$\text{For all } \mathbf{x} \in \mathbb{X}, \ \ u(\mathbf{x}, 0) = f_0(\mathbf{x}) \ \text{ and } \ \partial_t u(\mathbf{x}, 0) = f_1(\mathbf{x})$$

For example, suppose (6.3) is the Wave Equation on $\mathbb{X} = [0, L]$. Imagine $[0, L]$ as a vibrating string. Thus, f_0 describes the initial displacement of the string, and f_1 its initial momentum.

If $f_0 \not\equiv 0$, and $f_1 \equiv 0$, then the string is initially at rest, but is released from a displaced state —in other words, it is *plucked*. Hence, the initial value problem asks, "How does a guitar string sound when it is plucked?"

On the other hand, if $f_0 \equiv 0$, and $f_1 \not\equiv 0$, then the string is initially flat, but is imparted with nonzero momentum –in other words, it is *struck* (by the hammer in the piano). Hence, the initial value problem asks, "How does a piano string sound when it is struck?"

6.5 Boundary Value Problems

Prerequisites: §1.6, §2.3 **Recommended:** §6.4

If $\mathbb{X} \subset \mathbb{R}^D$ be is a finite domain, then $\partial\mathbb{X}$ denotes its **boundary**. The **interior** of $\mathbb{X}$ is the set $\mathsf{int}(\mathbb{X})$ of all points in $\mathbb{X}$ *not* on the boundary.

Johann Peter Gustav Lejeune Dirichlet
Born: February 13, 1805 in Düren, (now Germany)
Died: May 5, 1859 in Göttingen, Hanover

Example 6.5:

(a) If $\mathbb{I} = [0,1] \subset \mathbb{R}$ is the **unit interval**, then $\partial\mathbb{I} = \{0,1\}$ is a two-point set, and $\mathsf{int}(\mathbb{I}) = (0,1)$.

(b) If $\mathbb{X} = [0,1]^2 \subset \mathbb{R}^2$ is the **unit square**, then $\mathsf{int}(\mathbb{X}) = (0,1)^2$. and

$$\partial\mathbb{X} \quad = \quad \{(x,y) \in \mathbb{X} \ ; \ x = 0 \text{ or } x = 1 \text{ or } y = 0 \text{ or } y = 1\}.$$

(c) In polar coordinates on $\mathbb{R}^2$, let $\mathbb{D} = \{(r,\theta) \ ; \ r \leq 1, \theta \in [-\pi,\pi)\}$ be the **unit disk**. Then $\partial\mathbb{D} = \{(1,\theta) \ ; \ \theta \in [-\pi,\pi)\}$ is the **unit circle**, and $\mathsf{int}(\mathbb{D}) = \{(r,\theta) \ ; \ r < 1, \ \theta \in [-\pi,\pi)\}$.

(d) In spherical coordinates on $\mathbb{R}^3$, let $\mathbb{B} = \left\{\mathbf{x} \in \mathbb{R}^3 \ ; \ \|\mathbf{x}\| \leq 1\right\}$ be the 3-dimensional **unit ball** in $\mathbb{R}^3$. Then $\partial\mathbb{B} = \mathbb{S} := \{\left\{\mathbf{x} \in \mathbb{R}^D \ ; \ \|\mathbf{x}\| = 1\right\}$ is the **unit sphere**, and $\mathsf{int}(\mathbb{B}) = \left\{\mathbf{x} \in \mathbb{R}^D \ ; \ \|\mathbf{x}\| < 1\right\}$.

(e) In cylindrical coordinates on $\mathbb{R}^3$, let $\mathbb{X} = \{(r,\theta,z) \ ; \ r \leq R, \ \ 0 \leq z \leq L\}$ be the **finite cylinder** in $\mathbb{R}^3$. Then $\partial\mathbb{X} = \{(r,\theta,z) \ ; \ r = R \text{ or } z = 0 \text{ or } z = L\}$. ♢

A **boundary value problem** is a problem of the following kind: *Find* $u : \mathbb{X} \longrightarrow \mathbb{R}$ *so that*

1. *u satisfies some PDE at all* $\mathbf{x}$ *in the* **interior** *of* $\mathbb{X}$.

2. *u also satisfies some other equation (maybe a differential equation) for all* $\mathbf{x}$ *on the* **boundary** *of* $\mathbb{X}$.

3. (Optional) *u also some initial condition, as described in §6.4.*

The condition u must satisfy on the boundary of $\mathbb{X}$ is called a **boundary condition**. We will consider four kinds of boundary conditions: *Dirichlet*, *Neumann*, *Mixed*, and *Periodic*; each has a particular physical interpretation, and yields particular kinds of solutions for a partial differential equation.

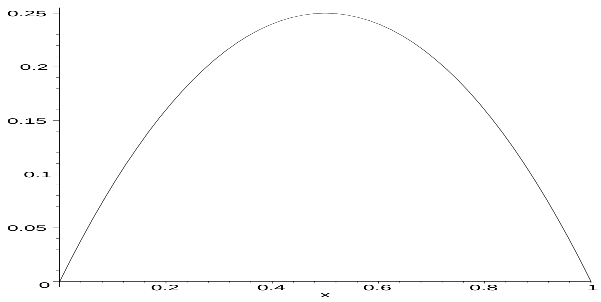


Figure 6.1: $f(x) = x(1-x)$ satisfies homogeneous Dirichlet boundary conditions on the interval $[0,1]$.

6.5(a) Dirichlet boundary conditions

Let $\mathbb{X}$ be a domain, and let $u : \mathbb{X} \longrightarrow \mathbb{R}$ be a function. We say that u satisfies **homogeneous Dirichlet boundary conditions (HDBC)** on $\mathbb{X}$ if:

$$\text{For all } \mathbf{x} \in \partial\mathbb{X}, \quad u(\mathbf{x}) \equiv 0.$$

Physical interpretation:

Heat Equation or Laplace Equation: In this case, u represents a temperature distribution. We imagine that the domain $\mathbb{X}$ represents some physical object, whose boundary $\partial\mathbb{X}$ is made out of metal or some other material which conducts heat almost perfectly. Hence, we can assume that *the temperature on the boundary is always equal to the temperature of the surrounding environment.*

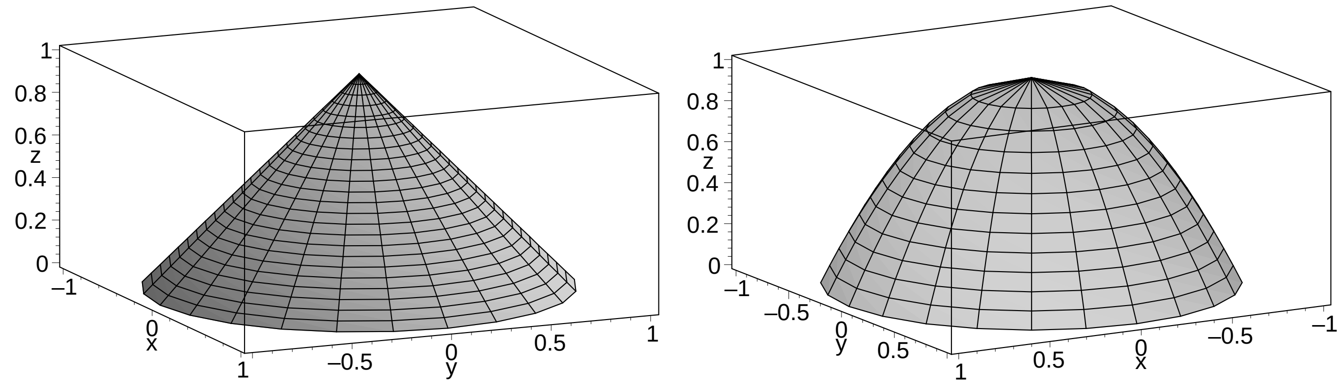
We further assume that this environment has a constant temperature T_E (for example, $\mathbb{X}$ is immersed in a 'bath' of some uniformly mixed fluid), which remains constant during the experiment (for example, the fluid is present in large enough quantities that the heat flowing into/out of $\mathbb{X}$ does not measurably change it). We can then assume that the ambient temperature is $T_E \equiv 0$, by simply subtracting a constant temperature of T_E off the inside and the outside. (This is like changing from measuring temperature in degrees Kelvin to measuring in degrees Celsius; you're just adding 273^o to both sides, which makes no mathematical difference.)

Wave Equation: In this case, u represents the vibrations of some vibrating medium (eg. a violin string or a drum skin). Homogeneous Dirichlet boundary conditions mean that the medium is *fixed* on the boundary $\partial\mathbb{X}$ (eg. a violin string is clamped at its endpoints; a drumskin is pulled down tightly around the rim of the drum).

The set of *infinitely differentiable* functions from $\mathbb{X}$ to $\mathbb{R}$ whic satisfy homogeneous Dirichlet Boundary Conditions will be denoted $\mathcal{C}_0^\infty(\mathbb{X};\mathbb{R})$ or $\mathcal{C}_0^\infty(\mathbb{X})$. Thus, for example

$$\mathcal{C}_0^\infty[0,L] \;=\; \Big\{ f : [0,L] \longrightarrow \mathbb{R}; \;\; f \text{ is smooth, and } f(0) \;=\; 0 \;=\; f(L) \Big\}$$

The set of *continuous* functions from $\mathbb{X}$ to $\mathbb{R}$ whic satisfy homogeneous Dirichlet Boundary Conditions will be denoted $\mathcal{C}_0(\mathbb{X};\mathbb{R})$ or $\mathcal{C}_0(\mathbb{X})$.



(A) **(B)**

Figure 6.2: **(A)** $f(r,\theta) = 1-r$ satisfies homogeneous Dirichlet boundary conditions on the disk $\mathbb{D} = \{(r,\theta)\ ;\ r \leq 1\}$, but is not smooth at zero. **(B)** $f(r,\theta) = 1-r^2$ satisfies homogeneous Dirichlet boundary conditions on the disk $\mathbb{D} = \{(r,\theta)\ ;\ r \leq 1\}$, and is smooth everywhere.

Example 6.6:

(a) Suppose $\mathbb{X} = [0,1]$, and $f : \mathbb{X} \longrightarrow \mathbb{R}$ is defined by $f(x) = x(1-x)$. Then $f(0) = 0 = f(1)$, and f is smooth, so $f \in \mathcal{C}_0^\infty[0,1]$. (See Figure 6.1).

(b) Let $\mathbb{X} = [0,\pi]$.

1. For any $n \in \mathbb{N}$, let $\mathbf{S}_n(x) = \sin(n \cdot x)$. Then $\mathbf{S}_n \in \mathcal{C}_0^\infty[0,\pi]$.
2. If $f(x) = 5\sin(x) - 3\sin(2x) + 7\sin(3x)$, then $f \in \mathcal{C}_0^\infty[0,\pi]$. More generally, any finite sum $\sum_{n=1}^{N} B_n\mathbf{S}_n(x)$ is in $\mathcal{C}_0^\infty[0,\pi]$.
3. If $f(x) = \sum_{n=1}^{\infty} B_n\mathbf{S}_n(x)$ is a *uniformly convergent* Fourier sine series[4], then $f \in \mathcal{C}_0^\infty[0,\pi]$.

(c) Let $\mathbb{D} = \{(r,\theta)\ ;\ r \leq 1\}$ is the unit disk. Let $f : \mathbb{D} \longrightarrow \mathbb{R}$ be the 'cone' in Figure 6.2(A), defined: $f(r,\theta) = (1-r)$. Then f is continuous, and $f \equiv 0$ on the boundary of the disk, so f satisfies Dirichlet boundary conditions. Thus, $f \in \mathcal{C}_0(\mathbb{D})$. However, f is not smooth (it is singular at zero), so $f \notin \mathcal{C}_0^\infty(\mathbb{D})$.

(d) Let $f : \mathbb{D} \longrightarrow \mathbb{R}$ be the 'dome' in Figure 6.2(B), defined $f(r,\theta) = 1-r^2$. Then $f \in \mathcal{C}_0^\infty(\mathbb{D})$.

(e) Let $\mathbb{X} = [0,\pi] \times [0,\pi]$ be the square of sidelength π.

1. For any $(n,m) \in \mathbb{N}^2$, let $\mathbf{S}_{(n,m)}(x,y) = \sin(n \cdot x) \cdot \sin(m \cdot y)$. Then $\mathbf{S}_{(n,m)} \in \mathcal{C}_0^\infty(\mathbb{X})$. (see Figure 10.2 on page 182).

[4] See § 8.2 on page 152.

2. If $f(x) = 5\sin(x)\sin(2y) - 3\sin(2x)\sin(7y) + 7\sin(3x)\sin(y)$, then $f \in \mathcal{C}_0^\infty(\mathbb{X})$. More generally, any finite sum $\sum_{n=1}^{N}\sum_{m=1}^{M} B_{n,m}\mathbf{S}_{n,m}(x)$ is in $\mathcal{C}_0^\infty(\mathbb{X})$.

3. If $f = \sum_{n,m=1}^{\infty} B_{n,m}\mathbf{S}_{n,m}$ is a *uniformly convergent* two dimensional Fourier sine series[5], then $f \in \mathcal{C}_0^\infty(\mathbb{X})$.

Exercise 6.2 Verify examples (b) to (e) ◇

Exercise 6.3 (a) Show that $\mathcal{C}_0^\infty(\mathbb{X})$ is a vector space. (b) Show that $\mathcal{C}_0(\mathbb{X})$ is a vector space.

Arbitrary **nonhomogeneous Dirichlet boundary conditions** are imposed by fixing some function $b : \partial\mathbb{X} \longrightarrow \mathbb{R}$, and then requiring:

$$u(\mathbf{x}) = b(\mathbf{x}), \quad \text{for all } \mathbf{x} \in \partial\mathbb{X}. \qquad (6.5)$$

For example, the **classical Dirichlet Problem** is to find $u : \mathbb{X} \longrightarrow \mathbb{R}$ satisfying the Dirichlet condition (6.5), and so that u also satisfies Laplace's Equation: $\triangle u(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathsf{int}(\mathbb{X})$. This models a stationary temperature distribution on $\mathbb{X}$, where the temperature is *fixed* on the boundary (eg. the boundary is a perfect conductor, so it takes the temperature of the surrounding medium).

For example, if $\mathbb{X} = [0, L]$, and $b(0)$ and $b(L)$ are two constants, then the **Dirichlet Problem** is to find $u : [0, L] \longrightarrow \mathbb{R}$ so that

$$u(0) = b(0), \quad u(L) = b(L), \quad \text{and} \quad \partial_x^2 u(x) = 0, \quad \text{for } 0 < x < L. \qquad (6.6)$$

That is, the temperature at the left-hand endpoint is fixed at $b(0)$, and at the right-hand endpoint is fixed at $b(1)$. The unique solution to this problem is $u(x) = (b(L) - b(0))x + a$.

6.5(b) Neumann Boundary Conditions

Suppose $\mathbb{X}$ is a domain with boundary $\partial\mathbb{X}$, and $u : \mathbb{X} \longrightarrow \mathbb{R}$ is some function. Then for any boundary point $\mathbf{x} \in \partial\mathbb{X}$, we use "$\partial_\perp u(\mathbf{x})$" to denote the **outward normal** derivative[6] of u on the boundary. Physically, $\partial_\perp u(\mathbf{x})$ is the *rate of change in u as you leave $\mathbb{X}$ by passing through $\partial\mathbb{X}$ in a perpendicular direction.*

Example 6.7:

(a) If $\mathbb{X} = [0, 1]$, then $\partial_\perp u(0) = -\partial_x u(0)$ and $\partial_\perp u(1) = \partial_x u(1)$.

(b) Suppose $\mathbb{X} = [0, 1]^2 \subset \mathbb{R}^2$ is the unit square, and $(x, y) \in \partial\mathbb{X}$. There are four cases:

- If $x = 0$, then $\partial_\perp u(x, y) = -\partial_x u(x, y)$.

[5] See § 10.1 on page 180.

[6] This is sometimes indicated as $\frac{\partial u}{\partial \mathbf{n}}$ or $\frac{\partial u}{\partial \nu}$, or as "$\nabla u \bullet \mathbf{n}$".

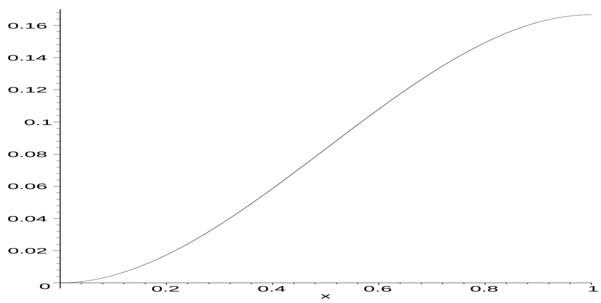


Figure 6.3: $f(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$ (homogeneous Neumann boundary conditions on the interval $[0, 1]$.)

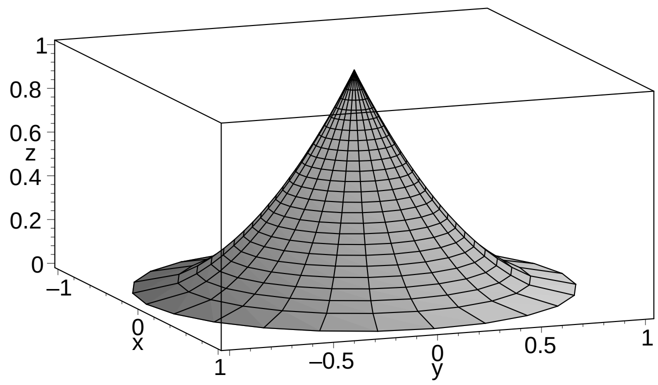


Figure 6.4: $f(r, \theta) = (1 - r)^2$ (homogeneous Neumann boundary conditions on the disk $\mathbb{D} = \{(r, \theta) \ ; \ r \leq 1\}$, but is singular at zero.)

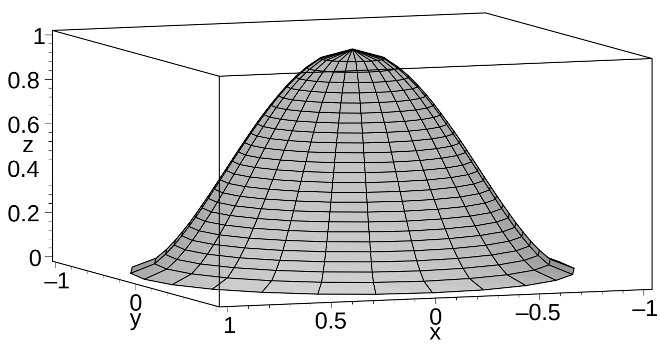


Figure 6.5: $f(r, \theta) = (1 - r^2)^2$ (homogeneous Neumann boundary conditions on the disk; smooth everywhere.)

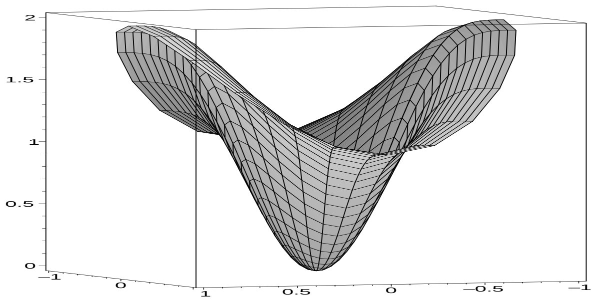


Figure 6.6: $f(r,\theta) = (1+\cos(\theta)^2)\cdot(1-(1-r^2)^4)$ (does *not* satisfy homogeneous Neumann boundary conditions on the disk; not constant on the boundary.)

- If $x = 1$, then $\partial_\perp u(x,y) = \partial_x u(x,y)$.
- If $y = 0$, then $\partial_\perp u(x,y) = -\partial_y u(x,y)$.
- If $y = 1$, then $\partial_\perp u(x,y) = \partial_y u(x,y)$.

(If more than one of these conditions is true —for example, at $(0,0)$ —then (x,y) is a corner, and $\partial_\perp u(x,y)$ is not well-defined).

(c) Let $\mathbb{D} = \{(r,\theta)\ ;\ r < 1\}$ be the unit disk in the plane. Then $\partial\mathbb{D}$ is the set $\{(1,\theta)\ ;\ \theta \in [-\pi,\pi)\}$, and for any $(1,\theta) \in \partial\mathbb{D}$, $\quad \partial_\perp u(1,\theta) \ = \ \partial_r u(1,\theta)$.

(d) Let $\mathbb{D} = \{(r,\theta)\ ;\ r < R\}$ be the disk of radius R. Then $\partial\mathbb{D} = \{(R,\theta)\ ;\ \theta \in [-\pi,\pi)\}$, and for any $(R,\theta) \in \partial\mathbb{D}$, $\quad \partial_\perp u(R,\theta) \ = \ \partial_r u(R,\theta)$.

(e) Let $\mathbb{B} = \{(r,\phi,\theta)\ ;\ r < 1\}$ be the unit ball in $\mathbb{R}^3$. Then $\partial\mathbb{B} = \{(r,\phi,\theta)\ ;\ r = 1\}$ is the unit sphere. If $u(r,\phi,\theta)$ is a function in polar coordinates, then for any boundary point $\mathbf{s} = (1,\phi,\theta)$, $\quad \partial_\perp u(\mathbf{s}) \ = \ \partial_r u(\mathbf{s})$.

(f) Suppose $\mathbb{X} = \{(r,\theta,z)\ ;\ r \leq R,\ \ 0 \leq z \leq L\}$, is the **finite cylinder**, and $(r,\theta,z) \in \partial\mathbb{X}$. There are three cases:

- If $r = R$, then $\partial_\perp u(r,\theta,z) \ = \ \partial_r u(r,\theta,z)$.
- If $z = 0$, then $\partial_\perp u(r,\theta,z) \ = \ -\partial_z u(r,\theta,z)$.
- If $z = L$, then $\partial_\perp u(r,\theta,z) \ = \ \partial_z u(r,\theta,z)$.

$\diamondsuit$

We say that u satisfies **homogeneous Neumann boundary condition** if

$$\partial_\perp u(\mathbf{x}) \ = \ 0 \ \text{ for all } \mathbf{x} \in \partial\mathbb{X}. \tag{6.7}$$

Physical Interpretation:

Heat (or Diffusion): Suppose u represents a temperature distribution. Recall that Fourier's Law of Heat Flow (§ 2.1 on page 20) says that $\nabla u(\mathbf{x})$ is the speed and direction in which heat is flowing at $\mathbf{x}$. Recall that $\partial_\perp u(\mathbf{x})$ is the component of $\nabla u(\mathbf{x})$ which is perpendicular to $\partial\mathbb{X}$. Thus, Homogeneous Neumann BC means that $\nabla u(\mathbf{x})$ is *parallel* to the boundary for all $\mathbf{x} \in \partial\mathbb{X}$. In other words *no heat is crossing the boundary*. This means that the boundary is a *perfect insulator*.

If u represents the concentration of a diffusing substance, then $\nabla u(\mathbf{x})$ is the flux of this substance at $\mathbf{x}$. Homogeneous Neumann Boundary conditions mean that the boundary is an *impermeable barrier* to this substance.

Heat/diffusion is normally the intended interpretation when Homogeneous Neumann BC appear in conjuction with the Heat Equation, and sometimes the Laplace equation.

Electrostatics Suppose u represents an electric potential. Thus $\nabla u(\mathbf{x})$ is the *electric field* at $\mathbf{x}$. Homogeneous Neumann BC means that $\nabla u(\mathbf{x})$ is *parallel* to the boundary for all $\mathbf{x} \in \partial\mathbb{X}$; i.e. no field lines penetrate the boundary. Elecrostatics is often the intended interpretation when Homogeneous Neumann BC appear in conjuction with the Laplace Equation or Poisson Equation:

The set of *continuous* functions from $\mathbb{X}$ to $\mathbb{R}$ which satisfy homogeneous Neumann boundary conditions will be denoted $\mathcal{C}_\perp(\mathbb{X})$. The set of *infinitely differentiable* functions from $\mathbb{X}$ to $\mathbb{R}$ which satisfy homogeneous Neumann boundary conditions will be denoted $\mathcal{C}_\perp^\infty(\mathbb{X})$. Thus, for example

$$\mathcal{C}_\perp^\infty[0,L] \quad = \quad \Big\{ f : [0,L] \longrightarrow \mathbb{R}; \quad f \text{ is smooth, and } f'(0) \;=\; 0 \;=\; f'(L) \Big\}$$

Example 6.8:

(a) Let $\mathbb{X} = [0,1]$, and let $f : [0,1] \longrightarrow \mathbb{R}$ be defined by $f(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$ (See Figure 6.3). Then $f'(0) = 0 = f'(1)$, and f is smooth, so $f \in \mathcal{C}_\perp^\infty[0,1]$.

(b) Let $\mathbb{X} = [0,\pi]$.

1. For any $n \in \mathbb{N}$, let. $\mathbf{C}_n(x) = \cos(n \cdot x)$. Then $\mathbf{C}_n \in \mathcal{C}_\perp^\infty[0,\pi]$.
2. If $f(x) = 5\cos(x) - 3\cos(2x) + 7\cos(3x)$, then $f \in \mathcal{C}_\perp^\infty[0,\pi]$. More generally, any finite sum $\displaystyle\sum_{n=1}^{N} A_n \mathbf{C}_n(x)$ is in $\mathcal{C}_\perp^\infty[0,\pi]$.
3. If $f(x) = \displaystyle\sum_{n=1}^{\infty} A_n \mathbf{C}_n(x)$ is a *uniformly convergent* Fourier cosine series[7], and the derivative series $f'(x) = -\displaystyle\sum_{n=1}^{\infty} n A_n \mathbf{S}_n(x)$ is *also* uniformly convergent, then $f \in \mathcal{C}_\perp^\infty[0,\pi]$.

[7] See § 8.2 on page 152.

(c) Let $\mathbb{D} = \{(r,\theta) \; ; \; r \leq 1\}$ be the unit disk.

1. Let $f : \mathbb{D} \longrightarrow \mathbb{R}$ be the "witch's hat" of Figure 6.4, defined: $f(r,\theta) := (1-r)^2$. Then $\partial_\perp f \equiv 0$ on the boundary of the disk, so f satisfies Neumann boundary conditions. Also, f is continuous on $\mathbb{D}$; hence $f \in \mathcal{C}_\perp(\mathbb{D})$. However, f is not smooth (it is singular at zero), so $f \notin \mathcal{C}^\infty_\perp(\mathbb{D})$.
2. Let $f : \mathbb{D} \longrightarrow \mathbb{R}$ be the "bell" of Figure 6.5, defined: $f(r,\theta) := (1-r^2)^2$. Then $\partial_\perp f \equiv 0$ on the boundary of the disk, and f is smooth everywhere on $\mathbb{D}$, so $f \in \mathcal{C}^\infty_\perp(\mathbb{D})$.
3. Let $f : \mathbb{D} \longrightarrow \mathbb{R}$ be the "flower vase" of Figure 6.6, defined $f(r,\theta) := (1+\cos(\theta)^2) \cdot (1-(1-r^2)^4)$. Then $\partial_\perp f \equiv 0$ on the boundary of the disk, and f is smooth everywhere on $\mathbb{D}$, so $f \in \mathcal{C}^\infty_\perp(\mathbb{D})$. Note that, in this case, the angular derivative is nonzero, so f is not constant on the boundary of the disk.

(d) Let $\mathbb{X} = [0,\pi] \times [0,\pi]$ be the square of sidelength π.

1. For any $(n,m) \in \mathbb{N}^2$, let $\mathbf{C}_{(n,m)}(x) = \cos(nx)\cdot\cos(my)$. Then $\mathbf{C}_{(n,m)} \in \mathcal{C}^\infty_\perp(\mathbb{X})$. (see Figure 10.2 on page 182).
2. If $f(x) = 5\cos(x)\cos(2y) - 3\cos(2x)\cos(7y) + 7\cos(3x)\cos(y)$, then $f \in \mathcal{C}^\infty_\perp(\mathbb{X})$. More generally, any finite sum $\sum_{n=1}^{N}\sum_{m=1}^{M} B_{n,m}\mathbf{C}_{n,m}(x)$ is in $\mathcal{C}^\infty_\perp(\mathbb{X})$.
3. More generally, if $f = \sum_{n,m=0}^{\infty} A_{n,m}\mathbf{C}_{n,m}$ is a *uniformly convergent* two dimensional Fourier cosine series[8], and the derivative series

$$\partial_x f(x,y) = -\sum_{n,m=0}^{\infty} nA_{\mathbf{n}} \sin(nx)\cdot\cos(my)$$

$$\partial_y f(x,y) = -\sum_{n,m=0}^{\infty} mA_{\mathbf{n}} \cos(nx)\cdot\sin(my)$$

are *also* uniformly convergent, then $f \in \mathcal{C}^\infty_\perp[0,L]^D$.

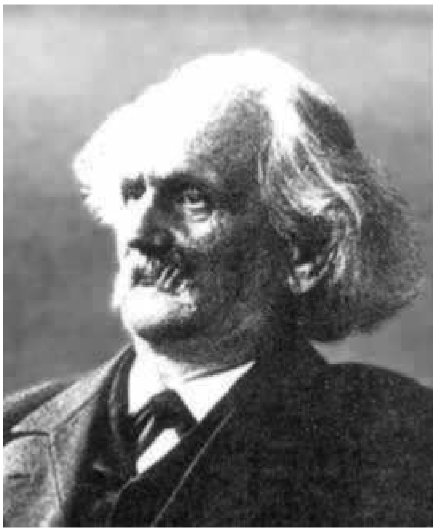
Exercise 6.4 Verify examples (b) to (d) ♢

Arbitrary **nonhomogeneous Neumann Boundary conditions** are imposed by fixing a function $b : \partial\mathbb{X} \longrightarrow \mathbb{R}$, and then requiring

$$\partial_\perp u(\mathbf{x}) = b(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \partial\mathbb{X}. \tag{6.8}$$

For example, the **classical Neumann Problem** is to find $u : \mathbb{X} \longrightarrow \mathbb{R}$ satisfying the Neumann condition (6.8), and so that u also satisfies Laplace's Equation: $\triangle u(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathsf{int}(\mathbb{X})$. This models a stationary temperature distribution on $\mathbb{X}$ where the temperature *gradient* is fixed on the boundary (eg. heat energy is entering or leaving through the boundary at some prescribed rate).

[8]See § 10.1 on page 180.



Carl Gottfried Neumann
Born: May 7, 1832 in Königsberg, (now Kaliningrad)
Died: March 27, 1925 in Leipzig, Germany

Physical Interpretation:

Heat Equation or Laplace Equation: Here u represents the concentration of some diffusing material. Recall that Fourier's Law (§ 2.1 on page 20) says that $\nabla u(\mathbf{x})$ is the flux of this material at $\mathbf{x}$. The nonhomogeneous Neumann Boundary condition $\nabla u(\mathbf{x}) = b(\mathbf{x})$ means that material is being 'pumped' across the boundary at a constant rate described by the function $b(\mathbf{x})$.

Laplace Equation or Poisson Equation: Here, u represents an electric potential. Thus $\nabla u(\mathbf{x})$ is the *electric field* at $\mathbf{x}$. Nonhomogeneous Neumann boundary conditions mean that the field vector perpendicular to the boundary is determined by the function $b(\mathbf{x})$.

6.5(c) Mixed (or Robin) Boundary Conditions

These are a combination of Dirichlet and Neumann-type conditions obtained as follows: Fix functions $b : \partial\mathbb{X} \longrightarrow \mathbb{R}$, and $h, h_\perp : \partial\mathbb{X} \longrightarrow \mathbb{R}$. Then $(h, h_\perp, b)$-**mixed boundary conditions** are given:

$$h(\mathbf{x}) \cdot u(\mathbf{x}) \; + \; h_\perp(\mathbf{x}) \cdot \partial_\perp u(\mathbf{x}) \;=\; b(x) \quad \text{for all } \mathbf{x} \in \partial\mathbb{X}. \tag{6.9}$$

For example:

- **Dirichlet Conditions** corresponds to $h \equiv 1$ and $h_\perp \equiv 0$.
- **Neumann Conditions** corresponds to $h \equiv 0$ and $h_\perp \equiv 1$.
- *No* boundary conditions corresponds to $h \equiv h_\perp \equiv 0$.
- **Newton's Law of Cooling** reads:

$$\partial_\perp u \;=\; c \cdot (u - T_E) \tag{6.10}$$

This describes a situation where the boundary is an *imperfect conductor* (with conductivity constant c), and is immersed in a bath with ambient temperature T_E. Thus, heat leaks in or out of the boundary at a rate proportional to c times the difference between the internal temperature u and the external temperature T_E. Equation (6.10) can be rewritten:

$$c \cdot u \; - \; \partial_{\perp} u \;\; = \;\; b,$$

where $b = c \cdot T_E$. This is the mixed boundary equation (6.9), with $h \equiv c$ and $h_{\perp} \equiv -1$.

- **Homogeneous** mixed boundary conditions take the form:

$$h \cdot u \; + \; h_{\perp} \cdot \partial_{\perp} u \;\; \equiv \;\; 0.$$

The set of functions in $\mathcal{C}^{\infty}(\mathbb{X})$ satisfying this property will be denoted $\mathcal{C}^{\infty}_{h,h_{\perp}}(\mathbb{X})$. Thus, for example, if $\mathbb{X} = [0, L]$, and $h(0)$, $h_{\perp}(0)$, $h(L)$ and $h_{\perp}(L)$ are four constants, then

$$\mathcal{C}^{\infty}_{h,h_{\perp}}[0, L] = \left\{ \begin{array}{ll} f : [0, L] \longrightarrow \mathbb{R}; & f \text{ is differentiable, } \; h(0)f(0) \; - \; h_{\perp}(0)f'(0) = 0 \\ & \qquad\qquad \text{and } \; h(L)f(L) \; + \; h_{\perp}(L)f'(L) = 0. \end{array} \right\}$$

- **Note** that there is some redundancy this formulation. Equation (6.9) is equivalent to

$$k \cdot h(\mathbf{x}) \cdot u(\mathbf{x}) \; + \; k \cdot h_{\perp}(\mathbf{x}) \cdot \partial_{\perp} u(\mathbf{x}) \;\; = \;\; k \cdot b(x)$$

for any constant $k \neq 0$. Normally we chose k so that at least one of the coefficients h or $h_{\perp}$ is equal to 1.

- Some authors (eg. Pinsky [Pin98]) call this **general boundary conditions**, and, for mathematical convenience, write this as

$$\cos(\alpha) u \;\; + \;\; L \cdot \sin(\alpha) \partial_{\perp} u \;\; = \;\; T. \qquad (6.11)$$

where and α and T are parameters. Basically, the "$\cos(\alpha)$, $\sin(\alpha)$" coefficients of (6.11) are just a mathematical "gadget" to concisely express any weighted combination of Dirichlet and Neumann conditions. An expression of type (6.9) can be transformed into one of type (6.11) as follows: Let

$$\alpha = \arctan\left(\frac{h_{\perp}}{L \cdot h}\right)$$

(if $h = 0$, then set $\alpha = \frac{\pi}{2}$) and let

$$T = b \frac{\cos(\alpha) + L \sin(\alpha)}{h + h_{\perp}}.$$

Going the other way is easier; simply define

$$h = \cos(\alpha), \quad h_{\perp} = L \cdot \sin(\alpha) \;\; \text{and} \;\; T = b.$$

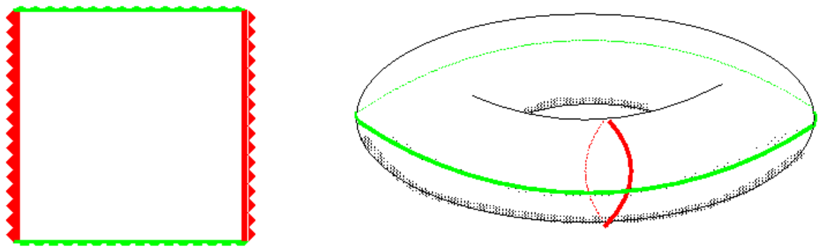


Figure 6.7: If we 'glue' the opposite edges of a square together, we get a torus.

6.5(d) Periodic Boundary Conditions

Periodic boundary conditions means that function u "looks the same" on opposite edges of the domain. For example, if we are solving a PDE on the **interval** $[-\pi, \pi]$, then periodic boundary conditions are imposed by requiring

$$u(-\pi) = u(\pi) \quad \text{and} \quad u'(-\pi) = u'(\pi).$$

Interpretation #1: Pretend that u is actually a small piece of an infinitely extended, periodic function $\widetilde{u} : \mathbb{R} \longrightarrow \mathbb{R}$, where, for any $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, we have:

$$\widetilde{u}(x + 2n\pi) = u(x).$$

Thus u must have the same value —and the same derivative —at x and $x + 2n\pi$, for any $x \in \mathbb{R}$. In particular, u must have the same value and derivative at $-\pi$ and π. This explains the name "periodic boundary conditions".

Interpretation #2: Suppose you 'glue together' the left and right ends of the interval $[-\pi, \pi]$ (ie. glue $-\pi$ to π). Then the interval looks like a a circle (where $-\pi$ and π actually become the 'same' point). Thus u must have the same value —and the same derivative —at $-\pi$ and π.

Example 6.9:

(a) $u(x) = \sin(x)$ and $v(x) = \cos(x)$ have periodic boundary conditions.

(b) For any $n \in \mathbb{N}$, the functions $\mathbf{S}_n(x) = \sin(nx)$ and $\mathbf{C}_n(x) = \cos(nx)$ have periodic boundary conditions.

(c) $\sin(3x) + 2\cos(4x)$ has periodic boundary conditions.

(d) If $u_1(x)$ and $u_2(x)$ have periodic boundary conditions, and c_1, c_2 are any constants, then $u(x) = c_1 u_1(x) + c_2 u_2(x)$ also has periodic boundary conditions.

Exercise 6.5 Verify these examples. ◇

On the **square** $[-\pi, \pi] \times [-\pi, \pi]$, periodic boundary conditions are imposed by requiring:

(P1) $u(x, -\pi) = u(x, \pi)$ and $\partial_y u(x, -\pi) = \partial_y u(x, \pi)$, for all $x \in [-\pi, \pi]$.

(P2) $u(-\pi, y) = u(\pi, y)$ and $\partial_x u(-\pi, y) = \partial_x u(\pi, y)$ for all $y \in [-\pi, \pi]$.

Interpretation #1: Pretend that u is actually a small piece of an infinitely extended, doubly periodic function $\widetilde{u} : \mathbb{R}^2 \longrightarrow \mathbb{R}$, where, for every $(x, y) \in \mathbb{R}^2$, and every $n, m \in \mathbb{Z}$, we have:

$$\widetilde{u}(x + 2n\pi, y + 2m\pi) \quad = \quad u(x, y).$$

<u>**Exercise 6.6**</u> Explain how conditions **(P1)** and **(P1)** arise naturally from this interpretation.

Interpretation #2: Glue the top edge of the square to the bottom edge, and the right edge to the left edge. In other words, pretend that the square is really a **torus** (Figure 6.7).

Example 6.10:

(a) $u(x, y) = \sin(x)\sin(y)$ and $v(x, y) = \cos(x)\cos(y)$ have periodic boundary conditions. So do $w(x, y) = \sin(x)\cos(y)$ and $w(x, y) = \cos(x)\sin(y)$

(b) For any $(n, m) \in \mathbb{N}^2$, the functions $\mathbf{S}_{n,m}(x) = \sin(nx)\sin(my)$ and $\mathbf{C}_{n,m}(x) = \cos(nx)\cos(mx)$ have periodic boundary conditions.

(c) $\sin(3x)\sin(2y) + 2\cos(4x)\cos(7y)$ has periodic boundary conditions.

(d) If $u_1(x, y)$ and $u_2(x, y)$ have periodic boundary conditions, and c_1, c_2 are any constants, then $u(x, y) = c_1u_1(x, y) + c_2u_2(x, y)$ also has periodic boundary conditions.

<u>**Exercise 6.7**</u> Verify these examples. ♢

On the D-dimensional **cube** $[-\pi, \pi]^D$, we require, for $d = 1, 2, \ldots, D$ and all $x_1, \ldots, x_D \in [-\pi, \pi]$, that

$$\begin{aligned} u(x_1, \ldots, x_{d-1}, -\pi, x_{d+1}, \ldots, x_D) &= u(x_1, \ldots, x_{d-1}, \pi, x_{d+1}, \ldots, x_D) \\ \text{and } \partial_d u(x_1, \ldots, x_{d-1}, -\pi, x_{d+1}, \ldots, x_D) &= \partial_d u(x_1, \ldots, x_{d-1}, \pi, x_{d+1}, \ldots, x_D). \end{aligned}$$

Again, the idea is that we are identifying $[-\pi, \pi]^D$ with the D-dimensional **torus**. The space of all functions satisfying these conditions will be denoted $\mathcal{C}^\infty_{\text{per}}[-\pi, \pi]^D$. Thus, for example,

$$\begin{aligned} \mathcal{C}^\infty_{\text{per}}[-\pi, \pi] &= \Big\{ f : [-\pi, \pi] \longrightarrow \mathbb{R};\ \ f \text{ is differentiable}, \ \ f(-\pi) = f(\pi) \text{ and } f'(-\pi) = f'(\pi) \Big\} \\ \mathcal{C}^\infty_{\text{per}}[-\pi, \pi]^2 &= \Big\{ f : [-\pi, \pi] \times [-\pi, \pi] \longrightarrow \mathbb{R};\ \ f \text{ is differentiable}, \quad \text{and satisfies } \textbf{(P1)} \text{ and } \textbf{(P2)} \text{ above} \Big\} \end{aligned}$$

6.6 Uniqueness of Solutions

Prerequisites: §2.2, §3.2, §2.3, §6.4, §6.5

Differential equations are interesting primarily because they can be used to express the physical laws governing a particular phenomenon (e.g. heat flow, wave motion, electrostatics, etc.). By specifying particular initial conditions and boundary conditions, we try to mathematically encode the physical conditions, constraints and external influences which are present in a particular situation. A solution to the differential equation which satisfies these boundary conditions thus constitutes a *prediction* about what will occur under these physical conditions.

However, this program can only succeed if there is a *unique* solution to a given equation with particular boundary conditions. Clearly, if there are many mathematically correct solutions, then we cannot make a clear prediction about which of them (if any) will really occur. Sometimes we can reject some solutions as being 'unphysical' (e.g. they contain unacceptable infinities, or predict negative values for a necessarily positive quantity like density). However, this notion of 'unphysicality' really just represents further mathematical constraints which we are implicitly imposing on the solution (and which we probably should have stated explicitly at the very beginning). If multiple solutions still exist, we must impose further constraints[9] until we get a unique solution.

The predictive power of a mathematical model is extremely limited unless it yields a unique solution.[10] Because of this, the question of *uniqueness of solutions* is extremely important in the general theory of differential equations (both ordinary and partial). We do not have the time to develop the theoretical background to prove the uniqueness of solutions of the linear partial differential equations we will consider in this book. However, we will at least state some of the important uniqeness results, since these are critical for the relevance of the solution methods in the following chapters.

Let $\mathcal{S} \subset \mathbb{R}^D$. We say that $\mathcal{S}$ is a **smooth graph** if there is some open subset $\mathbb{U} \subset \mathbb{R}^{D-1}$, and some function $f : \mathbb{U} \longrightarrow \mathbb{R}$, and some $d \in [1...D]$, such that $\mathcal{S}$ 'looks like' the graph of the function f, plotted over the domain $\mathbb{U}$, with the value of f plotted in the dth coordinate. In other words:

$$\mathcal{S} \quad = \quad \{(u_1, \ldots, u_{d-1}, y, u_d, \ldots, u_{D-1}) \; ; \; (u_1, \ldots, u_{D-1}) \in \mathbb{U} \text{ and } y = f(u_1, \ldots, u_{D-1})\} .$$

Intuitively, this means that $\mathcal{S}$ looks like a smooth surface (oriented 'roughly perpendicular' to the dth dimension). More generally, if $\mathcal{S} \subset \mathbb{R}^D$, we say that $\mathcal{S}$ is a **smooth hypersurface** if, for each $\mathbf{s} \in \mathcal{S}$, there exists some $\epsilon > 0$ such that $\mathbb{B}(\mathbf{s}, \epsilon) \cap \mathcal{S}$ is a smooth graph.

Example 6.11:

(a) Let $\mathbb{P} \subset \mathbb{R}^D$ be any $(D-1)$-dimensional hyperplane; then $\mathbb{P}$ is a smooth hypersurface.

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[9] i.e. construct a more realistic model which mathematically encodes more information about physical reality.

[10] Of course, it isn't true that a model with non-unique solutions has *no* predictive power. After all, it already performs a very useful task of telling you what *can't* happen.

(b) Let $\mathbb{S}^1 := \{\mathbf{s} \in \mathbb{R}^2 \ ; \ |\mathbf{s}| = 1\}$ be the unit circle in $\mathbb{R}^2$. Then $\mathbb{S}^1$ is a smooth hypersurface in $\mathbb{R}^2$.

(c) Let $\mathbb{S}^2 := \{\mathbf{s} \in \mathbb{R}^3 \ ; \ |\mathbf{s}| = 1\}$ be the unit sphere in $\mathbb{R}^3$. Then $\mathbb{S}^2$ is a smooth hypersurface in $\mathbb{R}^3$.

(d) Let $\mathbb{S}^{D-1} := \{\mathbf{s} \in \mathbb{R}^D \ ; \ |\mathbf{s}| = 1\}$ be the unit hypersphere in $\mathbb{R}^D$. Then $\mathbb{S}^{D-1}$ is a smooth hypersurface in $\mathbb{R}^D$.

(e) Let $\mathcal{S} \subset \mathbb{R}^D$ be any smooth hypersurface, and let $\mathbb{U} \subset \mathbb{R}^D$ be an open set. Then $\mathcal{S} \cap \mathbb{U}$ is also a smooth hypersurface (if it is nonempty).

Exercise 6.8 Verify these examples. ♢

A subset $\mathbb{X} \subset \mathbb{R}^D$ has **piecewise smooth boundary** if $\mathbb{X}$ is closed, $\mathsf{int}(\mathbb{X})$ is nonempty, and $\partial\mathbb{X}$ is a finite union of the closures of disjoint hypersurfaces. In other words,

$$\partial\mathbb{X} \quad = \quad \overline{\mathcal{S}_1} \cup \overline{\mathcal{S}_2} \cup \cdots \cup \overline{\mathcal{S}_n}$$

where $\mathcal{S}_1, \ldots, \mathcal{S}_n \subset \mathbb{R}^D$ are smooth hypersurfaces, and $\mathcal{S}_j \cap \mathcal{S}_k = \emptyset$ whenever $j \neq k$.

Example 6.12: Every domain in Example 6.5 on page 92 has a piecewise smooth boundary. (**Exercise 6.9** Verify this.) ♢

Indeed, every domain we will consider in this book will have a piecewise smooth boundary. Clearly this covers almost any domain which is likely to arise in any physically realistic model. Hence, it suffices to obtain uniqueness results for such domains. To get uniqueness, we generally must impose initial/boundary conditions of some kind.

Lemma 6.13: Uniqueness of the zero solution for the Laplace Equation; Homogeneous BC.

Let $\mathbb{X} \subset \mathbb{R}^D$ be a domain with a piecewise smooth boundary. Suppose $u : \mathbb{X} \longrightarrow \mathbb{R}$ satisfies <u>both</u> of the following conditions:

[i] (Regularity) $u \in \mathcal{C}^1(\mathbb{X})$ *and* $u \in \mathcal{C}^2(\mathsf{int}(\mathbb{X}))$ (i.e. u is continuously differentiable on $\mathbb{X}$ and twice-continuously differentiable on the interior of $\mathbb{X}$)*;*

[ii] (Laplace Equation) $\triangle u = 0$*;*

Then various homogeneous boundary conditions constrain the solution as follows:

(a) (Homogeneous Dirichlet BC) *If $u(\mathbf{x}) \ = \ 0$ for all $\mathbf{x} \in \partial\mathbb{X}$, then u <u>must</u> be the constant 0 function: i.e. $u(\mathbf{x}) = 0$, for all $\mathbf{x} \in \mathbb{X}$.*

(b) (Homogeneous Neumann BC) *If $\partial_\perp u(\mathbf{x}) \ = \ 0$ for all $\mathbf{x} \in \partial\mathbb{X}$, then u <u>must</u> be a constant: i.e. $u(\mathbf{x}) \ = \ C$, for all $\mathbf{x} \in \mathbb{X}$.*

(c) (Homogeneous Robin BC) *Suppose $h(\mathbf{x})u(\mathbf{x}) + h_\perp(\mathbf{x})\partial_\perp u(\mathbf{x}) \ = \ 0$ for all $\mathbf{x} \in \partial\mathbb{X}$, where $h, h_\perp : \partial\mathbb{X} \longrightarrow [0, \infty)$ are two other nonnegative functions. Suppose that h is nonzero somewhere on $\partial\mathbb{X}$. Then u <u>must</u> be the constant 0 function: i.e. $u(\mathbf{x}) = 0$, for all $\mathbf{x} \in \mathbb{X}$.*

Proof: See [CB87, §93] for the case $D = 3$. For the general case, see [Eva91, Theorem 5 on p.28 of §2.2]. □

One of the great things about *linear* differential equations is that we can then enormously simplify the problem of solution uniqueness. First we show that the only solution satisfying *homogeneous* boundary conditions (and, if applicable, *zero* initial conditions) is the constant zero function (as in Lemma 6.13 above). Then it is easy to deduce uniqueness for boundary conditions (and arbitrary initial conditions), as follows:

Theorem 6.14: Uniqueness of solutions for the Poisson Equation; Nonhomogeneous BC.

Let $\mathbb{X} \subset \mathbb{R}^D$ *be a domain with a piecewise smooth boundary. Let* $q : \mathbb{X} \longrightarrow \mathbb{R}$ *be a continuous function* (describing an electric charge or heat source), *and let* $b : \partial\mathbb{X} \times [0, \infty) \longrightarrow \mathbb{R}$ *be another continuous function* (describing a boundary condition). *Then there is* at most one *solution function* $u : \mathbb{X} \times [0, \infty] \longrightarrow \mathbb{R}$ *satisfying* both *of the following conditions:*

[i] (Regularity) $u \in \mathcal{C}^1(\mathbb{X})$ *and* $u \in \mathcal{C}^2(\mathsf{int}\,(\mathbb{X}))$*;*

[ii] (Poisson Equation) $\triangle u = q$*;*

...and satisfying either *of the following nonhomogeneous boundary conditions:*

(a) (Nonhomogeneous Dirichlet BC) $u(\mathbf{x}) = b(\mathbf{x})$ *for all* $\mathbf{x} \in \partial\mathbb{X}$.

(b) (Nonhomogeneous Robin BC) $h(\mathbf{x}))u(\mathbf{x}) + h_\perp(\mathbf{x})\partial_\perp u(\mathbf{x}) = b(\mathbf{x})$ *for all* $\mathbf{x} \in \partial\mathbb{X}$*, where* $h, h_\perp : \partial\mathbb{X} \longrightarrow [0, \infty)$ *are two other nonnegative functions, and* h *is nontrivial.*

Furthermore, if u_1 *and* u_2 *are two functions satisfying* [i] *and* [ii] *and also satisfying:*

(c) (Nonhomogeneous Neumann BC) $\partial_\perp u(\mathbf{x}) = b(\mathbf{x})$ *for all* $\mathbf{x} \in \partial\mathbb{X}$.

....then $u_1 = u_2 + C$*, where* C *is a constant.*

Proof: Suppose u_1 and u_2 were two functions satisfying [i] and [ii] and one of **(a)** or **(b)**. Let $u = u_1 - u_2$. Then u satisfies [i] and [ii] and one of **(a)** or **(c)** in Lemma 6.13, Thus, $u \equiv 0$. But this means that $u_1 \equiv u_2$ —i.e. they are really the same function. Hence, there can be at most one solution. The proof for **(c)** is **Exercise 6.10** . □

The uniqueness results for the Heat Equation and Wave Equation are similar.

Lemma 6.15: Uniqueness of the zero solution for the Heat Equation

Let $\mathbb{X} \subset \mathbb{R}^D$ *be a domain with a piecewise smooth boundary. Suppose that* $u : \mathbb{X} \times [0, \infty] \longrightarrow \mathbb{R}$ *satisfies* all three *of the following conditions:*

[i] (Regularity) u *is continuously differentiable on* $\mathbb{X} \times (0, \infty)$*;*

[ii] (Heat Equation) $\partial_t u = \triangle u$*;*

[iii] (Initial condition) $u(\mathbf{x}, 0) = 0$ *for all* $\mathbf{x} \in \mathbb{X}$*;*

...and that u *also satisfies* <u>*any one*</u> *of the following homogeneous boundary conditions:*

(a) (Homogeneous Dirichlet BC) $u(\mathbf{x}, t) = 0$ *for all* $\mathbf{x} \in \partial\mathbb{X}$ *and* $t \geq 0$.

(b) (Homogeneous Neumann BC) $\partial_\perp u(\mathbf{x}, t) = 0$ *for all* $\mathbf{x} \in \partial\mathbb{X}$ *and* $t \geq 0$.

(c) (Homogeneous Robin BC) $h(\mathbf{x}, t)u(\mathbf{x}, t) + h_\perp(\mathbf{x}, t)\partial_\perp u(\mathbf{x}, t) = 0$ *for all* $\mathbf{x} \in \partial\mathbb{X}$ *and* $t \geq 0$*, where* $h, h_\perp : \partial\mathbb{X} \times [0, \infty) \longrightarrow [0, \infty)$ *are two other nonnegative functions.*

Then u <u>*must*</u> *be the constant* 0 *function:* $u \equiv 0$.

Proof: See [CB87, §90] for the case $D = 3$. For the general case, see [Eva91, Theorem 7 on p.58 of §2.3]. □

Theorem 6.16: Uniqueness of solutions for the Heat Equation

Let $\mathbb{X} \subset \mathbb{R}^D$ *be a domain with a piecewise smooth boundary. Let* $\mathcal{I} : \mathbb{X} \longrightarrow \mathbb{R}$ *be a continuous function* (describing an initial condition)*, and let* $b : \partial\mathbb{X} \times [0, \infty) \longrightarrow \mathbb{R}$ *be another continuous function* (describing a boundary condition). *Then there is* <u>*at most one*</u> *solution function* $u : \mathbb{X} \times [0, \infty] \longrightarrow \mathbb{R}$ *satisfying* <u>*all three*</u> *of the following conditions:*

[i] (Regularity) u *is continuously differentiable on* $\mathbb{X} \times (0, \infty)$*;*

[ii] (Heat Equation) $\partial_t u = \triangle u$*;*

[iii] (Initial condition) $u(\mathbf{x}, 0) = \mathcal{I}(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{X}$*;*

...and satisfying <u>*any one*</u> *of the following nonhomogeneous boundary conditions:*

(a) (Nonhomogeneous Dirichlet BC) $u(\mathbf{x}, t) = b(\mathbf{x}, t)$ *for all* $\mathbf{x} \in \partial\mathbb{X}$ *and* $t \geq 0$.

(b) (Nonhomogeneous Neumann BC) $\partial_\perp u(\mathbf{x}, t) = b(\mathbf{x}, t)$ *for all* $\mathbf{x} \in \partial\mathbb{X}$ *and* $t \geq 0$.

(c) (Nonhomogeneous Robin BC) $h(\mathbf{x}, t)u(\mathbf{x}, t) + h_\perp(\mathbf{x}, t)\partial_\perp u(\mathbf{x}, t) = b(\mathbf{x}, t)$ *for all* $\mathbf{x} \in \partial\mathbb{X}$ *and* $t \geq 0$*, where* $h, h_\perp : \partial\mathbb{X} \times [0, \infty) \longrightarrow [0, \infty)$ *are two other nonnegative functions.*

Proof: Suppose u_1 and u_2 were two functions satisfying all of [i], [ii], and [iii], and one of **(a)**, **(b)**, or **(c)**. Let $u = u_1 - u_2$. Then u satisfies all of [i], [ii], and [iii] in Lemma 6.15, and one of **(a)**, **(b)**, or **(c)** in Lemma 6.15. Thus, $u \equiv 0$. But this means that $u_1 \equiv u_2$ —i.e. they are really the same function. Hence, there can be at most one solution. □

Lemma 6.17: Uniqueness of the zero solution for the Wave Equation

Let $\mathbb{X} \subset \mathbb{R}^D$ be a domain with a piecewise smooth boundary. Suppose $u : \mathbb{X} \times [0, \infty] \longrightarrow \mathbb{R}$ satisfies all five of the following conditions:

(a) (Regularity) $u \in \mathcal{C}^2(\mathbb{X} \times (0, \infty))$;

(b) (Wave Equation) $\partial_t^2 u = \triangle u$;

(c) (Zero Initial position) $u(\mathbf{x}, 0) = 0$, *for all* $\mathbf{x} \in \mathbb{X}$;

(d) (Zero Initial velocity) $\partial_t u(\mathbf{x}, 0) = 0$ *for all* $\mathbf{x} \in \mathbb{X}$;

(e) (Homogeneous Dirichlet BC) $u(\mathbf{x}, t) = 0$ *for all* $\mathbf{x} \in \partial\mathbb{X}$ *and* $t \geq 0$.

Then u must be the constant 0 function: $u \equiv 0$.

Proof: See [CB87, §92] for the case $D = 1$. For the general case, see [Eva91, Theorem 5 on p.83 of §2.4]. □

Theorem 6.18: Uniqueness of solutions for the Wave Equation

Let $\mathbb{X} \subset \mathbb{R}^D$ be a domain with a piecewise smooth boundary. Let $\mathcal{I}_0, \mathcal{I}_1 : \mathbb{X} \longrightarrow \mathbb{R}$ be continuous functions (describing initial position and velocity). *Let $b : \partial\mathbb{X} \times [0, \infty) \longrightarrow \mathbb{R}$ be another continuous function* (describing a boundary condition). *Then there is at most one solution function $u : \mathbb{X} \times [0, \infty] \longrightarrow \mathbb{R}$ satisfying all five of the following conditions:*

(a) (Regularity) $u \in \mathcal{C}^2(\mathbb{X} \times (0, \infty))$;

(b) (Wave Equation) $\partial_t^2 u = \triangle u$;

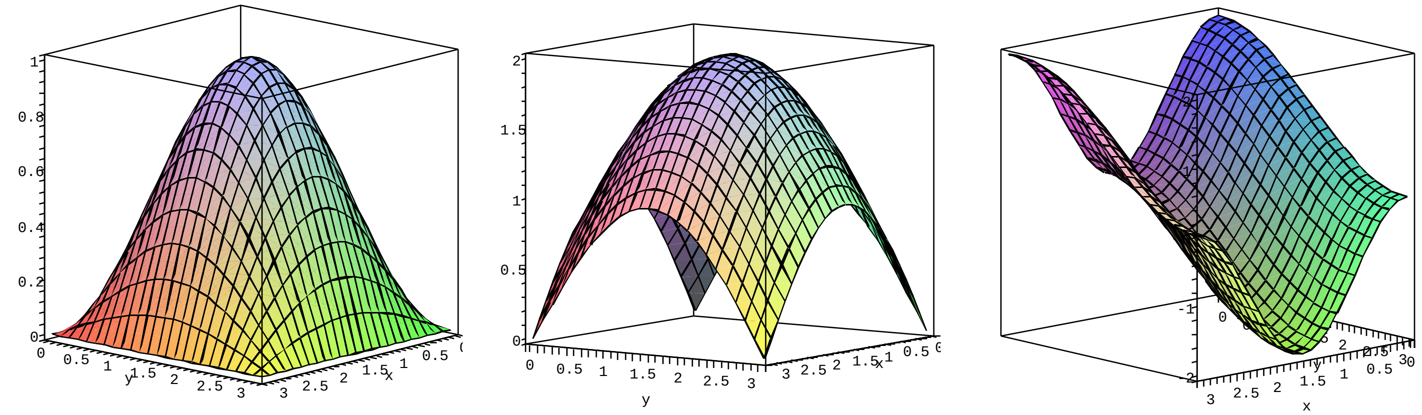
(c) (Initial position) $u(\mathbf{x}, 0) = \mathcal{I}_0(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{X}$.

(d) (Initial velocity) $\partial_t u(\mathbf{x}, 0) = \mathcal{I}_1(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{X}$.

(e) (Nonhomogeneous Dirichlet BC) $u(\mathbf{x}, t) = b(\mathbf{x}, t)$ *for all* $\mathbf{x} \in \partial\mathbb{X}$ *and* $t \geq 0$.

Proof: Suppose u_1 and u_2 were two functions satisfying all of **(a)**-**(e)**. Let $u = u_1 - u_2$. Then u satisfies all of **(a)**-**(e)**, in Lemma 6.17. Thus, $u \equiv 0$. But this means that $u_1 \equiv u_2$ —i.e. they are really the same function. Hence, there can be at most one solution. □

Remark: Notice Theorems 6.14, 6.16, and 6.18 apply under much more general conditions than any of the solution methods we will actually develop in this book (i.e. they work for almost any 'reasonable' domain, and we even allow the boundary conditions to vary in time). This is a recurring theme in differential equation theory; it is generally possible to prove 'qualitative' results (e.g. about existence, uniqueness, or general properties of solutions) in much more general settings than it is possible to get 'quantitative' results (i.e. explicit formulae for solutions). Indeed, for most *nonlinear* differential equations, qualitative results are pretty much all you can ever get.



(A) $f(x,y) = \sin(x)\sin(y)$ **(B)** $g(x,y) = \sin(x)+\sin(y)$ **(C)** $h(x,y) = \cos(2x)+\cos(y)$.

Figure 6.8: Problems #3a, #3b and #3c

6.7 Practice Problems

1. Each of the following functions is defined on the interval $[0,\pi]$, in Cartesian coordinates. For each function, decide: Does it satisfy homogeneous *Dirichlet* BC? Homogeneous *Neumann* BC? Homogeneous *Robin*[11] BC? Periodic BC? Justify your answers.

   (a) $u(x) = \sin(3x)$.
   (b) $u(x) = \sin(x)+3\sin(2x)-4\sin(7x)$.
   (c) $u(x) = \cos(x)+3\sin(3x)-2\cos(6x)$.
   (d) $u(x) = 3+\cos(2x)-4\cos(6x)$.
   (e) $u(x) = 5+\cos(2x)-4\cos(6x)$.

2. Each of the following functions is defined on the interval $[-\pi,\pi]$, in Cartesian coordinates. For each function, decide: Does it satisfy homogeneous *Dirichlet* BC? Homogeneous *Neumann* BC? Homogeneous *Robin*[11] BC? Periodic BC? Justify your answers.

   (a) $u(x) = \sin(x)+5\sin(2x)-2\sin(3x)$.
   (b) $u(x) = 3\cos(x)-3\sin(2x)-4\cos(2x)$.
   (c) $u(x) = 6+\cos(x)-3\cos(2x)$.

3. Each of the following functions is defined on the box $[0,\pi]^2$. in Cartesian coordinates. For each function, decide: Does it satisfy homogeneous *Dirichlet* BC? Homogeneous *Neumann* BC? Homogeneous *Robin*[11] BC? Periodic BC? Justify your answers.

[11] Here, 'Robin' B.C. means *nontrivial* Robin B.C. —ie. *not* just homogenous Dirichlet or Neumann.

(a) $f(x,y) = \sin(x)\sin(y)$ (Figure 6.8(A))
(b) $g(x,y) = \sin(x)+\sin(y)$ (Figure 6.8(B))
(c) $h(x,y) = \cos(2x)+\cos(y)$ (Figure 6.8(C))
(d) $u(x,y) = \sin(5x)\sin(3y)$.
(e) $u(x,y) = \cos(-2x)\cos(7y)$.

4. Each of the following functions is defined on the unit disk $\mathbb{D} = \{(r,\theta)\ ;\ 0 \leq r \leq 1, \text{ and } \theta \in [0,2\pi)\}$ in polar coordinates. For each function, decide: Does it satisfy homogeneous *Dirichlet* BC? Homogeneous *Neumann* BC? Homogeneous *Robin*[11] BC? Justify your answers.

   (a) $u(r,\theta) = (1-r^2)$.
   (b) $u(r,\theta) = 1-r^3$.
   (c) $u(r,\theta) = 3+(1-r^2)^2$.
   (d) $u(r,\theta) = \sin(\theta)(1-r^2)^2$.
   (e) $u(r,\theta) = \cos(2\theta)(e-e^r)$.

5. Each of the following functions is defined on the 3-dimensional unit ball

$$\mathbb{B} = \left\{(r,\theta,\varphi)\ ;\ 0 \leq r \leq 1,\ \theta \in [0,2\pi), \quad \text{and} \quad \varphi \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right\}$$

   in spherical coordinates. For each function, decide: Does it satisfy homogeneous *Dirichlet* BC? Homogeneous *Neumann* BC? Homogeneous *Robin*[11] BC? Justify your answers.

   (a) $u(r,\theta,\varphi) = (1-r)^2$.
   (b) $u(r,\theta,\varphi) = (1-r)^3+5$.

Notes: ....

....

....

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III Fourier Series on Bounded Domains

It is a well-known fact that any complex sound is a combination of simple 'pure tones' of different frequencies. For example, a musical *chord* is a superposition of three (or more) musical notes, each with a different frequency. In fact, a musical note itself is not really a single frequency at all; a note consists of a 'fundamental' frequency, plus a cascade of higher frequency 'harmonics'. The energy distribution of these harmonics is part of what gives each musical instrument its distinctive sound. The decomposition of a sound into a combination of separate frequencies is sometimes called its *power spectrum.* A crude graphical representation of this power spectrum is visible on most modern stereo systems (the little jiggling red bars).

Fourier theory is based on the idea that a real-valued function is like a sound, which can be represented as a superposition of 'pure tones' (i.e. sine waves and/or cosine waves) of distinct frequencies. This powerful idea allows a precise mathematical formulation of the above discussion of 'cascades of harmonics', etc. But it does much more. Fourier theory provides a 'coordinate system' for expressing functions, and within this coordinate system, we can express the solutions for many partial differential equations in a simple and elegant way. Fourier theory is also an essential tool in probability theory, signal analysis, the ergodic theory of dynamical systems, and the representation theory of Lie groups (although we will not discuss these applications in this book).

The idea of Fourier theory is simple, but to make this idea rigorous enough to be useful, we must deploy some formidable mathematical machinery. So we will begin by developing the necessary background concerning inner products, orthogonality, and the convergence of functions.

7 Background: Some Functional Analysis

7.1 Inner Products (Geometry)

Prerequisites: §5.1

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^D$, with $\mathbf{x} = (x_1, \ldots, x_D)$ and $\mathbf{y} = (y_1, \ldots, y_D)$. The **inner product**[1] of $\mathbf{x}, \mathbf{y}$ is defined:

$$\langle \mathbf{x}, \mathbf{y} \rangle \quad := \quad x_1 y_1 + x_2 y_2 + \ldots + x_D y_D.$$

The inner product describes the geometric relationship between $\mathbf{x}$ and $\mathbf{y}$, via the formula:

$$\langle \mathbf{x}, \mathbf{y} \rangle \quad := \quad \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot \cos(\theta)$$

where $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ are the lengths of vectors $\mathbf{x}$ and $\mathbf{y}$, and θ is the angle between them. (**Exercise 7.1** Verify this). In particular, if $\mathbf{x}$ and $\mathbf{y}$ are *perpendicular*, then $\theta = \pm\frac{\pi}{2}$, and then $\langle \mathbf{x}, \mathbf{y} \rangle = 0$; we then say that $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal**. For example, $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are orthogonal in $\mathbb{R}^2$, while

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

are all orthogonal to one another in $\mathbb{R}^4$. Indeed, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ also have unit norm; we call any such collection an **orthonormal set** of vectors. Thus, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is an orthonormal set, but $\{\mathbf{x}, \mathbf{y}\}$ is not.

The **norm** of a vector satisfies the equation:

$$\|\mathbf{x}\| \;=\; \left(x_1^2 + x_2^2 + \ldots + x_D^2\right)^{1/2} \;=\; \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}.$$

If $\mathbf{x}_1, \ldots, \mathbf{x}_N$ are a collection of mutually orthogonal vectors, and $\mathbf{x} = \mathbf{x}_1 + \ldots + \mathbf{x}_N$, then we have the generalized **Pythagorean formula**:

$$\|\mathbf{x}\|^2 \;=\; \|\mathbf{x}_1\|^2 + \|\mathbf{x}_2\|^2 + \ldots + \|\mathbf{x}_N\|^2$$

(**Exercise 7.2** Verify the Pythagorean formula.)

An **orthonormal basis** of $\mathbb{R}^D$ is any collection of mutually orthogonal vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_D\}$, all of norm 1, so that, for any $\mathbf{w} \in \mathbb{R}^D$, if we define $\omega_d = \langle \mathbf{w}, \mathbf{v}_d \rangle$ for all $d \in [1..D]$, then:

$$\mathbf{w} = \omega_1 \mathbf{v}_1 + \omega_2 \mathbf{v}_2 + \ldots + \omega_D \mathbf{v}_D$$

In other words, the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_D\}$ defines a *coordinate system* for $\mathbb{R}^D$, and in this coordinate system, the vector $\mathbf{w}$ has coordinates $(\omega_1, \omega_2, \ldots, \omega_D)$.

[1]This is sometimes this is called the **dot product**, and denoted "$\mathbf{x} \bullet \mathbf{y}$".

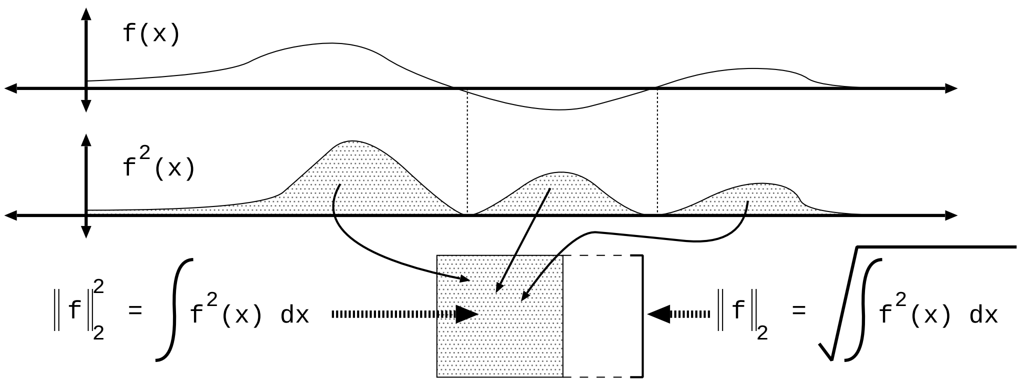


Figure 7.1: The $\mathbf{L}^2$ norm of f: $\|f\|_2 = \sqrt{\int_{\mathbb{X}} |f(x)|^2 \, dx}$

In this case, the Pythagorean Formula becomes **Parseval's Equality:**

$$\|\mathbf{w}\|^2 = \omega_1^2 + \omega_2^2 + \ldots + \omega_D^2$$

(**Exercise 7.3** Deduce Parseval's equality from the Pythagorean formula.)

Example 7.1:

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^D$.

(b) If $\mathbf{v}_1 = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal basis of $\mathbb{R}^2$.

If $\mathbf{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, then $\omega_1 = \sqrt{3} + 2$ and $\omega_2 = 2\sqrt{3} - 1$, so that

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \omega_1 \mathbf{v}_1 + \omega_2 \mathbf{v}_2 = \left(\sqrt{3} + 2\right) \cdot \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} + \left(2\sqrt{3} - 1\right) \cdot \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}.$$

Thus, $\|\mathbf{w}\|_2^2 = 2^2 + 4^2 = 20$, and also, by Parseval's equality, $20 = \omega_1^2 + \omega_2^2 = \left(\sqrt{3} + 2\right)^2 + \left(1 - 2\sqrt{3}\right)^2$. (**Exercise 7.4** Verify these claims.) ♢

7.2 $\mathbf{L}^2$ space (finite domains)

All of this generalizes to spaces of functions. Suppose $\mathbb{X} \subset \mathbb{R}^D$ is some bounded domain, and let $M = \int_{\mathbb{X}} 1 \, d\mathbf{x}$ be the *volume*[2] of the domain $\mathbb{X}$. The second column of Table 7.1 provides examples of M for various domains.

[2]Or *length*, if $D = 1$, or *area* if $D = 2$....

Domain				M		Inner Product
Unit interval	$\mathbb{X} = [0,1]$	$\subset$	$\mathbb{R}$	length	$M = 1$	$\langle f,g\rangle = \int_0^1 f(x)\cdot g(x)\ dx$
	$\mathbb{X} = [0,\pi]$	$\subset$	$\mathbb{R}$	length	$M = \pi$	$\langle f,g\rangle = \frac{1}{\pi}\int_0^\pi f(x)\cdot g(x)\ dx$
Unit square	$\mathbb{X} = [0,1]\times[0,1]$	$\subset$	$\mathbb{R}^2$	area	$M = 1$	$\langle f,g\rangle = \int_0^1\int_0^1 f(x,y)\cdot g(x,y)\ dx\ dy$
$\pi\times\pi$ square	$\mathbb{X} = [0,\pi]\times[0,\pi]$	$\subset$	$\mathbb{R}^2$	area	$M = \pi^2$	$\langle f,g\rangle = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi f(x,y)\cdot g(x,y)\ dx\ dy$
Unit Disk (polar coords)	$\mathbb{X} = \{(r,\theta)\ ;\ r\leq 1\}$	$\subset$	$\mathbb{R}^2$	area	$M = \pi$	$\langle f,g\rangle = \frac{1}{\pi}\int_0^1\int_{-\pi}^\pi f(r,\theta)\cdot g(r,\theta)\ r\cdot d\theta\ dr$
Unit cube	$\mathbb{X} = [0,1]\times[0,1]\times[0,1]$	$\subset$	$\mathbb{R}^3$	volume	$M = 1$	$\langle f,g\rangle = \int_0^1\int_0^1\int_0^1 f(x,y,z)\cdot g(x,y,z)\ dx\ dy\ dz$

Table 7.1: Inner products on various domains.

If $f,g : \mathbb{X} \longrightarrow \mathbb{R}$ are integrable functions, then the **inner product** of f and g is defined:

$$\langle f,g\rangle \ := \ \frac{1}{M}\int_{\mathbb{X}} f(\mathbf{x})\cdot g(\mathbf{x})\ d\mathbf{x}.$$

Example 7.2:

(a) Suppose $\mathbb{X} = [0,3] = \{x\in\mathbb{R}\ ;\ 0\leq x\leq 3\}$. Then $M = 3$. If $f(x) = x^2+1$ and $g(x) = x$ for all $x\in[0,3]$, then

$$\langle f,g\rangle \quad = \quad \frac{1}{3}\int_0^3 f(x)g(x)\ dx \quad = \quad \frac{1}{3}\int_0^3 (x^3+x)\ dx \quad = \quad \frac{27}{4}+\frac{3}{2}.$$

(b) The third column of Table 7.1 provides examples of $\langle f,g\rangle$ for various other domains. $\diamondsuit$

The $\mathbf{L}^2$**-norm** of an integrable function $f : \mathbb{X} \longrightarrow \mathbb{R}$ is defined

$$\|f\|_2 \quad := \quad \langle f,f\rangle^{1/2} \quad = \quad \left(\frac{1}{M}\int_{\mathbb{X}} f^2(\mathbf{x})\ d\mathbf{x}\right)^{1/2}.$$

(See Figure 7.1. Of course, this integral may not converge.) The set of all integrable functions on $\mathbb{X}$ with finite $\mathbf{L}^2$-norm is denoted $\mathbf{L}^2(\mathbb{X})$, and is called $\mathbf{L}^2$**-space**. For example, any bounded, continuous function $f : \mathbb{X} \longrightarrow \mathbb{R}$ is in $\mathbf{L}^2(\mathbb{X})$.

Example 7.3: (a) Suppose $\mathbb{X} = [0,3]$, as in Example 7.2, and let $f(x) = x+1$. Then $f \in \mathbf{L}^2[0,3]$, because

$$\|f\|_2^2 \;=\; \langle f,f\rangle \;=\; \frac{1}{3}\int_0^3 (x+1)^2\,dx$$
$$=\; \frac{1}{3}\int_0^3 x^2+2x+1\,dx \;=\; \frac{1}{3}\left(\frac{x^3}{3}+x^2+x\right)_{x=0}^{x=3} \;=\; 7,$$

hence $\|f\|_2 = \sqrt{7} < \infty$.

(b) Let $\mathbb{X} = (0,1]$, and suppose $f \in \mathcal{C}^\infty(0,1]$ is defined $f(x) := 1/x$. Then $\|f\|_2 = \infty$, so$f \notin \mathbf{L}^2(0,1]$. ♢

Remark: Some authors define the inner product as $\langle f,g\rangle := \int_{\mathbb{X}} f(\mathbf{x})\cdot g(\mathbf{x})\,d\mathbf{x}$, and define the $\mathbf{L}^2$-norm as $\|f\|_2 := \left(\int_{\mathbb{X}} f^2(\mathbf{x})\,d\mathbf{x}\right)^{1/2}$. In other words, these authors do not divide by the volume M of the domain. This yields a mathematically equivalent theory. The advantage of our definition is greater computational convenience in some situations. (For example, if $\mathbb{1}_{\mathbb{X}}$ is the constant 1-valued function, then in our definition, $\|\mathbb{1}_{\mathbb{X}}\|_2 = 1$.) When comparing formulae from different books, you should always check their respective definitions of $\mathbf{L}^2$ norm.

$\mathbf{L}^2$ space on an infinite domain: Suppose $\mathbb{X} \subset \mathbb{R}^D$ is a region of *infinite* volume (or length, area, etc.). For example, maybe $\mathbb{X} = [0,\infty)$ is the *positive half-line*, or perhaps $\mathbb{X} = \mathbb{R}^D$. In this case, $M = \infty$, so it doesn't make any sense to divide by M. If $f,g : \mathbb{X} \longrightarrow \mathbb{R}$ are integrable functions, then the **inner product** of f and g is defined:

$$\langle f,g\rangle \;:=\; \int_{\mathbb{X}} f(\mathbf{x})\cdot g(\mathbf{x})\,d\mathbf{x}$$

Example 7.4: Suppose $\mathbb{X} = \mathbb{R}$. If $f(x) = e^{-|x|}$ and $g(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 7 \\ 0 & \quad\text{otherwise} \end{cases}$, then

$$\langle f,g\rangle \;=\; \int_{-\infty}^{\infty} f(x)g(x)\,dx \;=\; \int_0^7 e^{-x}\,dx \;=\; -(e^{-7}-e^0) \;=\; 1-\frac{1}{e^7}. \quad ♢$$

The **$\mathbf{L}^2$-norm** of an integrable function $f : \mathbb{X} \longrightarrow \mathbb{R}$ is defined

$$\|f\|_2 \;=\; \langle f,f\rangle^{1/2} \;=\; \left(\int_{\mathbb{X}} f^2(\mathbf{x})\,d\mathbf{x}\right)^{1/2}.$$

Again, this integral may not converge. Indeed, even if f is bounded and continuous everywhere, this integral may still equal infinity. The set of all integrable functions on $\mathbb{X}$ with finite $\mathbf{L}^2$-norm is denoted $\mathbf{L}^2(\mathbb{X})$, and called **$\mathbf{L}^2$-space**. (You may recall that on page 55 of §4.1, we discussed how $\mathbf{L}^2$-space arises naturally in quantum mechanics as the space of 'physically meaningful' wavefunctions.)

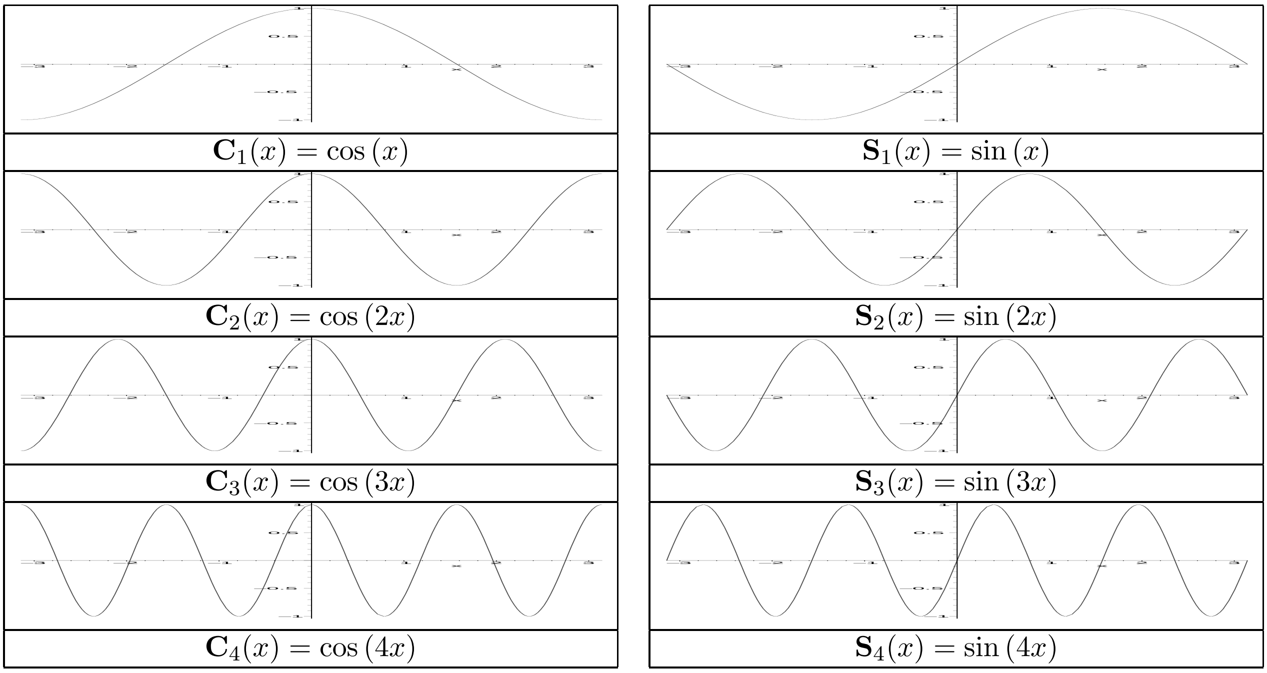


Figure 7.2: $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, and $\mathbf{C}_4$; $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, and $\mathbf{S}_4$

7.3 Orthogonality

Prerequisites: §7.1

Two functions $f, g \in \mathbf{L}^2(\mathbb{X})$ are **orthogonal** if $\langle f, g\rangle = 0$.

Example 7.5: Treat sin and cos as elements of $\mathbf{L}^2[-\pi, \pi]$. Then they are orthogonal:

$$\langle \sin, \cos\rangle \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(x)\cos(x)\, dx \;=\; 0. \qquad (\textbf{Exercise 7.5}). \qquad \diamondsuit$$

An **orthogonal set** of functions is a set $\{f_1, f_2, f_3, \ldots\}$ of elements in $\mathbf{L}^2(\mathbb{X})$ so that $\langle f_j, f_k\rangle = 0$ whenever $j \neq k$. If, in addition, $\|f_j\|_2 = 1$ for all j, then we say this is an **orthonormal set** of functions. Fourier analysis is based on the orthogonality of certain families of trigonometric functions. Example 7.5 was an example of this, which generalizes as follows....

Proposition 7.6: Trigonometric Orthogonality on $[-\pi, \pi]$

For every $n \in \mathbb{N}$, define $\mathbf{S}_n(x) = \sin(nx)$ and $\mathbf{C}_n(x) = \cos(nx)$. (See Figure 7.2).

The set $\{\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \ldots; \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \ldots\}$ is an **orthogonal set** *of functions for $\mathbf{L}^2[-\pi, \pi]$. In other words:*

(a) $\langle \mathbf{S}_n, \mathbf{S}_m\rangle \;=\; \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\, dx \;=\; 0$, *whenever $n \neq m$.*

(b) $\langle \mathbf{C}_n, \mathbf{C}_m \rangle \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\; dx \;=\; 0$, *whenever* $n \neq m$.

(c) $\langle \mathbf{S}_n, \mathbf{C}_m \rangle \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\; dx = 0$, *for any* n *and* m.

(d) *However, these functions are not* orthonormal, *because they do* not *have unit norm. Instead, for any* $n \neq 0$,

$$\|\mathbf{C}_n\|_2 \;=\; \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(nx)^2\; dx} \;=\; \frac{1}{\sqrt{2}}, \quad \textit{and} \quad \|\mathbf{S}_n\|_2 \;=\; \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(nx)^2\; dx} \;=\; \frac{1}{\sqrt{2}}.$$

Proof: **Exercise 7.6** Hint: Use the trigonometric identities: $2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta)+\sin(\alpha-\beta)$, $2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta)-\cos(\alpha+\beta)$, and $2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta)+\cos(\alpha-\beta)$. □

Remark: Notice that $\mathbf{C}_0(x) = 1$ is just the *constant* function.

It is important to remember that the statement, "f and g are orthogonal" depends upon the *domain* $\mathbb{X}$ which we are considering. For example, compare the following theorem to the preceeding one...

Proposition 7.7: Trigonometric Orthogonality on $[0, L]$

Let $L > 0$, *and, for every* $n \in \mathbb{N}$, *define* $\mathbf{S}_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ *and* $\mathbf{C}_n(x) = \cos\left(\frac{n\pi x}{L}\right)$.

(a) *The set* $\{\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \ldots\}$ *is an* **orthogonal set** *of functions for* $\mathbf{L}^2[0, L]$. *In other words:*
$\langle \mathbf{C}_n, \mathbf{C}_m \rangle \;=\; \frac{1}{L}\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\; dx \;=\; 0$, *whenever* $n \neq m$.
However, these functions are **not** *orthonormal, because they do* not *have unit norm. Instead, for any* $n \neq 0$, $\|\mathbf{C}_n\|_2 \;=\; \sqrt{\frac{1}{L}\int_0^L \cos\left(\frac{n\pi}{L}x\right)^2\; dx} \;=\; \frac{1}{\sqrt{2}}$.

(b) *The set* $\{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \ldots\}$ *is an* **orthogonal set** *of functions for* $\mathbf{L}^2[0, L]$. *In other words:*
$\langle \mathbf{S}_n, \mathbf{S}_m \rangle \;=\; \frac{1}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\; dx \;=\; 0$, *whenever* $n \neq m$.
However, these functions are **not** *orthonormal, because they do* **not** *have unit norm. Instead, for any* $n \neq 0$, $\|\mathbf{S}_n\|_2 \;=\; \sqrt{\frac{1}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right)^2\; dx} \;=\; \frac{1}{\sqrt{2}}$.

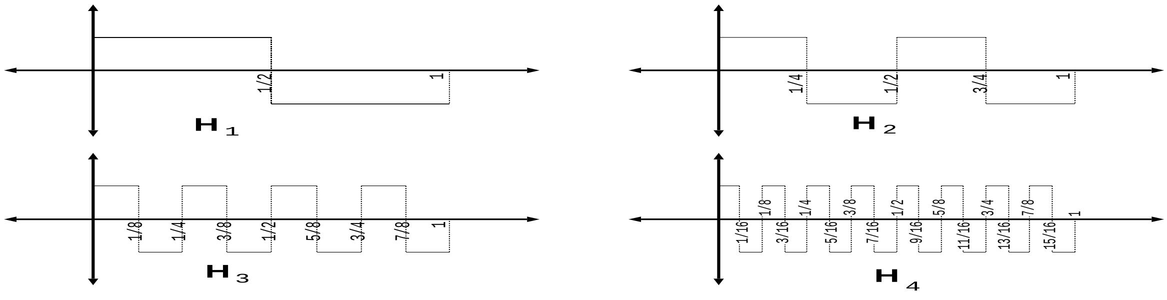


Figure 7.3: Four Haar basis elements: $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3, \mathbf{H}_4$

(c) *The functions* $\mathbf{C}_n$ *and* $\mathbf{S}_m$ *are* **not** *orthogonal to one another on* $[0, L]$*. Instead:*

$$\langle \mathbf{S}_n, \mathbf{C}_m \rangle \quad = \quad \frac{1}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\, dx \quad = \quad \begin{cases} 0 & \text{if } n+m \text{ is even} \\ \dfrac{2n}{\pi(n^2-m^2)} & \text{if } n+m \text{ is odd.} \end{cases}$$

Proof: Exercise 7.7 . □

Remark: The trigonometric functions are just one of several important orthogonal sets of functions. Different orthogonal sets are useful for different domains or different applications. For example, in some cases, it is convenient to use a collection of orthogonal *polynomial* functions. Several orthogonal polynomial families exist, including the *Legendre Polynomials* (see § 15.4 on page 283), the *Chebyshev polynomials* (see Exercise 14.2(e) on page 236 of §14.2(a)), the *Hermite polynomials* and the *Laguerre polynomials*. See [Bro89, Chap.3] for a good introduction.

In the study of partial differential equations, the following fact is particularly important:

> *Let* $\mathbb{X} \subset \mathbb{R}^D$ *be any domain. If* $f, g : \mathbb{X} \longrightarrow \mathbb{C}$ *are two eigenfunctions of the Laplacian with different eigenvalues, then* f *and* g *are orthogonal in* $\mathbf{L}^2(\mathbb{X})$*.*

(See Proposition 7.28 on page 137 of §7.6 for a precise statement of this.) Because of this, we can get orthogonal sets whose members are eigenfunctions of the Laplacian (see Theorem 7.31 on page 139 of §7.6). These orthogonal sets are the 'building blocks' with which we can construct solutions to a PDE satisfying prescribed initial conditions or boundary conditions. This is the basic strategy behind the solution methods of Chapters 11-14.2.

Exercise 7.8 Figure 7.3 portrays the **The Haar Basis**. We define $\mathbf{H}_0 \equiv 1$, and for any natural number $N \in \mathbb{N}$, we define the Nth **Haar function** $\mathbf{H}_N : [0,1] \longrightarrow \mathbb{R}$ by:

$$\mathbf{H}_N(x) = \begin{cases} 1 & \text{if } \dfrac{2n}{2^N} \leq x < \dfrac{2n+1}{2^N}, \quad \text{for some } n \in [0...2^{N-1}); \\ -1 & \text{if } \dfrac{2n+1}{2^N} \leq x < \dfrac{2n+2}{2^N}, \quad \text{for some } n \in [0...2^{N-1}). \end{cases}$$

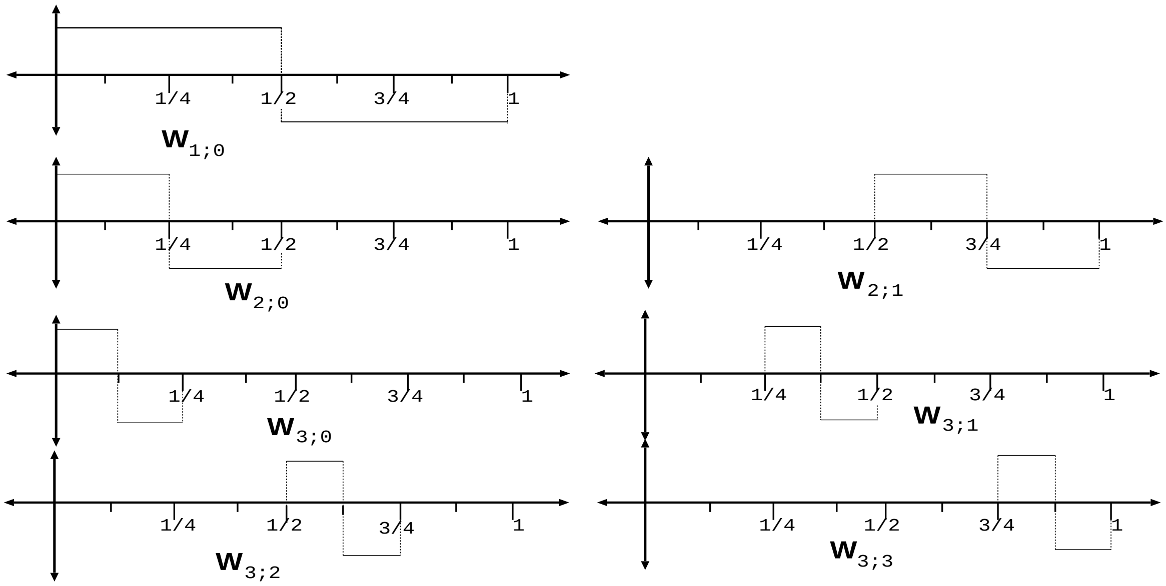


Figure 7.4: Seven Wavelet basis elements: $\mathbf{W}_{1,0}$; $\mathbf{W}_{2,0}$, $\mathbf{W}_{2,1}$; $\mathbf{W}_{3,0}$, $\mathbf{W}_{3,1}, \mathbf{W}_{3,2}$, $\mathbf{W}_{3,3}$

(a) Show that the set $\{\mathbf{H}_0, \mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3, \ldots\}$ is an orthonormal set in $\mathbf{L}^2[0,1]$.
(b) There is another way to define the Haar Basis. First recall that any number $x \in [0,1]$ has a unique **binary expansion** of the form

$$x \quad = \quad \frac{x_1}{2} + \frac{x_2}{4} + \frac{x_3}{8} + \frac{x_4}{16} + \cdots + \frac{x_n}{2^n} + \cdots$$

where $x_1, x_2, x_3, x_4, \ldots$ are all either 0 or 1. Show that, for any $n \geq 1$,

$$\mathbf{H}_n(x) \quad = \quad (-1)^{x_n} \quad = \quad \begin{cases} 1 & \text{if} \quad x_n = 0; \\ -1 & \text{if} \quad x_n = 1. \end{cases}$$

__**Exercise 7.9**__ Figure 7.4 portrays a **Wavelet Basis.** We define $\mathbf{W}_0 \equiv 1$, and for any $N \in \mathbb{N}$ and $n \in \left[0...2^{N-1}\right)$, we define

$$\mathbf{W}_{n;N}(x) = \begin{cases} 1 & \text{if } \dfrac{2n}{2^N} \leq \; x \; < \dfrac{2n+1}{2^N}; \\ -1 & \text{if } \dfrac{2n+1}{2^N} \leq \; x \; < \dfrac{2n+2}{2^N}; \\ 0 & \text{otherwise.} \end{cases}$$

Show that the the set

$$\{\mathbf{W}_0; \mathbf{W}_{1,0}; \mathbf{W}_{2,0}, \mathbf{W}_{2,1}; \mathbf{W}_{3,0}, \mathbf{W}_{3,1}, \mathbf{W}_{3,2}, \mathbf{W}_{3,3}; \mathbf{W}_{4,0}, \ldots, \mathbf{W}_{4,7}; \mathbf{W}_{5,0}, \ldots, \mathbf{W}_{5,15}; \ldots\}$$

is an *orthogonal* set in $\mathbf{L}^2[0,1]$, but is *not* orthonormal: for any N and n, we have $\|\mathbf{W}_{n;N}\|_2 = \dfrac{1}{2^{(N-1)/2}}$.

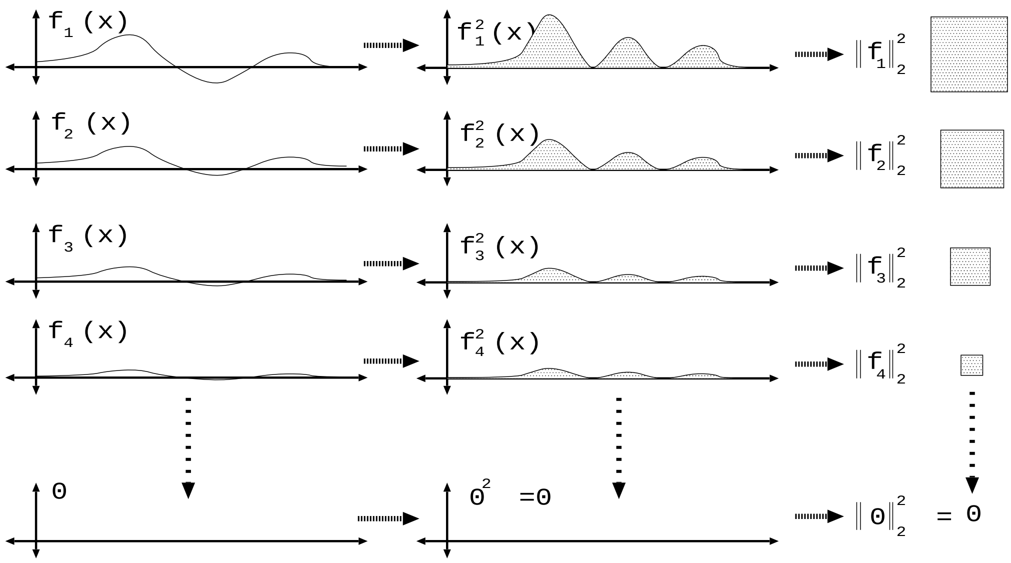


Figure 7.5: The sequence $\{f_1, f_2, f_3, \ldots\}$ converges to the constant 0 function in $\mathbf{L}^2(\mathbb{X})$.

7.4 Convergence Concepts

Prerequisites: §5.1

If $\{x_1, x_2, x_3, \ldots\}$ is a sequence of numbers, we know what it means to say "$\lim_{n\to\infty} x_n = x$". We can think of convergence as a kind of "approximation". Heuristically speaking, if the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x, then, for very large n, the number x_n is *approximately equal to* x.

If $\{f_1, f_2, f_3, \ldots\}$ was a sequence of functions, and f was some other function, then we might want to say that "$\lim_{n\to\infty} f_n = f$". We again imagine convergence as a kind of "approximation". Heuristically speaking, if the sequence $\{f_n\}_{n=1}^{\infty}$ converges to f, then, for very large n, the function f_n is a *good approximation* of f.

However, there are several ways we can interpret "good approximation", and these in turn lead to several different notions of "convergence". Thus, convergence of *functions* is a much more subtle concept that convergence of *numbers*. We will deal with *three* kinds of convergence here: $\mathbf{L}^2$-convergence, **pointwise** convergence, and **uniform** convergence.

7.4(a) $\mathbf{L}^2$ convergence

If $f, g \in \mathbf{L}^2(\mathbb{X})$, then the $\mathbf{L}^2$**-distance** between f and g is just

$$\|f-g\|_2 = \left(\frac{1}{M}\int_{\mathbb{X}} |f(\mathbf{x}) - g(\mathbf{x})|^2 \; d\mathbf{x}\right)^{1/2}, \text{ where } M = \begin{cases} \int_{\mathbb{X}} 1 \; d\mathbf{x} & \text{if } \mathbb{X} \text{ is a finite domain;} \\ 1 & \text{if } \mathbb{X} \text{ is an infinite domain.} \end{cases}$$

If we think of f as an "approximation" of g, then $\|f - g\|_2$ measures the *root-mean-squared error* of this approximation.

Lemma 7.8: $\|\bullet\|_2$ *is a* **norm**. *That is:*

(a) *For any* $f : \mathbb{X} \longrightarrow \mathbb{R}$ *and* $r \in \mathbb{R}$, $\|r \cdot f\|_2 = |r| \cdot \|f\|_2$.

(b) (Triangle Inequality) *For any* $f, g : \mathbb{X} \longrightarrow \mathbb{R}$, $\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$.

Proof: **Exercise 7.10** $\Box$

If $\{f_1, f_2, f_3, \ldots\}$ is a sequence of successive approximations of f, then we say the sequence **converges to f in $\mathbf{L}^2$** if $\lim_{n\to\infty} \|f_n - f\|_2 = 0$ (sometimes this is called *convergence in mean*). See Figure 7.5. We then write $f = \mathbf{L}^2\text{–}\lim_{n\to\infty} f_n$.

Example 7.9: In each of the following examples, let $\mathbb{X} = [0, 1]$.

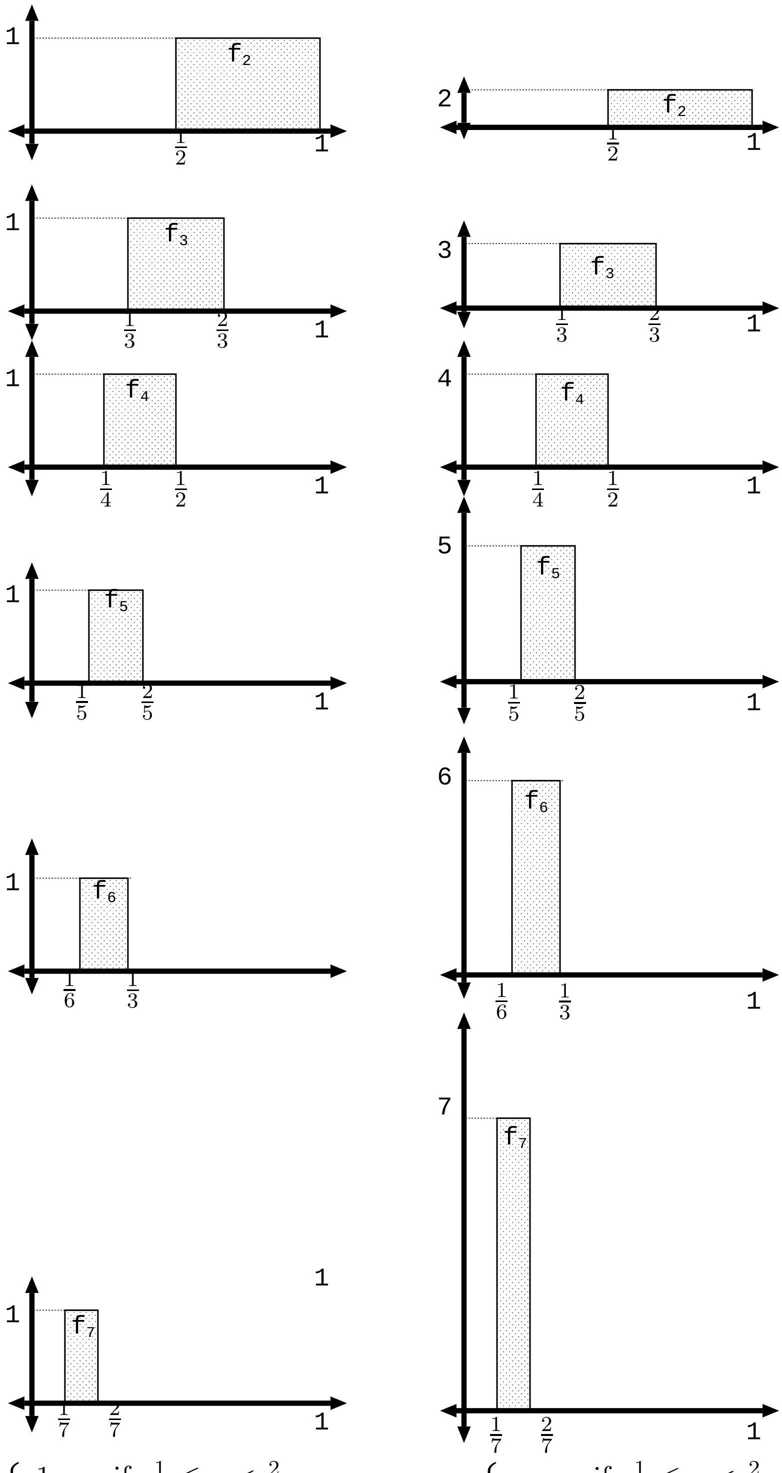
(a) Suppose $f_n(x) = \begin{cases} 1 & \text{if } 1/n < x < 2/n \\ 0 & \text{otherwise} \end{cases}$ (Figure 7.6A). Then $\|f_n\|_2 = \frac{1}{\sqrt{n}}$ (**Exercise 7.11**). Hence, $\lim_{n\to\infty} \|f_n\|_2 = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, so the sequence $\{f_1, f_2, f_3, \ldots\}$ converges to the constant 0 function in $\mathbf{L}^2[0, 1]$.

(b) For all $n \in \mathbb{N}$, let $f_n(x) = \begin{cases} n & \text{if } 1/n < x < 2/n; \\ 0 & \text{otherwise} \end{cases}$ (Figure 7.6B). Then $\|f_n\|_2 = \sqrt{n}$ (**Exercise 7.12**). Hence, $\lim_{n\to\infty} \|f_n\|_2 = \lim_{n\to\infty} \sqrt{n} = \infty$, so the sequence $\{f_1, f_2, f_3, \ldots\}$ does *not* converge to zero in $\mathbf{L}^2[0, 1]$.

(c) For each $n \in \mathbb{N}$, let $f_n(x) = \begin{cases} 1 & \text{if } \left|\frac{1}{2} - x\right| < \frac{1}{n}; \\ 0 & \text{otherwise} \end{cases}$. Then the sequence $\{f_n\}_{n=1}^{\infty}$ converges to 0 in $\mathbf{L}^2$. (**Exercise 7.13**)

(d) For all $n \in \mathbb{N}$, let $f_n(x) = \frac{1}{1 + n \cdot \left|x - \frac{1}{2}\right|}$. Figure 7.7 portrays elements f_1, f_{10}, f_{100}, and f_{1000}; these picture strongly suggest that the sequence is converging to the constant 0 function in $\mathbf{L}^2[0, 1]$. The proof of this is **Exercise 7.14** .

(e) Recall the **Wavelet** functions from Example 7.8(b). For any $N \in \mathbb{N}$ and $n \in \left[0..2^{N-1}\right)$, we had $\|\mathbf{W}_{N,n}\|_2 = \frac{1}{2^{(N-1)/2}}$. Thus, the sequence of wavelet basis elements converges to the constant 0 function in $\mathbf{L}^2[0, 1]$. $\diamondsuit$



(A) $f_n(x) = \begin{cases} 1 & \text{if } \frac{1}{n} \leq x < \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$ **(B)** $f_n(x) = \begin{cases} n & \text{if } \frac{1}{n} \leq x < \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$

Figure 7.6: **(A)** Examples 7.9(a), 7.11(a), and 7.15(a); **(B)** Examples 7.9(b) and 7.11(b).

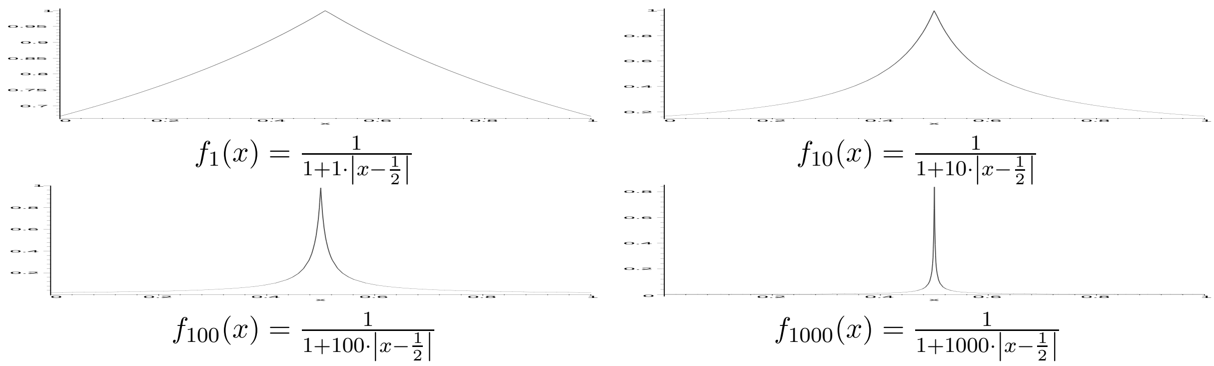


Figure 7.7: Examples 7.9(c) and 7.11(c): If $f_n(x) = \frac{1}{1+n\cdot\left|x-\frac{1}{2}\right|}$, then the sequence $\{f_1, f_2, f_3, \ldots\}$ converges to the constant 0 function in $\mathbf{L}^2[0,1]$.

Note that, if we define $g_n = f - f_n$ for all $n \in \mathbb{N}$, then

$$\left(f_n \xrightarrow[n\to\infty]{} f \text{ in } \mathbf{L}^2 \right) \iff \left(g_n \xrightarrow[n\to\infty]{} 0 \text{ in } \mathbf{L}^2 \right)$$

Hence, to understand $\mathbf{L}^2$-convergence in general, it is sufficient to understand $\mathbf{L}^2$-convergence to the constant 0 function.

7.4(b) Pointwise Convergence

Convergence in $\mathbf{L}^2$ only means that the *average* approximation error gets small. It does *not* mean that $\lim_{n\to\infty} f_n(\mathbf{x}) = f(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{X}$. If *this* equation is true, then we say that the sequence $\{f_1, f_2, \ldots\}$ converges **pointwise** to f (see Figure 7.8). We then write $f \equiv \lim_{n\to\infty} f_n$.

Pointwise convergence is generally considered stronger than $\mathbf{L}^2$ convergence because of the following result:

Theorem 7.10: *Let $\mathbb{X} \subset \mathbb{R}^D$ be a bounded domain, and let $\{f_1, f_2, \ldots\}$ be a sequence of functions in $\mathbf{L}^2(\mathbb{X})$.*

1. *All the functions are uniformly bounded —that is, there is some $M > 0$ so that $|f_n(x)| < M$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{X}$.*

2. *The sequence $\{f_1, f_2, \ldots\}$ converges* **pointwise** *to some function $f \in \mathbf{L}^2(\mathbb{X})$.*

Then the sequence $\{f_1, f_2, \ldots\}$ also converges to f in $\mathbf{L}^2(\mathbb{X})$. □

Example 7.11: In each of the following examples, let $\mathbb{X} = [0,1]$.

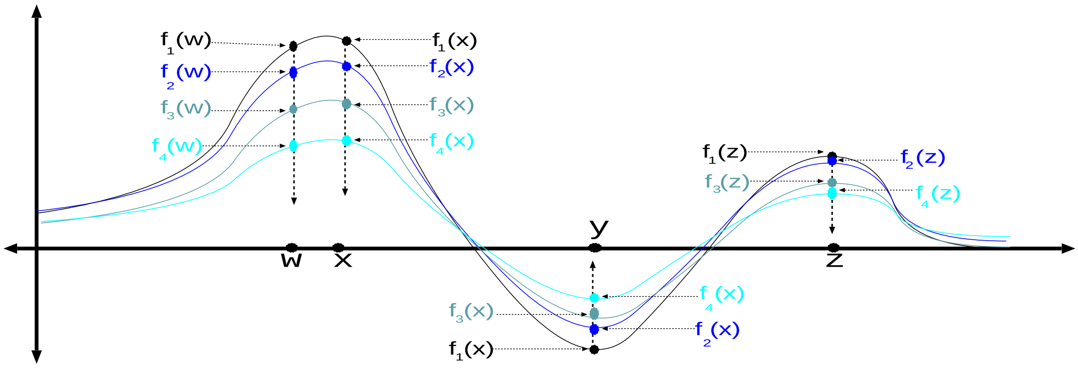


Figure 7.8: The sequence $\{f_1, f_2, f_3, \ldots\}$ converges **pointwise** to the constant 0 function. Thus, if we pick some random points $w, x, y, z \in \mathbb{X}$, then we see that $\lim_{n\to\infty} f_n(w) = 0$, $\lim_{n\to\infty} f_n(x) = 0$, $\lim_{n\to\infty} f_n(y) = 0$, and $\lim_{n\to\infty} f_n(z) = 0$.

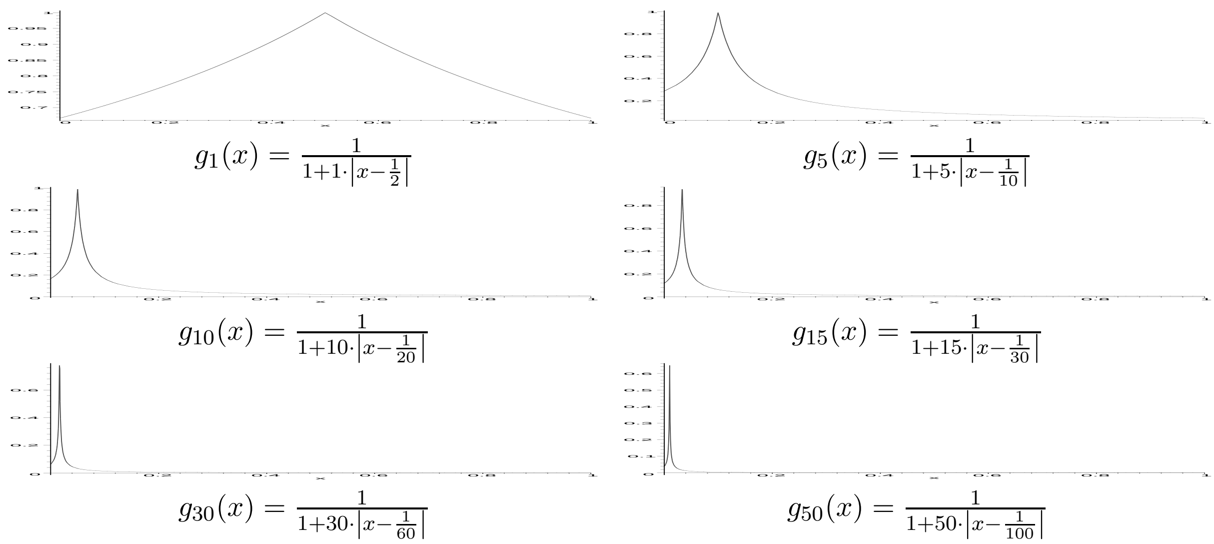


Figure 7.9: Examples 7.11(d) and 7.15(d): If $g_n(x) = \frac{1}{1+n\cdot\left|x-\frac{1}{2n}\right|}$, then the sequence $\{g_1, g_2, g_3, \ldots\}$ converges pointwise to the constant 0 function on $[0, 1]$.

(a) As in Example 7.9(a), for each $n \in \mathbb{N}$, let $f_n(x) = \begin{cases} 1 & \text{if } 1/n < x < 2/n; \\ 0 & \text{otherwise} \end{cases}$. (Fig.7.6A).

The sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the constant 0 function on $[0,1]$. Also, as predicted by Theorem 7.10, the sequence $\{f_n\}_{n=1}^{\infty}$ converges to the constant 0 function in $\mathbf{L}^2$ (see Example 7.9(a)).

(b) As in Example 7.9(b), for each $n \in \mathbb{N}$, let $f_n(x) = \begin{cases} n & \text{if } 1/n < x < 2/n; \\ 0 & \text{otherwise} \end{cases}$ (Fig.7.6B). Then this sequence converges *pointwise* to the constant 0 function, but does *not* converge to zero in $\mathbf{L}^2[0,1]$. This illustrates the importance of the *boundedness* hypothesis in Theorem 7.10.

(c) As in Example 7.9(c), for each $n \in \mathbb{N}$, let $f_n(x) = \begin{cases} 1 & \text{if } \left|\frac{1}{2} - x\right| < \frac{1}{n}; \\ 0 & \text{otherwise} \end{cases}$. Then the sequence $\{f_n\}_{n=1}^{\infty}$ does *not* converges to 0 in pointwise, although it *does* converge in $\mathbf{L}^2$.

(d) Recall the functions $f_n(x) = \dfrac{1}{1 + n \cdot \left|x - \frac{1}{2}\right|}$ from Example 7.9(d). This sequence of functions converges to zero in $\mathbf{L}^2[0,1]$, however, it does *not* converge to zero pointwise (**Exercise 7.15**).

(e) For all $n \in \mathbb{N}$, let $g_n(x) = \dfrac{1}{1 + n \cdot \left|x - \frac{1}{2n}\right|}$. Figure 7.9 on the preceding page portrays elements g_1, g_5, g_{10}, g_{15}, g_{30}, and g_{50}; These picture strongly suggest that the sequence is converging pointwise to the constant 0 function on $[0,1]$. The proof of this is **Exercise 7.16** .

(f) Recall from Example 7.9(e) that the sequence of Wavelet basis elements $\{\mathbf{W}_{N;n}\}$ converges to zero in $\mathbf{L}^2[0,1]$. Note, however, that it does *not* converge to zero pointwise (**Exercise 7.17**). ◇

Note that, if we define $g_n = f - f_n$ for all $n \in \mathbb{N}$, then

$$\left(f_n \xrightarrow[n\to\infty]{} f \text{ pointwise} \right) \iff \left(g_n \xrightarrow[n\to\infty]{} 0 \text{ pointwise} \right)$$

Hence, to understand pointwise convergence in general, it is sufficient to understand pointwise convergence to the constant 0 function.

7.4(c) Uniform Convergence

There is an even stronger form of convergence. The **uniform norm** of a function f is defined:

$$\|f\|_\infty = \sup_{\mathbf{x}\in\mathbb{X}} \left|f(\mathbf{x})\right|$$

This measures the farthest deviation of the function f from zero (see Figure 7.10).

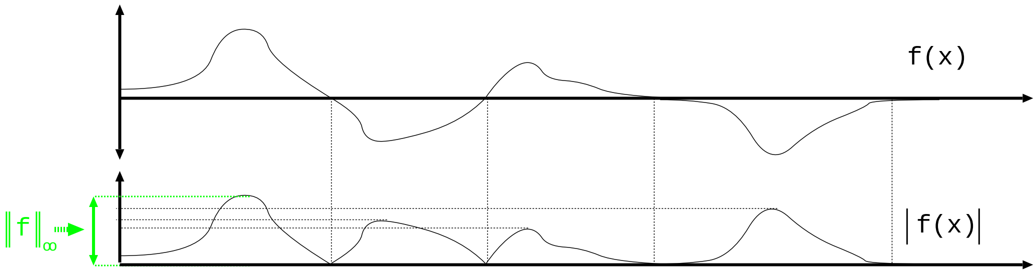


Figure 7.10: The uniform norm of f is given: $\|f\|_\infty = \sup_{x\in\mathbb{X}} |f(x)|$.

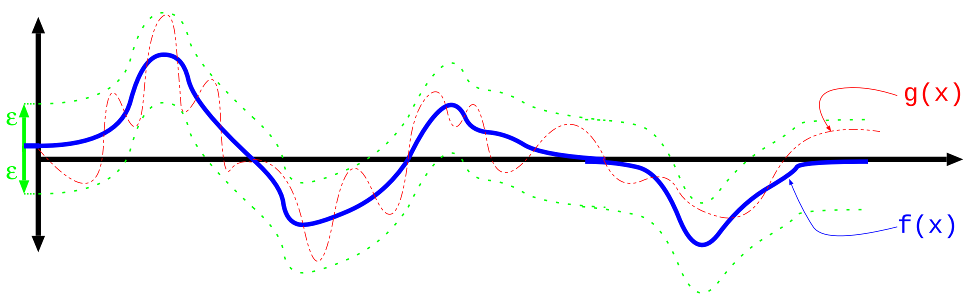
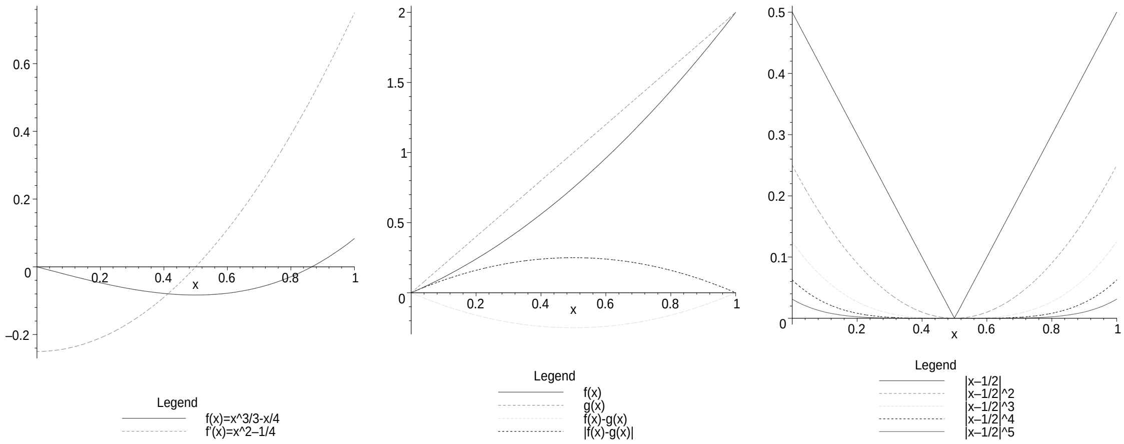


Figure 7.11: If $\|f-g\|_\infty < \epsilon$, this means that $g(x)$ is confined within an ϵ-tube around f for all x.



(A) **(B)** **(C)**

Figure 7.12: **(A)** The uniform norm of $f(x) = \frac{1}{3}x^3 - \frac{1}{4}x$ (Example 7.12). **(B)** The uniform distance between $f(x) = x(x+1)$ and $g(x) = 2x$ (Example 7.14). **(C)** $g_n(x) = \left|x - \frac{1}{2}\right|^n$, for $n = 1, 2, 3, 4, 5$ (Example (2b))

Example 7.12: Suppose $\mathbb{X} = [0,1]$, and $f(x) = \frac{1}{3}x^3 - \frac{1}{4}x$ (as in Figure 7.12A). The minimal point of f is $x = \frac{1}{2}$, where $f(\frac{1}{2}) = \frac{-1}{12}$. The maximal point of f is $x = 1$, where $f(1) = \frac{1}{12}$. Thus, $|f(x)|$ takes a maximum value of $\frac{1}{12}$ at either point, so that $\|f\|_\infty = \sup_{0 \le x \le 1} \left|\frac{1}{3}x^3 - \frac{1}{4}x\right| = \frac{1}{12}$. ◇

Lemma 7.13: $\|\bullet\|_\infty$ *is a* **norm**. *That is:*

(a) *For any* $f : \mathbb{X} \longrightarrow \mathbb{R}$ *and* $r \in \mathbb{R}$, $\|r \cdot f\|_\infty = |r| \cdot \|f\|_\infty$.

(b) (Triangle Inequality) *For any* $f, g : \mathbb{X} \longrightarrow \mathbb{R}$, $\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.

Proof: Exercise 7.18 □

The **uniform distance** between two functions f and g is then given by:

$$\|f - g\|_\infty = \sup_{\mathbf{x} \in \mathbb{X}} \left| f(\mathbf{x}) - g(\mathbf{x}) \right|$$

One way to interpret this is portrayed in Figure 7.11. Define a "tube" of width ϵ around the function f. If $\|f - g\|_\infty < \epsilon$, this means that $g(x)$ is confined within this tube for all x.

Example 7.14: Let $\mathbb{X} = [0,1]$, and suppose $f(x) = x(x+1)$ and $g(x) = 2x$ (as in Figure 7.12B). For any $x \in [0,1]$,

$$|f(x) - g(x)| = \left|x^2 + x - 2x\right| = \left|x^2 - x\right| = x - x^2.$$

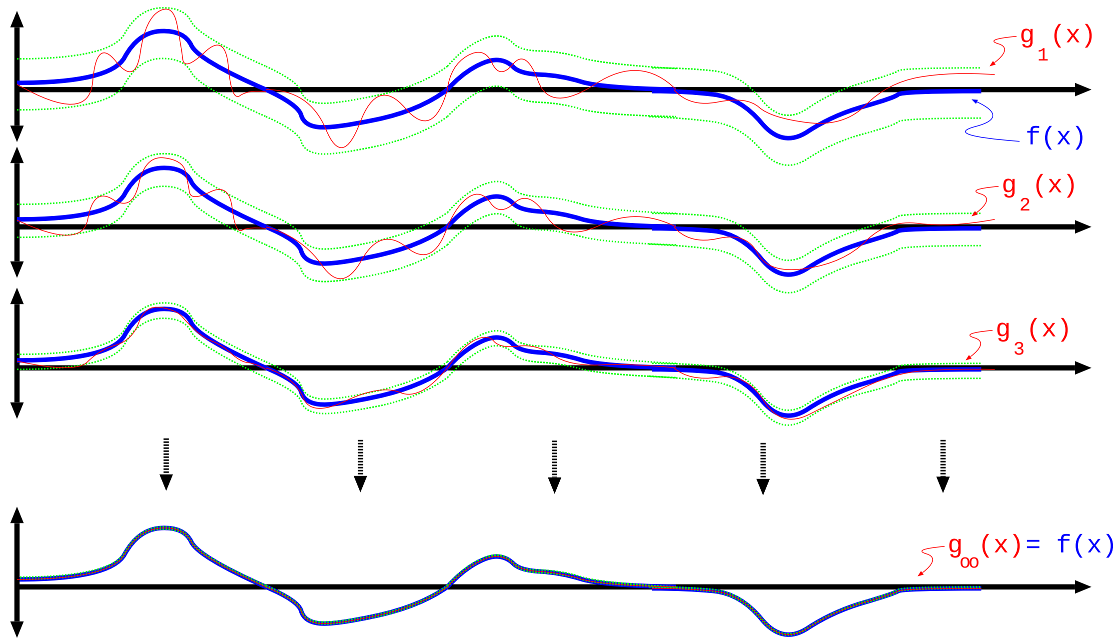


Figure 7.13: The sequence $\{g_1, g_2, g_3, \ldots\}$ converges **uniformly** to f.

(because it is nonnegative). This expression takes its maximum at $x = \frac{1}{2}$ (to see this, take the derivative), and its value at $x = \frac{1}{2}$ is $\frac{1}{4}$. Thus, $\|f - g\|_\infty = \sup_{x \in \mathbb{X}} \left| x(x-1) \right| = \frac{1}{4}$. ♢

Let $\{g_1, g_2, g_3, \ldots\}$ be functions from $\mathbb{X}$ to $\mathbb{R}$, and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be some other function. The sequence $\{g_1, g_2, g_3, \ldots\}$ **converges uniformly** to f if $\lim_{n\to\infty} \|g_n - f\|_\infty = 0$. We then write $f = \operatorname{unif-lim}_{n\to\infty} g_n$. This means not only that $\lim_{n\to\infty} g_n(\mathbf{x}) = f(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{X}$, but furthermore, that the functions g_n converge to f everywhere at the same "speed". This is portrayed in Figure 7.13. For any $\epsilon > 0$, we can define a "tube" of width ϵ around f, and, no matter how small we make this tube, the sequence $\{g_1, g_2, g_3, \ldots\}$ will eventually enter this tube and remain there. To be precise: there is some N so that, for all $n > N$, the function g_n is confined within the ϵ-tube around f —ie. $\|f - g_n\|_\infty < \epsilon$.

Example 7.15: In each of the following examples, let $\mathbb{X} = [0, 1]$.

(a) Suppose, as in Example 7.11(a) on page 123, and Figure 7.6B on page 122, that

$$g_n(x) = \begin{cases} 1 & \text{if } \frac{1}{n} < x < \frac{2}{n}; \\ 0 & \text{otherwise.} \end{cases}$$

Then the sequence $\{g_1, g_2, \ldots\}$ converges *pointwise* to the constant zero function, but does *not* converge to zero uniformly on $[0, 1]$. (**Exercise 7.19** Verify these claims.).

(b) If $g_n(x) = \left|x - \frac{1}{2}\right|^n$ (see Figure 7.12C), then $\|g_n\|_\infty = \frac{1}{2^n}$ (**Exercise 7.20**). Thus, the sequence $\{g_1, g_2, \ldots\}$ converges to zero uniformly on $[0,1]$, because $\lim_{n\to\infty} \|g_n\|_\infty = \lim_{n\to\infty} \frac{1}{2^n} = 0$.

(c) If $g_n(x) = 1/n$ for all $x \in [0,1]$, then the sequence $\{g_1, g_2, \ldots\}$ converges to zero uniformly on $[0,1]$ (**Exercise 7.21**).

(d) Recall the functions $g_n(x) = \frac{1}{1+n\cdot\left|x-\frac{1}{2n}\right|}$ from Example 7.11(e) (Figure 7.9 on page 124). The sequence $\{g_1, g_2, \ldots\}$ converges *pointwise* to the constant zero function, but does *not* converge to zero uniformly on $[0,1]$. (**Exercise 7.22** Verify these claims.). ♢

Note that, if we define $g_n = f - f_n$ for all $n \in \mathbb{N}$, then

$$\left(f_n \xrightarrow[n\to\infty]{} f \text{ uniformly} \right) \iff \left(g_n \xrightarrow[n\to\infty]{} 0 \text{ uniformly} \right)$$

Hence, to understand uniform convergence in general, it is sufficient to understand uniform convergence to the constant 0 function.

Important: $\left(\text{Uniform convergence} \right) \implies \left(\text{Pointwise convergence} \right)$.

Also, if the sequence of functions is uniformly bounded and $\mathbb{X}$ is compact, we have

$$\left(\text{Pointwise convergence} \right) \implies \left(\text{Convergence in } \mathbf{L}^2 \right)$$

However, the opposite implications are *not* true. In general:

$$\left(\text{Convergence in } \mathbf{L}^2 \right) \not\Longrightarrow \left(\text{Pointwise convergence} \right) \not\Longrightarrow \left(\text{Uniform convergence} \right)$$

Thus, uniform convergence is the 'best' kind of convergence; it has the most useful consequences, but is also the most difficult to achieve (in many cases, we must settle for pointwise or $\mathbf{L}^2$ convergence instead). For example, the following consequence of uniform convergence is extremely useful:

Proposition 7.16: *Let $\{f_1, f_2, f_3, \ldots\}$ be continuous functions from $\mathbb{X}$ to $\mathbb{R}$, and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be some other function. If $f_n \xrightarrow[n\to\infty]{} f$ uniformly, then f is also continuous on $\mathbb{X}$.*

Proof: **Exercise 7.23** (Slightly challenging; for students with some analysis background) □

Note that Proposition 7.16 is *false* if we replace 'uniformly' with 'pointwise' or 'in $\mathbf{L}^2$.' Sometimes, however, uniform convergence is a little too much to ask for, and we must settle for a slightly weaker form of convergence.

Let $\mathbb{X} \subset \mathbb{R}^D$ be some subset (not necessarily closed). Let $\{g_1, g_2, g_3, \ldots\}$ be functions from $\mathbb{X}$ to $\mathbb{R}$, and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be some other function. The sequence $\{g_1, g_2, g_3, \ldots\}$ **converges semiuniformly** to f if:

(a) $\{g_1, g_2, g_3, \ldots\}$ converges *pointwise* to f on $\mathbb{X}$; i.e. $f(x) = \lim_{n\to\infty} g_n(x)$ for all $x \in \mathbb{X}$.

(b) $\{g_1, g_2, g_3, \ldots\}$ converges *uniformly* to f on any *closed subset* of $\mathbb{X}$. In other words, if $\mathbb{Y} \subset \mathbb{X}$ is any closed set, then

$$\lim_{n\to\infty} \left(\sup_{y\in\mathbb{Y}} |f(y) - g_n(y)| \right) \quad = \quad 0.$$

Heuristically speaking, this means that the sequence $\{g_n\}_{n=1}^{\infty}$ is 'trying' to converge to f uniformly on $\mathbb{X}$, but it is maybe getting 'stuck' at some of the boundary points of $\mathbb{X}$.

Example 7.17: Let $\mathbb{X} := (0, 1)$. Recall the functions $g_n(x) = \frac{1}{1+n\cdot\left|x-\frac{1}{2n}\right|}$ from Figure 7.9 on page 124. By Example 7.15(d) on page 129, we know that this sequence *doesn't* converge uniformly to 0 on $(0, 1)$. However, it does converge *semi*uniformly to 0. First, we know it converges pointwise on $(0, 1)$, by Example 7.11(e) on page 125. Second, if $0 < a < b < 1$, it is easy to check that $\{g_n\}_{n=1}^{\infty}$ converges to f uniformly on the closed interval $[a, b]$ (**Exercise 7.24**). It follows that $\{g_n\}_{n=1}^{\infty}$ converges to f uniformly on any closed subset of $(0, 1)$. ♢

Note: If $\mathbb{X}$ itself is a closed set, then semiuniform convergence is equivalent to uniform convergence, because condition **(b)** means that the sequence $\{g_n\}_{n=1}^{\infty}$ converges uniformly (because $\mathbb{X}$ is a closed subset of itself).

However, if $\mathbb{X}$ is *not* closed, then the two convergence forms are not equivalent. In general,

$$\Big(\text{Uniform convergence}\Big) \Rightarrow \Big(\text{Semiuniform convergence}\Big) \Rightarrow \Big(\text{Pointwise convergence}\Big).$$

However, the the opposite implications are *not* true in general.

7.4(d) Convergence of Function Series

Let $\{f_1, f_2, f_3, \ldots\}$ be functions from $\mathbb{X}$ to $\mathbb{R}$. The **function series** $\sum_{n=1}^{\infty} f_n$ is the formal infinite summation of these functions; we would like to think of this series as defining another function from $\mathbb{X}$ to $\mathbb{R}$. . Intuitively, the symbol "$\sum_{n=1}^{\infty} f_n$" should represent the function which arises as the limit $\lim_{N\to\infty} F_N$, where, for each $N \in \mathbb{N}$, $F_N(x) := \sum_{n=1}^{N} f_n(x) \;=\; f_1(x) + f_2(x) + \cdots + f_N(x)$ is the Nth *partial sum*. To make this previse, we must specify the sense in which the partial sums $\{F_1, F_2, \ldots\}$ converge. If $F : \mathbb{X} \longrightarrow \mathbb{R}$ is this putative limit function, then we say that the series $\sum_{n=1}^{\infty} f_n$....

- ...converges **in $\mathbf{L}^2$** to F if $F \;=\; \mathbf{L}^2\text{–}\lim_{N\to\infty} \sum_{n=1}^{N} f_n$. We then write $F \underset{\text{L2}}{\approx} \sum_{n=1}^{\infty} f_n$.

- ...converges **pointwise** to F if, for each $x \in \mathbb{X}$, $F(x) = \lim_{N\to\infty} \sum_{n=1}^{N} f_n(x)$. We then write $F \equiv \sum_{n=1}^{\infty} f_n$.

- ...converges **uniformly** to F if $F = \text{unif}-\lim_{N\to\infty} \sum_{n=1}^{N} f_n$. We then write $F \underset{\text{unif}}{\equiv} \sum_{n=1}^{\infty} f_n$.

The next three results provide useful conditions for the uniform convergence of an infinite summation of functions; these will be important in our study of Fourier series and other eigenfunction expansions in Chapters 8 to 10:

Proposition 7.18: Weierstrass M-test

Let $\{f_1, f_2, f_3, \ldots\}$ be functions from $\mathbb{X}$ to $\mathbb{R}$. For every $n \in \mathbb{N}$, let $M_n := \|f_n\|_\infty$. Then

$$\left(\text{The series } \sum_{n=1}^{\infty} f_n \text{ converges uniformly}\right) \iff \left(\sum_{n=1}^{\infty} M_n < \infty\right)$$

Proof: "$\Longleftarrow$" **Exercise 7.25** (a) Show that the series converges *pointwise* to some limit function $f : \mathbb{X} \longrightarrow \mathbb{R}$.

(b) For any $N \in \mathbb{N}$, show that $\left\| F - \sum_{n=1}^{N} f_n \right\|_\infty \leq \sum_{n=N+1}^{\infty} M_n$.

(c) Show that $\lim_{N\to\infty} \sum_{n=N+1}^{\infty} M_n = 0$.

"$\Longrightarrow$" **Exercise 7.26** . □

Proposition 7.19: Cauchy's Criterion

Let $\{f_1, f_2, f_3, \ldots\}$ be functions from $\mathbb{X}$ to $\mathbb{R}$. For every $N \in \mathbb{N}$, let $C_N := \sup_{M>N} \left\| \sum_{n=N}^{M} f_n \right\|_\infty$.

Then $$\left(\text{The series } \sum_{n=1}^{\infty} f_n \text{ converges uniformly}\right) \iff \left(\lim_{N\to\infty} C_N = 0\right).$$

Proof: See [CB87, §88]. □

Proposition 7.20: Abel's Test

Let $\mathbb{X} \subset \mathbb{R}^N$ and $\mathbb{Y} \subset \mathbb{R}^M$ be two domains. Let $\{f_1, f_2, f_3, \ldots\}$ be a sequence of functions from $\mathbb{X}$ to $\mathbb{R}$, such that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $\mathbb{X}$. Let $\{g_1, g_2, g_3, \ldots\}$ be another sequence of functions from $\mathbb{Y}$ to $\mathbb{R}$, and consider the sequence $\{h_1, h_2, \ldots\}$ of functions from $\mathbb{X} \times \mathbb{Y}$ to $\mathbb{R}$, defined by $h_n(x, y) := f_n(x) g_n(y)$. Suppose:

(a) *The sequence $\{g_n\}_{n=1}^{\infty}$ is uniformly bounded; i.e. there is some $M > 0$ such that $|g_n(y)| < M$ for all $n \in \mathbb{N}$ and $y \in \mathbb{Y}$.*

(b) *The sequence $\{g_n\}_{n=1}^{\infty}$ is monotonic; i.e. either $g_1(y) \leq g_2(y) \leq g_3(y) \leq \cdots$ for all $y \in \mathbb{Y}$, or $g_1(y) \geq g_2(y) \geq g_3(y) \geq \cdots$ for all $y \in \mathbb{Y}$.*

Then the series $\sum_{n=1}^{\infty} h_n$ converges uniformly on $\mathbb{X} \times \mathbb{Y}$. □

Proof: See [CB87, §88]. □

Further Reading:

Most of the 'mathematically rigorous' texts on partial differential equations (such as [CB87] or [Eva91, Appendix D]) contain detailed and thorough discussions of $\mathbf{L}^2$ space, orthogonal basis, and the various convergence concepts discussed in this chapter.[3] This is because almost all solutions to partial differential equations arise through some sort of infinite series or approximating sequence; hence it is essential to properly understand the various forms of function convergence and their relationships.

The convergence of sequences of functions is part of a subject called *real analysis*, and any advanced textbook on real analysis will contain extensive material on convergence. There are many other forms of function convergence we haven't even mentioned in this chapter, including $\mathbf{L}^p$ *convergence* (for any value of p between 1 and ∞), convergence *in measure*, convergence *almost everywhere*, and *weak** convergence. Different convergence modes are useful in different contexts, and the logical relationships between them are fairly subtle. See [Fol84, §2.4] for a good summary. Other standard references are [WZ77, Chap.8], [KF75, §28.4-§28.5; §37], [Rud87] or [Roy88].

The geometry of infinite-dimensional vector spaces is called *functional analysis*, and is logically distinct from the convergence theory for functions (although of course, most of the important infinite dimensional spaces are spaces of functions). Infinite-dimensional vector spaces fall into several broad classes, depending upon the richness of the geometric and topological structure, which include *Hilbert spaces* [such as $\mathbf{L}^2(\mathbb{X})$], *Banach Spaces* [such as $\mathcal{C}(\mathbb{X})$ or $\mathbf{L}^1(\mathbb{X})$] and *locally convex spaces*. An excellent introduction to functional analysis is [Con90]. Another standard reference is [Fol84, Chap.5]. Hilbert spaces are the mathematical foundation of quantum mechanics; see [Pru81, BEH94].

[3] However, many of the more 'applications-oriented' introductions to PDEs do *not* discuss these matters, or at least, not very precisely.

7.5 Orthogonal/Orthonormal Bases

Prerequisites: §7.1, §7.4(a) **Recommended:** §7.4(d)

An **orthogonal basis** for $\mathbf{L}^2(\mathbb{X})$ is an infinite collection of functions $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \ldots\}$ such that:

- $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \ldots\}$ form an *orthogonal set*, ie. $\langle \mathbf{f}_k, \mathbf{f}_j \rangle = 0$ whenever $k \neq j$.
- For any $\mathbf{g} \in \mathbf{L}^2(\mathbb{X})$, if we define $\gamma_n = \dfrac{\langle \mathbf{g}, \mathbf{f}_n \rangle}{\|\mathbf{f}_n\|_2^2}$, for all $n \in \mathbb{N}$, then $\mathbf{g} \underset{\text{L2}}{\approx} \sum_{n=1}^{\infty} \gamma_n \mathbf{f}_n$.

Recall that this means that $\lim_{N\to\infty} \left\| \mathbf{g} - \sum_{n=1}^{N} \gamma_n \mathbf{f}_n \right\|_2 = 0$. In other words, we can approximate $\mathbf{g}$ as closely as we want in $\mathbf{L}^2$ norm with a partial sum $\sum_{n=1}^{N} \gamma_n \mathbf{f}_n$, if we make N large enough.

An **orthonormal basis** for $\mathbf{L}^2(\mathbb{X})$ is an infinite collection of functions $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \ldots\}$ such that:

- $\|\mathbf{f}_k\|_2 = 1$ for every k.
- $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \ldots\}$ is an orthogonal basis for $\mathbf{L}^2(\mathbb{X})$. In other words, $\langle \mathbf{f}_k, \mathbf{f}_j \rangle = 0$ whenever $k \neq j$, and, for any $\mathbf{g} \in \mathbf{L}^2(\mathbb{X})$, if we define $\gamma_n = \langle \mathbf{g}, \mathbf{f}_n \rangle$ for all $n \in \mathbb{N}$, then $\mathbf{g} \underset{\text{L2}}{\approx} \sum_{n=1}^{\infty} \gamma_n \mathbf{f}_n$.

One consequence of this is

Theorem 7.21: Parseval's Equality

Let $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \ldots\}$ be an orthonormal basis for $\mathbf{L}^2(\mathbb{X})$, and let $\mathbf{g} \in \mathbf{L}^2(\mathbb{X})$. Let $\gamma_n = \langle \mathbf{g}, \mathbf{f}_n \rangle$ for all $n \in \mathbb{N}$. Then $\|\mathbf{g}\|_2^2 = \sum_{n=1}^{\infty} |\gamma_n|^2$. □

The idea of Fourier analysis is to find an *orthogonal basis* for an $\mathbf{L}^2$-space, using familiar trigonometric functions. We will return to this in Chapter 8.

7.6 Self-Adjoint Operators and their Eigenfunctions (∗)

Prerequisites: §5.2(d), §7.5, §6.5

A linear operator $F : \mathbb{R}^D \longrightarrow \mathbb{R}^D$ is **self-adjoint** if, for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^D$,

$$\langle F(\mathbf{x}),\ \mathbf{y} \rangle \ = \ \langle \mathbf{x},\ F(\mathbf{y}) \rangle\,.$$

Example 7.22: The matrix $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ defines a self-adjoint operator on $\mathbb{R}^2$, because for any $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in $\mathbb{R}^2$, we have

$$\begin{aligned} \langle F(\mathbf{x}), \mathbf{y} \rangle &= \left\langle \begin{bmatrix} x_1 - 2x_2 \\ x_2 - 2x_1 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle = y_1 \Big(x_1 - 2x_2 \Big) + y_2 \Big(x_2 - 2x_1 \Big) \\ &= x_1 \Big(y_1 - 2y_2 \Big) + x_2 \Big(y_2 - 2y_1 \Big) = \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 - 2y_2 \\ y_2 - 2y_1 \end{bmatrix} \right\rangle \\ &= \langle \mathbf{x}, \ F(\mathbf{y}) \rangle . \end{aligned}$$

◊

Theorem 7.23: *Let $F : \mathbb{R}^D \longrightarrow \mathbb{R}^D$ be a linear operator with matrix* $\mathbf{A}$. *Then F is self-adjoint if and only if* $\mathbf{A}$ *is symmetric* (i.e. $a_{ij} = a_{ji}$ for all j, i)

Proof: Exercise 7.27 . □

A linear operator $\mathsf{L} : \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty$ is **self-adjoint** if, for any two functions $f, g \in \mathcal{C}^\infty$,

$$\langle \mathsf{L}[f], \ g \rangle = \langle f, \ \mathsf{L}[g] \rangle$$

whenever both sides are well-defined[4].

Example 7.24: Multiplication Operators are Self-Adjoint.

Let $\mathbb{X} \subset \mathbb{R}^D$ be any bounded domain. Let $\mathcal{C}^\infty := \mathcal{C}^\infty(\mathbb{X}; \mathbb{R})$. Fix $q \in \mathcal{C}^\infty(\mathbb{X})$, and define the operator $\mathsf{Q} : \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty$ by $\mathsf{Q}(f) := q \cdot f$ for any $f \in \mathcal{C}^\infty$. Then Q is self-adjoint. To see this, let $f, g \in \mathcal{C}^\infty$. Then

$$\langle q \cdot f, \ g \rangle = \int_{\mathbb{X}} (q \cdot f) \cdot g \ dx = \int_{\mathbb{X}} f \cdot (q \cdot g) \ dx = \langle f, \ q \cdot g \rangle$$

(whenever these integrals are well-defined). ◊

Let $L > 0$, and consider the interval $[0, L]$. Recall that $\mathcal{C}^\infty[0, L]$ is the set of all smooth functions from $[0, L]$ into $\mathbb{R}$, and that....

$\mathcal{C}_0^\infty[0, L]$ is the space of all $f \in \mathcal{C}^\infty[0, L]$ satisfying homogeneous *Dirichlet* boundary conditions: $f(0) = 0 = f(L)$ (see §6.5(a)).

$\mathcal{C}_\perp^\infty[0, L]$ is the space of all $f \in \mathcal{C}^\infty[0, L]$ satisfying $f : [0, L] \longrightarrow \mathbb{R}$ satisfying homogeneous *Neumann* boundary conditions: $f'(0) = 0 = f'(L)$ (see §6.5(b)).

$\mathcal{C}_{\text{per}}^\infty[0, L]$ is the space of all $f \in \mathcal{C}^\infty[0, L]$ satisfying $f : [0, L] \longrightarrow \mathbb{R}$ satisfying *periodic* boundary conditions: $f(0) = f(L)$ and $f'(0) = f'(L)$ (see §6.5(d)).

[4]This is an important point. Often, one of these inner products (say, the left one) will *not* be well-defined, because the integral $\int_{\mathbb{X}} \mathsf{L}[f] \cdot g \ dx$ does not converge, in which case "self-adjointness" is meaningless.

$\mathcal{C}^\infty_{h,h_\perp}[0,L]$ is the space of all $f \in \mathcal{C}^\infty[0,L]$ satisfying homogeneous *mixed* boundary conditions, for any fixed real numbers $h(0)$, $h_\perp(0)$, $h(L)$ and $h_\perp(L)$ (see §6.5(c)).

When restricted to these function spaces, the one-dimensional Laplacian operator ∂_x^2 is self-adjoint....

Proposition 7.25: *Let $L > 0$, and consider the operator ∂_x^2 on $\mathcal{C}^\infty[0,L]$.*

(a) *∂_x^2 is self-adjoint when restricted to $\mathcal{C}^\infty_0[0,L]$.*

(b) *∂_x^2 is self-adjoint when restricted to $\mathcal{C}^\infty_\perp[0,L]$.*

(c) *∂_x^2 is self-adjoint when restricted to $\mathcal{C}^\infty_{\text{per}}[0,L]$.*

(d) *∂_x^2 is self-adjoint when restricted to $\mathcal{C}^\infty_{h,h_\perp}[0,L]$, for any $h(0)$, $h_\perp(0)$, $h(L)$ and $h_\perp(L)$ in $\mathbb{R}$.*

Proof: Let $f, g : [0,L] \longrightarrow \mathbb{R}$ be smooth functions. We apply integration by parts to get:

$$\left\langle \partial_x^2 f,\ g \right\rangle \;=\; \int_0^L f''(x)\cdot g(x)\,dx \;=\; f'(x)\cdot g(x)\Big|_{x=0}^{x=L} - \int_0^L f'(x)\cdot g'(x)\,dx \quad (7.1)$$

(whenever these integrals are well-defined). But now, if we apply Dirichlet, Neumann, or Periodic boundary conditions, we get:

$$f'(x)\cdot g(x)\Big|_{x=0}^{x=L} = f'(L)\cdot g(L) - f'(0)\cdot g(0) = \begin{cases} f'(L)\cdot 0 - f'(0)\cdot 0 = 0 & \text{(Dirichlet)} \\ 0\cdot g(L) - 0\cdot g(0) = 0 & \text{(Neumann)} \\ f'(0)\cdot g(0) - f'(0)\cdot g(0) = 0 & \text{(Periodic)} \end{cases}$$
$$= 0 \quad \text{in all cases.}$$

Thus, the first term in (7.1) is zero, so $\left\langle \partial_x^2 f,\ g \right\rangle = \int_0^L f'(x)\cdot g'(x)\,dx$.

But by the same reasoning, with f and g interchanged, $\int_0^L f'(x)\cdot g'(x)\,dx = \left\langle f,\ \partial_x^2 g \right\rangle$.

Thus, we've proved parts **(a)**, **(b)**, and **(c)**. To prove part **(d)**,

$$\begin{aligned} f'(x)\cdot g(x)\Big|_{x=0}^{x=L} &= f'(L)\cdot g(L) - f'(0)\cdot g(0) \\ &= f(L)\cdot \frac{h(L)}{h_\perp(L)}\cdot g(L) + f(0)\cdot\frac{h(0)}{h_\perp(0)}\cdot g(0) \\ &= f(L)\cdot g'(L) - f(0)\cdot g'(0) = f(x)\cdot g'(x)\Big|_{x=0}^{x=L}. \end{aligned}$$

Hence, substituting $f(x)\cdot g'(x)\Big|_{x=0}^{x=L}$ for $f'(x)\cdot g(x)\Big|_{x=0}^{x=L}$ in (7.1), we get: $\left\langle \partial_x^2 f,\ g \right\rangle = \int_0^L f''(x)\cdot g(x)\,dx = \int_0^L f(x)\cdot g''(x)\,dx = \left\langle f,\ \partial_x^2 g \right\rangle$. □

Proposition 7.25 generalizes to higher-dimensional Laplacians in the obvious way:

Theorem 7.26: *Let $L > 0$.*

(a) *The Laplacian operator $\triangle$ is self-adjoint on any of the spaces: $\mathcal{C}_0^\infty[0,L]^D$, $\mathcal{C}_\perp^\infty[0,L]^D$, $\mathcal{C}_{h,h_\perp}^\infty[0,L]^D$ or $\mathcal{C}_{\mathrm{per}}^\infty[0,L]^D$.*

(b) *More generally, if $\mathbb{X} \subset \mathbb{R}^D$ is any bounded domain with a smooth boundary[5], then the Laplacian operator $\triangle$ is self-adjoint on any of the spaces: $\mathcal{C}_0^\infty[0,L]^D$, $\mathcal{C}_\perp^\infty[0,L]^D$, or $\mathcal{C}_{h,h_\perp}^\infty[0,L]^D$.*

In other words, the Laplacian is self-adjoint whenever we impose homogeneous Dirichlet, Neumann, or mixed boundary conditions, or (when meaningful) periodic boundary conditions.

Proof: **(a)** **Exercise 7.28** Hint: The argument is similar to Proposition 7.25. Apply integration by parts in each dimension, and cancel the "boundary" terms using the boundary conditions.

(b) (Sketch) The strategy is similar to **(a)** but more complex. If $\mathbb{X}$ is an arbitrary bounded open domain in $\mathbb{R}^D$, then the analog of 'integration by parts' is an application of the Gauss-Green-Stokes Theorem. This allows us to reduce an integral over the interior of $\mathbb{X}$ to an integral on the boundary of $\mathbb{X}$; at this point we can use homogeneous boundary conditions to conclude that this boundary integral is zero. □

If L_1 and L_2 are two self-adjoint operators, then their sum $\mathsf{L}_1 + \mathsf{L}_2$ is also self-adjoint (**Exercise 7.29**).

Example 7.27: Let $s, q : [0,L] \longrightarrow \mathbb{R}$ be differentiable. The **Sturm-Liouville** operator

$$\mathsf{SL}_{s,q}[f] \quad := \quad s \cdot f'' + s' \cdot \partial f' + q \cdot f$$

is **self-adjoint** on any of the spaces $\mathcal{C}_0^\infty[0,L]$, $\mathcal{C}_\perp^\infty[0,L]$, $\mathcal{C}_{h,h_\perp}^\infty[0,L]$ or $\mathcal{C}_{\mathrm{per}}^\infty[0,L]$.

To see this, notice that

$$\mathsf{SL}_{s,q}[f] \;=\; (s \cdot f')' + (q \cdot f) \;=\; \mathsf{S}[f] + \mathsf{Q}[f], \tag{7.2}$$

where $\mathsf{Q}[f] = q \cdot f$ is just a multiplication operator, and $\mathsf{S}[f] = (s \cdot f')'$. We know that Q is self-adjoint from Example 7.24. We claim that S is also self-adjoint. To see this, note that:

$$\begin{aligned}\langle \mathsf{S}[f],\ g\rangle \;&=\; \int_0^L (s \cdot f')'(x) \cdot g(x)\, dx \\ &\underset{(*)}{=}\; s(x) \cdot f'(x) \cdot g(x)\Big|_{x=0}^{x=L} \;-\; s(x) \cdot f(x) \cdot g'(x)\Big|_{x=0}^{x=L} \;+\; \int_0^L s(x) \cdot f(x) \cdot g''(x)\, dx \\ &\underset{(\dagger)}{=}\; \int_0^L f(x) \cdot (s \cdot g')'(x)\, dx \qquad = \qquad \langle f,\ \mathsf{S}[g]\rangle\,.\end{aligned}$$

Here, $(*)$ is integration by parts twice over, and $(\dagger)$ follows from any of the cited boundary conditions as in Proposition 7.25 on the preceding page (**Exercise 7.30**). Thus, S is self-adjoint, so $\mathsf{SL}_{s,q} = \mathsf{S} + \mathsf{Q}$ is self-adjoint. ♢

Self-adjoint operators are nice because their eigenfunctions are orthogonal.

[5] See page 105 of §6.6.

Proposition 7.28: *Suppose* L *is a self-adjoint operator. If* f_1 *and* f_2 *are eigenfunctions of* L *with eigenvalues* $\lambda_1 \neq \lambda_2$*, then* f_1 *and* f_2 *are* **orthogonal**.

Proof: By hypothesis, $\mathsf{L}[f_k] = \lambda_k \cdot f_k$, for $k = 1, 2$. Thus,

$$\lambda_1 \cdot \langle f_1, f_2 \rangle \;=\; \langle \lambda_1 \cdot f_1,\; f_2 \rangle \;=\; \langle \mathsf{L}[f_1],\; f_2 \rangle \underset{(*)}{=} \langle f_1,\; \mathsf{L}[f_2] \rangle \;=\; \langle f_1,\; \lambda_2 \cdot f_2 \rangle \;=\; \lambda_2 \cdot \langle f_1, f_2 \rangle,$$

where $(*)$ follows from self-adjointness. Since $\lambda_1 \neq \lambda_2$, this can only happen if $\langle f_1, f_2 \rangle = 0$.
□

Example 7.29: Eigenfunctions of ∂_x^2

(a) Let ∂_x^2 act on $\mathcal{C}^\infty[0, L]$. Then *all real numbers* $\lambda \in \mathbb{R}$ are eigenvalues of ∂_x^2. For any $\mu \in \mathbb{R}$,

- If $\lambda = \mu^2 > 0$, the eigenfunctions are of the form $\phi(x) = A \sinh(\mu \cdot x) + B \cosh(\mu \cdot x)$ for any constants $A, B \in \mathbb{R}$.
- If $\lambda = 0$, the eigenfunctions are of the form $\phi(x) = Ax + B$ for any constants $A, B \in \mathbb{R}$.
- If $\lambda = -\mu^2 < 0$, the eigenfunctions are of the form $\phi(x) = A \sin(\mu \cdot x) + B \cos(\mu \cdot x)$ for any constants $A, B \in \mathbb{R}$.

Note: Because we have not imposed any boundary conditions, Proposition 7.25 does *not* apply; indeed ∂_x^2 is *not* a self-adjoint operator on $\mathcal{C}^\infty[0, L]$.

(b) Let ∂_x^2 act on $\mathcal{C}^\infty([0, L]; \mathbb{C})$. Then *all complex numbers* $\lambda \in \mathbb{C}$ are eigenvalues of ∂_x^2. For any $\mu \in \mathbb{C}$, with $\lambda = \mu^2$, the eigenvalue λ has eigenfunctions of the form $\phi(x) = A \exp(\mu \cdot x) + B \exp(-\mu \cdot x)$ for any constants $A, B \in \mathbb{C}$. (Note that the three cases of the previous example arise by taking $\lambda \in \mathbb{R}$.)

Again, Proposition 7.25 does *not* apply in this case, and ∂_x^2 is *not* a self-adjoint operator on $\mathcal{C}^\infty([0, L]; \mathbb{C})$.

(c) Now let ∂_x^2 act on $\mathcal{C}_0^\infty[0, L]$. Then the eigenvalues of ∂_x^2 are $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$ for every $n \in \mathbb{N}$, each of multiplicity 1; the corresponding eigenfunctions are all scalar multiples of $\mathbf{S}_n(x) = \sin\left(\frac{n\pi x}{L}\right)$.

(d) If ∂_x^2 acts on $\mathcal{C}_\perp^\infty[0, L]$, then the eigenvalues of ∂_x^2 are again $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$ for every $n \in \mathbb{N}$, each of multiplicity 1, but the corresponding eigenfunctions are now all scalar multiples of $\mathbf{C}_n(x) = \cos\left(\frac{n\pi x}{L}\right)$. Also, 0 is an eigenvalue, with eigenfunction $\mathbf{C}_0 = \mathbb{1}$.

(e) Let $h > 0$, and let ∂_x^2 act on $\mathcal{C} = \{f \in \mathcal{C}^\infty[0, L] \; ; \; f(0) = 0 \text{ and } h \cdot f(L) + f'(L) = 0\}$. Then the eigenfunctions of ∂_x^2 are all scalar multiples of

$$\Phi_n(x) = \sin(\mu_n \cdot x),$$

with eigenvalue $\lambda_n = -\mu_n^2$, where $\mu_n > 0$ is any real number so that

$$\tan(L \cdot \mu_n) \quad = \quad \frac{-\mu_n}{h}$$

This is a *transcendental equation* in the unknown μ_n. Thus, although there is an infinite sequence of solutions $\{\mu_0 < \mu_1 < \mu_2 < \ldots\}$, there is no closed-form algebraic expression for μ_n. At best, we can estimate μ_n through "graphical" methods.

(f) Let $h(0)$, $h_{\perp}(0)$, $h(L)$, and $h_{\perp}(L)$ be real numbers, and let ∂_x^2 act on $\mathcal{C}^{\infty}_{h,h_{\perp}}[0, L]$. Then the eigenfunctions of ∂_x^2 are all scalar multiples of

$$\Phi_n(x) = \sin\left(\theta_n + \mu_n \cdot x\right),$$

with eigenvalue $\lambda_n = -\mu_n^2$, where $\theta_n \in [0, 2\pi]$ and $\mu_n > 0$ are constants satisfying the transcendental equations:

$$\tan(\theta_n) \;=\; \mu_n \cdot \frac{h_{\perp}(0)}{h(0)} \qquad \text{and} \qquad \tan(\mu_n \cdot L + \theta_n) \;=\; -\mu_n \cdot \frac{h_{\perp}(L)}{h(L)}. \qquad (\textbf{Exercise7.31})$$

In particular, if $h_{\perp}(0) = 0$, then we must have $\theta = 0$. If $h(L) = h$ and $h_{\perp}(L) = 1$, then we return to Example (e).

(g) Let ∂_x^2 act on $\mathcal{C}^{\infty}_{\text{per}}[-L, L]$. Then the eigenvalues of ∂_x^2 are again $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$, for every $n \in \mathbb{N}$, each having multiplicity 2. The corresponding eigenfunctions are of the form $A \cdot \mathbf{S}_n(x) + B \cdot \mathbf{C}_n$ for any $A, B \in \mathbb{R}$. In particular, 0 is an eigenvalue, with eigenfunction $\mathbf{C}_0 = \mathbb{1}$.

(h) Let ∂_x^2 act on $\mathcal{C}^{\infty}_{\text{per}}([-L, L]; \mathbb{C})$. Then the eigenvalues of ∂_x^2 are again $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$, for every $n \in \mathbb{N}$, each having multiplicity 2. The corresponding eigenfunctions are of the form $A \cdot \mathbf{E}_n(x) + B \cdot \mathbf{E}_{-n}$ for any $A, B \in \mathbb{R}$, where $\mathbf{E}_n(x) = \exp\left(\frac{\pi \mathbf{i} n x}{L}\right)$. In particular 0 is an eigenvalue, with eigenfunction $\mathbf{E}_0 = \mathbb{1}$. ♢

Example 7.30: Eigenfunctions of $\triangle$

(a) Let $\triangle$ act on $\mathcal{C}_0^{\infty}[0, L]^D$. Then the eigenvalues of $\triangle$ are $\lambda_{\mathbf{m}} = -\left(\frac{\pi}{L}\right)^2 (m_1^2 + \ldots + m_d^2)$ for all $\mathbf{m} = (m_1, \ldots, m_D) \in \mathbb{N}^D$. The corresponding eigenspace is spanned by all functions

$$\mathbf{S}_{\mathbf{n}}(x_1, ..., x_D) \quad := \quad \sin\left(\frac{\pi n_1 x_1}{L}\right) \sin\left(\frac{\pi n_2 x_2}{L}\right) ... \sin\left(\frac{\pi n_D x_D}{L}\right)$$

such that $\|\mathbf{m}\| = \|\mathbf{n}\|$.

(b) Now let $\triangle$ act on $\mathcal{C}^\infty_\perp[0,L]^D$. Then the eigenvalues of $\triangle$ are again $\lambda_{\mathbf{m}}$ but the corresponding eigenspace is spanned by all functions

$$\mathbf{C_n}(x_1,...,x_D) \quad := \quad \cos\left(\frac{\pi n_1 x_1}{L}\right)\cos\left(\frac{\pi n_2 x_2}{L}\right)...\cos\left(\frac{\pi n_D x_D}{L}\right)$$

such that $\|\mathbf{m}\| = \|\mathbf{n}\|$. Also, 0 is an eigenvalue with eigenvector $\mathbf{C}_0 = \mathbb{1}$.

(c) Let $\triangle$ act on $\mathcal{C}^\infty_{\text{per}}[-L,L]^D$. Then the eigenvalues of $\triangle$ are again $\lambda_{\mathbf{m}}$, and the corresponding eigenspace contains all $\mathbf{C_m}$ and $\mathbf{S_m}$ with $\|\mathbf{m}\| = \|\mathbf{n}\|$. Also, 0 is an eigenvalue whose eigenvectors are all *constant* functions —ie. multiples of $\mathbf{C}_0 = \mathbb{1}$.

(d) Let $\triangle$ act on $\mathcal{C}^\infty_{\text{per}}\left([-L,L]^D;\ \mathbb{C}\right)$. Then the eigenvalues of $\triangle$ are again $\lambda_{\mathbf{n}}$. The corresponding eigenspace is spanned by all $\mathbf{E_n}(x_1,\ldots,x_D) := \exp\left(\frac{\pi \mathbf{i} n_1 x_1}{L}\right)\ldots\exp\left(\frac{\pi \mathbf{i} n_D x_D}{L}\right)$ with $\|\mathbf{m}\| = \|\mathbf{n}\|$. Also, 0 is an eigenvalue whose eigenvectors are all *constant* functions —ie. multiples of $\mathbf{E}_0 = \mathbb{1}$. ♢

The alert reader will notice that, in each of the above scenarios (except Examples 7.29(a) and 7.29(b), where ∂_x^2 is not self-adjoint), the set of eigenfunctions are not only orthogonal, but actually form an *orthogonal basis* for the corresponding $\mathbf{L}^2$-space. This is not a coincidence.

Theorem 7.31: *Let $L > 0$.*

(a) *Let $\mathcal{C}$ be any one of $\mathcal{C}^\infty_0[0,L]^D$, $\mathcal{C}^\infty_\perp[0,L]^D$, $\mathcal{C}^\infty_h[0,L]^D$ or $\mathcal{C}^\infty_{\text{per}}[0,L]^D$, and treat $\triangle$ as a linear operator on $\mathcal{C}$. Then there is an orthogonal basis $\{\Phi_0,\Phi_1,\Phi_2,\ldots\}$ for $\mathbf{L}^2[0,L]^D$ such that, for all n, $\Phi_n \in \mathcal{C}$ and Φ_n is an eigenfunction of $\triangle$.*

(b) *More generally, if $\mathbb{X} \subset \mathbb{R}^D$ is any bounded open domain, and $\mathcal{C} = \mathcal{C}^\infty_0[\mathbb{X}]$, then there is a set $\{\Phi_0,\Phi_1,\Phi_2,\ldots\} \subset \mathcal{C}$ of eigenfunctions for $\triangle$ in $\mathcal{C}$ which form an orthogonal basis for $\mathbf{L}^2[\mathbb{X}]$.*

Proof: **(a)** we have already established. The eigenfunctions of the Laplacian in these contexts are $\left\{\mathbf{C_n}\ ;\ \mathbf{n}\in\mathbb{Z}^D\right\}$ or $\left\{\mathbf{S_n}\ ;\ \mathbf{n}\in\mathbb{Z}^D\right\}$ or $\left\{\Phi_{\mathbf{n}}\ ;\ \mathbf{n}\in\mathbb{Z}^D\right\}$ or $\left\{\mathbf{R_n}\ ;\ \mathbf{n}\in\mathbb{Z}^D\right\}$, and results from Fourier Theory tell us that these form orthogonal bases.

(b) follows from Theorem 7.35 in §7.6(a) (below). Alternately, see [War83], chapter 6, p. 255; exercise 16(g), or [Cha93], Theorem 3.21, p. 156. □

Example 7.32:

(a) Let $\mathbb{B} = \left\{\mathbf{x}\in\mathbb{R}^D\ ;\ \|\mathbf{x}\| < R\right\}$ be the **ball** of radius R. Then there is a set $\{\Phi_0,\Phi_1,\Phi_2,\ldots\} \subset \mathcal{C}^\infty(\mathbb{B})$ of eigenfunctions of $\triangle$ on the ball which are all zero on the surface of the sphere, and which form an orthogonal basis for $\mathbf{L}^2(\mathbb{B})$.

(b) Let $\mathbb{A} = \left\{(x,y) \in \mathbb{R}^2 \; ; \; r^2 < x^2 + y^2 < R^2\right\}$ be the **annulus** of inner radius r and outer radius R in the plane. Then there is a set $\{\Phi_0, \Phi_1, \Phi_2, \ldots\} \subset \mathcal{C}^\infty(\mathbb{A})$ of eigenfunctions of $\triangle = \partial_x^2 + \partial_y^2$ on the annulus, which are all zero on the inner and outer perimeters, and which form an orthogonal basis for $\mathbf{L}^2(\mathbb{B})$. ♢

7.6(a) Appendix: Symmetric Elliptic Operators

Prerequisites: §6.2

Lemma 7.33: *Let $\mathbb{X} \subset \mathbb{R}^D$. If L is an elliptic differential operator on $\mathcal{C}^\infty(\mathbb{X})$, then there are functions $\omega_{cd} : \mathbb{X} \longrightarrow \mathbb{R}$ (for $c, d = 1...D$) and $\xi_1, \ldots, \xi_D : \mathbb{X} \longrightarrow \mathbb{R}$ so that L can be written in* **divergence form:**

$$\begin{aligned} \mathsf{L}[u] &= \sum_{c,d=1}^{D} \partial_c(\omega_{cd} \cdot \partial_d u) \;+\; \sum_{d=1}^{D} \xi_d \cdot \partial_d u \;+\; \alpha \cdot u, \\ &= \mathbf{div}\,[\mathbf{\Omega} \cdot \nabla\phi] \;+\; \langle \Xi,\ \nabla\phi \rangle \;+\; \alpha \cdot u, \end{aligned}$$

where $\Xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_D \end{bmatrix}$, *and* $\mathbf{\Omega} = \begin{bmatrix} \omega_{11} \ldots \omega_{1D} \\ \vdots \;\; \ddots \;\; \vdots \\ \omega_{D1} \ldots \omega_{DD} \end{bmatrix}$ *is a symmetric, positive-definite matrix.*

Proof: The idea is basically the same as equation (7.2). The details are an exercise. □

L is called **symmetric** if, in the divergence form, $\Xi \equiv 0$. For example, in the case when $\mathsf{L} = \triangle$, we have $\Omega = \mathbf{Id}$ and $\Xi = 0$, so $\triangle$ is symmetric.

Theorem 7.34: *If $\mathbb{X} \subset \mathbb{R}^D$ is an open bounded domain, then any symmetric elliptic differential operator on $\mathcal{C}_0^\infty(\mathbb{X})$ is* **self-adjoint.** □

Proof: This is a generalization of the integration-by-parts argument used before. See §6.5 of [Eva91]. □

Theorem 7.35: *Let $\mathbb{X} \subset \mathbb{R}^D$ be an open, bounded domain, and let L be any symmetric, elliptic differential operator on $\mathcal{C}_0^\infty(\mathbb{X})$. Then:*

1. *The eigenvalues of L form an infinite decreasing series $0 > \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \ldots$, with $\lim_{n\to\infty} \lambda_n = -\infty$.*
2. *There exists an orthogonal basis for $\mathbf{L}^2(\mathbb{X})$ of the form $\{\Phi_1, \Phi_2, \Phi_3, \ldots\}$, such that:*
    - *$\Phi_n \in \mathcal{C}_0^\infty(\mathbb{X})$ for all n*

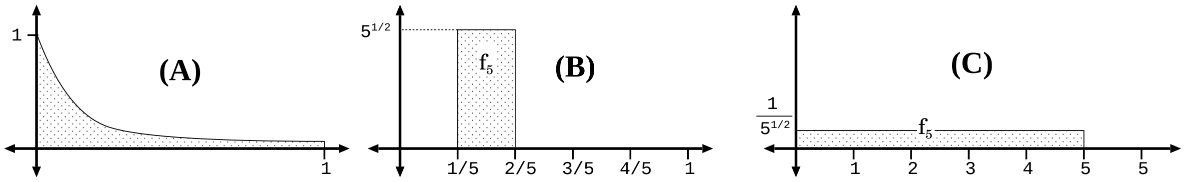


Figure 7.14: Problems for Chapter 7

- $\mathsf{L}[\Phi_n] = \lambda_n \cdot \Phi_n$.

Proof: See §6.5.1, Theorem 1, page 335 of [Eva91]. Alternately, employ the Spectral Theorem for unbounded self-adjoint operators (see [Con90], chapter X, section 4, p. 319). □

Further Reading:

An analogy of the Laplacian can be defined on any Riemannian manifold; it is often called the *Laplace-Beltrami operator*, and its eigenfunctions reveal much about the geometry of the manifold; see [War83, Chap.6] or [Cha93, §3.9]. In particular, the eigenfunctions of the Laplacian on *spheres* have been extensively studied. These are called *spherical harmonics*, and a sort of "Fourier theory" can be developed on spheres, analogous to multivariate Fourier theory on the cube $[0, L]^D$, but with the spherical harmonics forming the orthonormal basis [Tak94, Mül66]. Much of this theory generalizes to a broader family of manifolds called *symmetric spaces* [Ter85, Hel81]. The eigenfunctions of the Laplacian on symmetric spaces are closely related to the theory of *Lie groups* and their representations [CW68, Sug75], a subject which is sometimes called *noncommutative harmonic analysis* [Tay86].

The study of eigenfunctions and eigenvalues is sometimes called *spectral theory*, and the spectral theory of the Laplacian is a special case of the spectral theory of *self-adjoint operators* [Con90, Chap.X], a subject which of central importance many areas of mathematics, particularly quantum mechanics [Pru81, BEH94].

7.7 Practice Problems

1. Let $\mathbb{X} = (0, 1]$. For any $n \in \mathbb{N}$, define the function $f_n : (0, 1] \longrightarrow \mathbb{R}$ by $f_n(x) = \exp(-nx)$. (Fig. 7.14A)

   (a) Compute $\|f_n\|_2$ for all $n \in \mathbb{N}$.

   (b) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function in $\mathbf{L}^2(0, 1]$? Explain.

   (c) Compute $\|f_n\|_\infty$ for all $n \in \mathbb{N}$. **Hint:** *Look at the picture. Where is the value of $f_n(x)$ largest?*

   Warning: *Remember that* 0 *is not an element of* $(0, 1]$, *so you cannot just evaluate* $f_n(0)$.

(d) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function uniformly on $(0,1]$? Explain.

(e) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function pointwise on $(0,1]$? Explain.

2. Let $\mathbb{X} = [0,1]$. For any $n \in \mathbb{N}$, define $f_n : [0,1] \longrightarrow \mathbb{R}$ by $f_n(x) = \begin{cases} \sqrt{n} & \text{if } \frac{1}{n} \leq x < \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$. (Fig. 7.14B)

   (a) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function pointwise on $[0,1]$? Explain.

   (b) Compute $\|f_n\|_2$ for all $n \in \mathbb{N}$.

   (c) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function in $\mathbf{L}^2[0,1]$? Explain.

   (d) Compute $\|f_n\|_\infty$ for all $n \in \mathbb{N}$. **Hint:** *Look at the picture. Where is the value of $f_n(x)$ largest?*

   (e) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function uniformly on $[0,1]$? Explain.

3. Let $\mathbb{X} = \mathbb{R}$. For any $n \in \mathbb{N}$, define $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ by $f_n(x) = \begin{cases} \frac{1}{\sqrt{n}} & \text{if } 0 \leq x < n \\ 0 & \text{otherwise} \end{cases}$. (Fig. 7.14C)

   (a) Compute $\|f_n\|_\infty$ for all $n \in \mathbb{N}$.

   (b) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function uniformly on $\mathbb{R}$? Explain.

   (c) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function pointwise on $\mathbb{R}$? Explain.

   (d) Compute $\|f_n\|_2$ for all $n \in \mathbb{N}$.

   (e) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function in $\mathbf{L}^2(\mathbb{R})$? Explain.

4. Let $\mathbb{X} = (0,1]$. For all $n \in \mathbb{N}$, define $f_n : (0,1] \longrightarrow \mathbb{R}$ by $f_n(x) = \dfrac{1}{\sqrt[3]{nx}}$ (for all $x \in (0,1]$). (Figure 7.15A)

   (a) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function *pointwise* on $(0,1]$? Why or why not?

   (b) Compute $\|f_n\|_2$ for all $n \in \mathbb{N}$.

   (c) Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge to the constant 0 function in $\mathbf{L}^2(0,1]$? Why or why not?

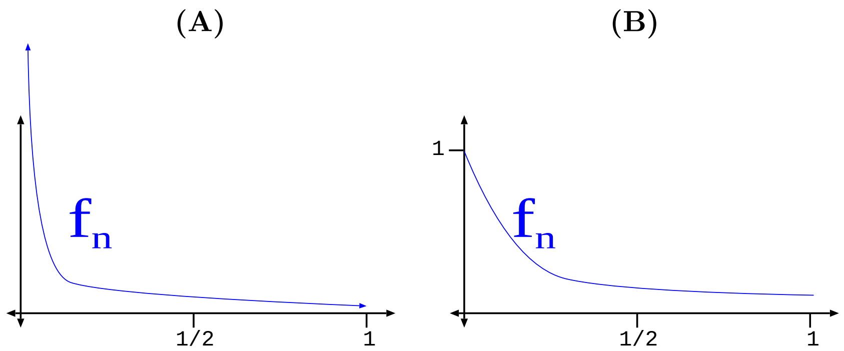


Figure 7.15: Problems for Chapter 7

(d) Compute $\|f_n\|_\infty$ for all $n \in \mathbb{N}$.
Hint: *Look at the picture. Where is the value of $f_n(x)$ largest?*
Warning: *Remember that 0 is not an element of $(0,1]$, so you cannot just evaluate $f_n(0)$ (which is not well-defined in any case).*

(e) Does the sequence $\{f_n\}_{n=1}^\infty$ converge to the constant 0 function uniformly on $(0,1]$? Explain.

5. Let $\mathbb{X} = [0,1]$. For all $n \in \mathbb{N}$, define $f_n : [0,1] \longrightarrow \mathbb{R}$ by $f_n(x) = \dfrac{1}{(nx+1)^2}$ (for all $x \in [0,1]$). (Figure 7.15B)

   (a) Does the sequence $\{f_n\}_{n=1}^\infty$ converge to the constant 0 function pointwise on $[0,1]$? Explain.

   (b) Compute $\|f_n\|_2$ for all $n \in \mathbb{N}$.

   (c) Does the sequence $\{f_n\}_{n=1}^\infty$ converge to the constant 0 function in $\mathbf{L}^2[0,1]$? Explain.

   (d) Compute $\|f_n\|_\infty$ for all $n \in \mathbb{N}$.
   Hint: *Look at the picture. Where is the value of $f_n(x)$ largest?*

   (e) Does the sequence $\{f_n\}_{n=1}^\infty$ converge to the constant 0 function uniformly on $[0,1]$? Explain.

6. In each of the following cases, you are given two functions $f, g : [0,\pi] \longrightarrow \mathbb{R}$. Compute the inner product $\langle f, g \rangle$.

   (a) $f(x) = \sin(3x)$, $\quad g(x) = \sin(2x)$.

   (b) $f(x) = \sin(nx)$, $\quad g(x) = \sin(mx)$, with $n \neq m$.

   (c) $f(x) = \sin(nx) = g(x)$ for some $n \in \mathbb{N}$. Question: What is $\|f\|_2$?

   (d) $f(x) = \cos(3x)$, $\quad g(x) = \cos(2x)$.

   (e) $f(x) = \cos(nx)$, $\quad g(x) = \cos(mx)$, with $n \neq m$.

   (f) $f(x) = \sin(3x)$, $\quad g(x) = \cos(2x)$.

7. In each of the following cases, you are given two functions $f, g : [-\pi, \pi] \longrightarrow \mathbb{R}$. Compute the inner product $\langle f, g \rangle$.

   (a) $f(x) = \sin(nx), \quad g(x) = \sin(mx)$, with $n \neq m$.

   (b) $f(x) = \sin(nx) = g(x)$ for some $n \in \mathbb{N}$. Question: What is $\|f\|_2$?

   (c) $f(x) = \cos(nx), \quad g(x) = \cos(mx)$, with $n \neq m$.

   (d) $f(x) = \sin(3x), \quad g(x) = \cos(2x)$.

Notes: ....

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8 Fourier Sine Series and Cosine Series

8.1 Fourier (co)sine Series on $[0, \pi]$

Prerequisites: §7.4(d), §7.5

Throughout this section, let $\mathbf{S}_n(x) := \sin(nx)$ for all natural numbers $n \geq 1$, and let $\mathbf{C}_n(x) := \cos(nx)$, for all natural numbers $n \geq 0$.

8.1(a) Sine Series on $[0, \pi]$

Recommended: §6.5(a)

Suppose $f \in \mathbf{L}^2[0, \pi]$ (ie. $f : [0, \pi] \longrightarrow \mathbb{R}$ is a function with $\|f\|_2 < \infty$). We define the **Fourier sine coefficients** of f:

$$B_n \;=\; \frac{\langle f, \mathbf{S}_n \rangle}{\|\mathbf{S}_n\|_2^2} \;=\; \boxed{\frac{2}{\pi} \int_0^\pi f(x) \sin(nx)\, dx}$$

The **Fourier sine series** of f is then the infinite summation of functions:

$$\boxed{\sum_{n=1}^{\infty} B_n \mathbf{S}_n(x)} \tag{8.1}$$

A function $f : [0, \pi] \longrightarrow \mathbb{R}$ is **continuously differentiable** on $[0, \pi]$ if $f'(x)$ exists for all $x \in (0, \pi)$, and furthermore, the function $f' : (0, \pi) \longrightarrow \mathbb{R}$ is itself bounded and continuous on $(0, \pi)$. Let $\mathcal{C}^1[0, \pi]$ be the space of all continuously differentiable functions.

Exercise 8.1 Show that any continuously differentiable function has finite $\mathbf{L}^2$-norm. In other words, $\mathcal{C}^1[0, \pi] \subset \mathbf{L}^2[0, \pi]$.

We say f is **piecewise continuously differentiable** (or **piecewise** $\mathcal{C}^1$) if there exist points $0 = j_0 < j_1 < j_2 < \cdots < j_{M+1} = \pi$ (for some $M \in \mathbb{N}$) so that f is bounded and continuously differentiable on each of the open intervals (j_m, j_{m+1}); these are called $\mathcal{C}^1$ **intervals** for f. In particular, any continuously differentiable function on $[0, \pi]$ is piecewise continuously differentiable (in this case, $M = 0$ and the set $\{j_1, \ldots, j_M\}$ is empty, so all of $(0, \pi)$ is a $\mathcal{C}^1$ interval).

Exercise 8.2 Show that any piecewise $\mathcal{C}^1$ function on $[0, \pi]$ is in $\mathbf{L}^2[0, \pi]$.

The Fourier sine series eqn.(8.1) usually converges to f in $\mathbf{L}^2$. If f is piecewise $\mathcal{C}^1$, then the Fourier sine series eqn.(8.1) also converges semiuniformly to f on every $\mathcal{C}^1$ interval:

Theorem 8.1: Fourier Sine Series Convergence on $[0, \pi]$

(a) *The set* $\{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \ldots\}$ *is an* <u>*orthogonal basis*</u> *for* $\mathbf{L}^2[0,\pi]$. *Thus, if* $f \in \mathbf{L}^2[0,\pi]$, *then the sine series* (8.1) *converges to* f *in* $\mathbf{L}^2$*-norm. That is:* $\lim_{N\to\infty} \left\| f - \sum_{n=1}^{N} B_n \mathbf{S}_n \right\|_2 = 0.$

Furthermore, the coefficient sequence $\{B_n\}_{n=1}^{\infty}$ *is the* <u>*unique*</u> *sequence of coefficients with this property. In other words, if* $\{B'_n\}_{n=1}^{\infty}$ *is some other sequence of coefficients such that* $f \underset{\mathrm{L2}}{\approx} \sum_{n=1}^{\infty} B'_n \mathbf{S}_n$, *then we must have* $B'_n = B_n$ *for all* $n \in \mathbb{N}$.

(b) *If* $f \in \mathcal{C}^1[0,\pi]$, *then the sine series* (8.1) *converges* <u>*pointwise*</u> *on* $(0,\pi)$.

More generally, if f *is piecewise* $\mathcal{C}^1$, *then the sine series* (8.1) *converges to* f <u>*pointwise*</u> *on each* $\mathcal{C}^1$ *interval for* f. *In other words, if* $\{j_1, \ldots, j_m\}$ *is the set of discontinuity points of* f *and/or* f', *and* $j_m < x < j_{m+1}$, *then* $f(x) = \lim_{N\to\infty} \sum_{n=1}^{N} B_n \sin(nx)$.

(c) $\Big($ *The sine series* (8.1) *converges to* f <u>*uniformly*</u> *on* $[0,\pi]$ $\Big) \iff \left(\sum_{n=1}^{\infty} |B_n| < \infty \right)$.

(d) *If* $f \in \mathcal{C}^1[0,\pi]$, *then:* $\Big($ *The sine series* (8.1) *converges to* f <u>*uniformly*</u> *on* $[0,\pi]$ $\Big)$

$\iff \Big($ f *satisfies homogeneous* <u>*Dirichlet*</u> *boundary conditions (ie.* $f(0) = f(\pi) = 0$*)* $\Big)$.

(e) *If* f *is piecewise* $\mathcal{C}^1$, *and* $\mathbf{K} \subset (j_m, j_{m+1})$ *is any closed subset of a* $\mathcal{C}^1$ *interval of* f, *then the series* (8.1) *converges* <u>*uniformly*</u> *to* f *on* $\mathbf{K}$.

Proof: For **(a)**, see [Kat76, p.29 of §1.5], or [Bro89, Theorem 2.3.10], or [CB87, §38].

(b) follows from **(e)**. For a direct proof of **(b)**, see [Bro89, Theorem 1.4.4, p.15] or [CB87, §31].

For **(d)**"$\Longleftarrow$", see [WZ77, Theorem 12.20, p.219] or [CB87, §35]. **(e)**, see [Fol84, Theorem 8.43, p.256].

(d)"$\Longrightarrow$" is <u>**Exercise 8.3**</u> .

(c) is <u>**Exercise 8.4**</u> (Hint: Use the Weierstrass M-test, Proposition 7.18 on page 131.) □

Example 8.2:

(a) If $f(x) = \sin(5x) - 2\sin(3x)$, then the Fourier sine series of f is just "$\sin(5x) - 2\sin(3x)$". In other words, the Fourier coefficients B_n are all zero, except that $B_3 = -2$ and $B_5 = 1$.

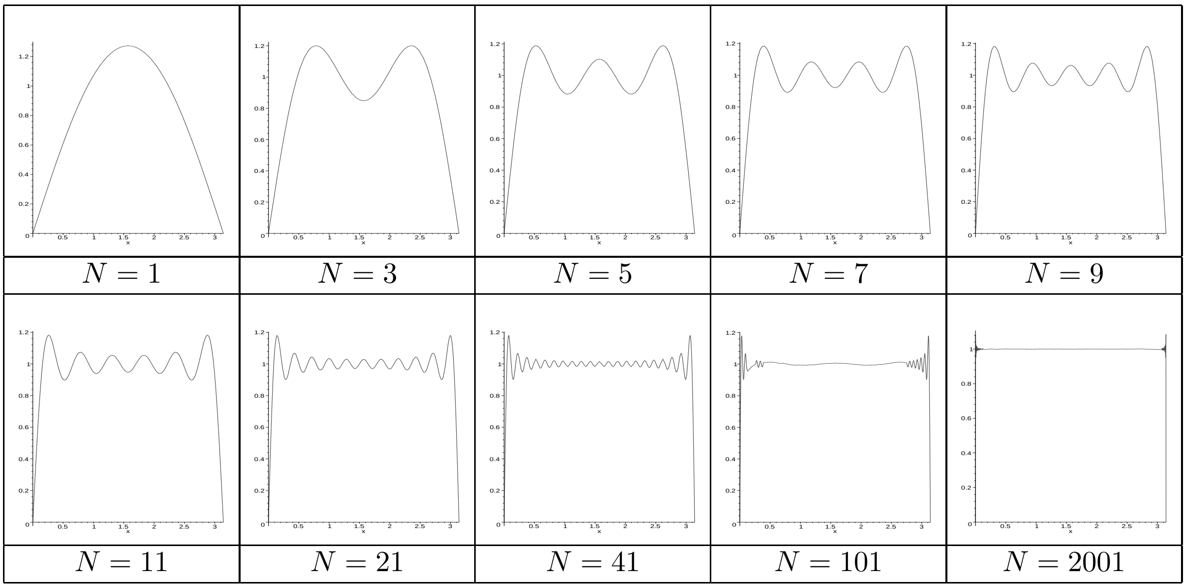

Figure 8.1: $\displaystyle\frac{4}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{N} \frac{1}{n}\sin(nx)$, for $N = 1, 3, 5, 7, 9, 11, 21, 41$, and 2001. Notice the Gibbs phenomenon in the plots for large N.

(b) Suppose $f(x) \equiv 1$. For all $n \in \mathbb{N}$,

$$B_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)\, dx = \frac{-2}{n\pi}\cos(nx)\Big|_{x=0}^{x=\pi} = \frac{2}{n\pi}\Big[1-(-1)^n\Big]$$
$$= \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is } \textbf{odd} \\ 0 & \text{if } n \text{ is } \textbf{even} \end{cases}.$$

Thus, the Fourier sine series is:

$$\frac{4}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n}\sin(nx) = \frac{4}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \cdots\right) \tag{8.2}$$

Theorem 8.1**(a)** says that $1 \underset{L2}{\approx} \displaystyle\frac{4}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n}\sin(nx)$. Figure 8.1 displays some partial sums of the series (8.2). The function $f \equiv 1$ is clearly continuously differentiable, so, by Theorem 8.1**(b)**, the Fourier sine series converges pointwise to 1 on the interior of the interval $[0, \pi]$. However, the series does *not* converge to f at the points 0 or π. This is betrayed by the violent oscillations of the partial sums near these points; this is an example of the **Gibbs phenomenon**.

Since the Fourier sine series does not converge at the endpoints 0 and π, we know automatically that it does not converge to f uniformly on $[0, \pi]$. However, we could have also deduced this fact by noticing that

$$\sum_{n=1}^{\infty} |B_n| \quad = \quad \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} \quad = \quad \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \quad = \quad \infty;$$

thus, we fail the condition in Theorem 8.1**(c)**.

The 'Gibbs Phenomenon' at the endpoints 0 and π occurs because f does *not* have homogeneous Dirichlet boundary conditions (because $f(0) = 1 = f(\pi)$), whereas every finite sum of $\sin(nx)$-type functions *does* have homogeneous Dirichlet BC. Thus, the series (8.2) is 'trying' to converge to f, but it is 'stuck' at the endpoints 0 and π. This is the idea behind Theorem 8.1**(d)**.

(c) If $f(x) = \cos(mx)$, then the Fourier sine series of f is: $\displaystyle \frac{4}{\pi} \sum_{\substack{n=1 \\ n+m \text{ odd}}}^{\infty} \frac{n}{n^2 - m^2} \sin(nx)$.

(**Exercise 8.5** Hint: Use Theorem 7.7 on page 117). ♢

Example 8.3: $\sinh(\alpha x)$

If $\alpha > 0$, and $f(x) = \sinh(\alpha x)$, then its Fourier sine series is given by:

$$\sinh(\alpha x) \quad \underset{\text{L2}}{\approx} \quad \frac{2\sinh(\alpha\pi)}{\pi} \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{\alpha^2 + n^2} \cdot \sin(nx)$$

To prove this, we must show that, for all $n > 0$,

$$B_n \quad = \quad \frac{2}{\pi} \int_0^{\pi} \sinh(\alpha x) \cdot \sin(nx)\, dx \quad = \quad \frac{2\sinh(\alpha\pi)}{\pi} \frac{n(-1)^{n+1}}{\alpha^2 + n^2}.$$

To begin with, let $I = \int_0^{\pi} \sinh(\alpha x) \cdot \sin(nx)\, dx$. Then, applying integration by parts:

$$\begin{aligned}
I &= \frac{-1}{n}\left[\sinh(\alpha x) \cdot \cos(nx)\, \Big|_{x=0}^{x=\pi} - \alpha \cdot \int_0^{\pi} \cosh(\alpha x) \cdot \cos(nx)\, dx\right] \\
&= \frac{-1}{n}\left[\sinh(\alpha\pi) \cdot (-1)^n - \frac{\alpha}{n} \cdot \left(\cosh(\alpha x) \cdot \sin(nx)\, \Big|_{x=0}^{x=\pi} - \alpha \int_0^{\pi} \sinh(\alpha x) \cdot \sin(nx)\, dx\right)\right] \\
&= \frac{-1}{n}\left[\sinh(\alpha\pi) \cdot (-1)^n - \frac{\alpha}{n} \cdot (0 - \alpha \cdot I)\right] \\
&= \frac{-\sinh(\alpha\pi) \cdot (-1)^n}{n} - \frac{\alpha^2}{n^2} I.
\end{aligned}$$

$$\begin{aligned}
\text{Hence:} \quad I &= \frac{-\sinh(\alpha\pi) \cdot (-1)^n}{n} - \frac{\alpha^2}{n^2} I; \\
\text{thus} \quad \left(1 + \frac{\alpha^2}{n^2}\right) I &= \frac{-\sinh(\alpha\pi) \cdot (-1)^n}{n};
\end{aligned}$$

$$\text{ie.} \quad \left(\frac{n^2+\alpha^2}{n^2}\right) I \;=\; \frac{\sinh(\alpha\pi)\cdot(-1)^{n+1}}{n};$$
$$\text{so that} \quad I \;=\; \frac{n\cdot\sinh(\alpha\pi)\cdot(-1)^{n+1}}{n^2+\alpha^2}.$$

Thus, $B_n \;=\; \frac{2}{\pi} I \;=\; \frac{2}{\pi}\frac{n\cdot\sinh(\alpha\pi)\cdot(-1)^{n+1}}{n^2+\alpha^2}.$

The function sinh is clearly continuously differentiable, so, Theorem 8.1**(b)** implies that the Fourier sine series converges to $\sinh(\alpha x)$ pointwise on the open interval $(0,\pi)$. However, the series does *not* converge uniformly on $[0,\pi]$ (**Exercise 8.6** Hint: What is $\sinh(\alpha\pi)$?). ♢

8.1(b) Cosine Series on $[0,\pi]$

Recommended: §6.5(b)

If $f \in \mathbf{L}^2[0,\pi]$, we define the **Fourier cosine coefficients** of f:

$$A_0 \;=\; \langle f, \mathbb{1}\rangle \;=\; \boxed{\frac{1}{\pi}\int_0^\pi f(x)\, dx} \quad \text{and} \quad A_n \;=\; \frac{\langle f, \mathbf{C}_n\rangle}{\|\mathbf{C}_n\|_2^2} \;=\; \boxed{\frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\, dx} \quad \text{for all } n > 0.$$

The **Fourier cosine series** of f is then the infinite summation of functions:

$$\boxed{\sum_{n=0}^{\infty} A_n \mathbf{C}_n(x)} \tag{8.3}$$

Theorem 8.4: Fourier Cosine Series Convergence on $[0,\pi]$

(a) *The set* $\{\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \ldots\}$ *is an orthogonal basis for* $\mathbf{L}^2[0,\pi]$. *Thus, if* $f \in \mathbf{L}^2[0,\pi]$, *then the cosine series* (8.3) *converges to* f *in* $\mathbf{L}^2$*-norm, i.e.* $\lim_{N\to\infty}\left\| f \;-\; \sum_{n=0}^{N} A_n\mathbf{C}_n \right\|_2 \;=\; 0.$

Furthermore, the coefficient sequence $\{A_n\}_{n=0}^{\infty}$ *is the unique sequence of coefficients with this property. In other words, if* $\{A'_n\}_{n=1}^{\infty}$ *is some other sequence of coefficients such that* $f \underset{\text{L2}}{\approx} \sum_{n=0}^{\infty} A'_n\mathbf{C}_n$, *then we must have* $A'_n = A_n$ *for all* $n \in \mathbb{N}$.

(b) *If* f *is piecewise* $\mathcal{C}^1$ *on* $[0,\pi]$, *then the cosine series* (8.3) *converges to* f *pointwise on each* $\mathcal{C}^1$ *interval for* f. *In other words, if* $\{j_1,\ldots,j_m\}$ *is the set of discontinuity points of* f *and/or* f', *and* $j_m < x < j_{m+1}$, *then* $f(x) \;=\; \lim_{N\to\infty}\sum_{n=0}^{N} A_n\cos(nx)$.

(c) $\Big($ *The cosine series* (8.3) *converges to* f *uniformly on* $[0,\pi]$ $\Big) \iff \left(\sum_{n=0}^{\infty}|A_n| \;<\; \infty\right).$

(d) *If $f \in \mathcal{C}^1[0,\pi]$, then the cosine series (8.3) converges uniformly on $[0,\pi]$.*

(e) *More generally, if f is piecewise $\mathcal{C}^1$, and $\mathbf{K} \subset (j_m, j_{m+1})$ is any closed subset of a $\mathcal{C}^1$ interval of f, then the series (8.3) converges uniformly to f on $\mathbf{K}$.*

(f) *If $f \in \mathcal{C}^1[0,\pi]$, then:*

$$\Big(f \text{ satisfies homogeneous Neumann boundary conditions (ie. } f'(0) = f'(\pi) = 0) \Big) \iff \left(\sum_{n=0}^{\infty} n\,|A_n| \quad < \quad \infty. \right).$$

....and in this case, the cosine series (8.3) also converges to f uniformly on $[0,\pi]$.

Proof: For **(a)**, see [Kat76, p.29 of §1.5] or [Bro89, Theorem 2.3.10], or [CB87, §38]..

(b) follows from **(e)**. For a direct proof of **(b)**, see [Bro89, Theorem 1.4.4, p.15] or [CB87, §31].

For **(d)**, see [WZ77, Theorem 12.20, p.219] or [CB87, §35]. For **(e)** see [Fol84, Theorem 8.43, p.256].

(f) is **Exercise 8.7** (Hint: Use Theorem 8.1(d) on page 145, and Theorem 8.20 on page 166.)

(c) is **Exercise 8.8** (Hint: Use the Weierstrass M-test, Proposition 7.18 on page 131.) □

Example 8.5:

(a) If $f(x) = \cos(13x)$, then the Fourier cosine series of f is just "$\cos(13x)$". In other words, the Fourier coefficients A_n are all zero, except that $A_{13} = 1$.

(b) Suppose $f(x) \equiv 1$. Then $f = \mathbf{C}_0$, so the Fourier cosine coefficients are: $A_0 = 1$, while $A_1 = A_2 = A_3 = \ldots 0$.

(c) Let $f(x) = \sin(mx)$. If m is *even*, then the Fourier cosine series of f is: $\displaystyle\frac{4}{\pi} \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{n}{n^2 - m^2} \cos(nx)$.

If m is *odd*, then the Fourier cosine series of f is: $\displaystyle\frac{2}{\pi m} + \frac{4}{\pi} \sum_{\substack{n=2\\ n \text{ even}}}^{\infty} \frac{n}{n^2 - m^2} \cos(nx)$.

(**Exercise 8.9** Hint: Use Theorem 7.7 on page 117). ◊

Example 8.6: $\cosh(x)$

Suppose $f(x) = \cosh(x)$. Then the Fourier cosine series of f is given by:

$$\cosh(x) \underset{L2}{\approx} \frac{\sinh(\pi)}{\pi} + \frac{2\sinh(\pi)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \cdot \cos(nx)}{n^2 + 1}.$$

To see this, first note that $A_0 = \frac{1}{\pi}\int_0^\pi \cosh(x)\, dx = \frac{1}{\pi}\sinh(x)\Big|_{x=0}^{x=\pi} = \frac{\sinh(\pi)}{\pi}$ (because $\sinh(0)=0$).

Next, let $I = \int_0^\pi \cosh(x)\cdot\cos(nx)\, dx$. Then

$$\begin{aligned} I &= \frac{1}{n}\left(\cosh(x)\cdot\sin(nx)\Big|_{x=0}^{x=\pi} - \int_0^\pi \sinh(x)\cdot\sin(nx)\, dx\right) \\ &= \frac{-1}{n}\int_0^\pi \sinh(x)\cdot\sin(nx)\, dx \\ &= \frac{1}{n^2}\left(\sinh(x)\cdot\cos(nx)\Big|_{x=0}^{x=\pi} - \int_0^\pi \cosh(x)\cdot\cos(nx)\, dx\right) \\ &= \frac{1}{n^2}\left(\sinh(\pi)\cdot\cos(n\pi) - I\right) = \frac{1}{n^2}\left((-1)^n\sinh(\pi) - I\right). \end{aligned}$$

Thus, $I = \frac{1}{n^2}\Big((-1)^n\cdot\sinh(\pi) - I\Big)$. Hence, $(n^2+1)I = (-1)^n\cdot\sinh(\pi)$. Hence, $I = \frac{(-1)^n\cdot\sinh(\pi)}{n^2+1}$. Thus, $A_n = \frac{2}{\pi}I = \frac{2}{\pi}\frac{(-1)^n\cdot\sinh(\pi)}{n^2+1}$. ♢

Remark: (a) In Theorems 8.1(d) and 8.4(d), we don't quite need f to be *differentiable* to guarantee uniform convergence of the Fourier (co)sine series. We say that f is *Hölder continuous* on $[0,\pi]$ with Hölder exponent α if there is some $M<\infty$ such that,

$$\text{For all } x,y\in[0,\pi],\quad \frac{|f(x)-f(y)|}{|x-y|^\alpha} \;\le\; M.$$

If f is Hölder continuous with $\alpha > \frac{1}{2}$, then the Fourier (co)sine series will converge uniformly to f; this is called *Bernstein's Theorem* [Fol84, Theorem 8.39]. (If f was differentiable, then f would be Hölder continuous with $\alpha \ge 1$, so Bernstein's Theorem immediately implies Theorems 8.1(d) and 8.4(d).)

(b) However, merely being *continuous* is *not* sufficient for uniform Fourier convergence, or even pointwise convergence. There is an example of a continuous function $f:[0,\pi]\longrightarrow\mathbb{R}$ whose Fourier series does *not* converge pointwise on $(0,\pi)$ —i.e. the series diverges at some points in $(0,\pi)$. See [WZ77, Theorem 12.35, p.227]. Thus, Theorems 8.1(b) and 8.4(b) are *false* if we replace 'differentiable' with 'continuous'.

(c) In Theorems 8.1(b) and 8.4(b), if x is a discontinuity point of f, then the Fourier (co)sine series converges to the average of the 'left-hand' and 'right-hand' limits of f at x, namely:

$$\frac{f(x-)+f(x+)}{2},\quad\text{where}\quad f(x-) := \lim_{y\nearrow x} f(y) \quad\text{and}\quad f(x+) := \lim_{y\searrow x} f(y)$$

(d) For other discussions of the Fourier convergence theorems, see [dZ86, Thm.6.1, p.72] or [Hab87, §3.2]

8.2 Fourier (co)sine Series on $[0, L]$

Prerequisites: §7.4, §7.5 **Recommended:** §8.1

Throughout this section, let $L > 0$ be some positive real number. Let $\mathbf{S}_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ for all natural numbers $n \geq 1$, and let $\mathbf{C}_n(x) = \cos\left(\frac{n\pi x}{L}\right)$, for all natural numbers $n \geq 0$. Notice that, if $L = \pi$, then $\mathbf{S}_n(x) = \sin(nx)$ and $\mathbf{C}_n(x) = \cos(nx)$, as in §8.1. The results in this section exactly parallel those in §8.1, except that we replace π with L to obtain slightly greater generality. In principle, every statement in this section is equivalent to the corresponding statement in §8.1, through the change of variables $y = x/\pi$ (it is a useful exercise to reflect on this as you read this section).

8.2(a) Sine Series on $[0, L]$

Recommended: §6.5(a), §8.1(a)

Fix $L > 0$, and let $[0, L]$ be an interval of length L. If $f \in \mathbf{L}^2[0, L]$, we define the **Fourier sine coefficients** of f:

$$B_n = \frac{\langle f, \mathbf{S}_n \rangle}{\|\mathbf{S}_n\|_2^2} = \boxed{\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx}$$

The **Fourier sine series** of f is then the infinite summation of functions:

$$\sum_{n=1}^{\infty} B_n \mathbf{S}_n(x) \tag{8.4}$$

A function $f : [0, L] \longrightarrow \mathbb{R}$ is **continuously differentiable** on $[0, L]$ if $f'(x)$ exists for all $x \in (0, L)$, and furthermore, the function $f' : (0, L) \longrightarrow \mathbb{R}$ is itself bounded and continuous on $(0, L)$. Let $\mathcal{C}^1[0, L]$ be the space of all continuously differentiable functions.

Exercise 8.10 Show that any continuously differentiable function has finite $\mathbf{L}^2$-norm. In other words, $\mathcal{C}^1[0, L] \subset \mathbf{L}^2[0, L]$.

We say $f : [0, L] \longrightarrow \mathbb{R}$ is **piecewise continuously differentiable** (or **piecewise $\mathcal{C}^1$**) if there exist points $0 = j_0 < j_1 < j_2 < \cdots < j_{M+1} = L$ so that f is bounded and continuously differentiable on each of the open intervals (j_m, j_{m+1}); these are called **$\mathcal{C}^1$ intervals** for f. In particular, any continuously differentiable function on $[0, L]$ is piecewise continuously differentiable (in this case, all of $(0, L)$ is a $\mathcal{C}^1$ interval).

Theorem 8.7: Fourier Sine Series Convergence on $[0, L]$

(a) *The set $\{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \ldots\}$ is an orthogonal basis for $\mathbf{L}^2[0, L]$. Thus, if $f \in \mathbf{L}^2[0, L]$, then the sine series (8.4) converges to f in $\mathbf{L}^2$-norm. That is:* $\lim_{N\to\infty}\left\|f - \sum_{n=1}^{N} B_n\mathbf{S}_n\right\|_2 = 0.$ *Furthermore, $\{B_n\}_{n=1}^{\infty}$ is the unique sequence of coefficients with this property.*

(b) *If $f \in \mathcal{C}^1[0,L]$, then the sine series (8.4) converges pointwise on $(0,L)$. More generally, if f is piecewise $\mathcal{C}^1$, then the sine series (8.4) converges to f pointwise on each $\mathcal{C}^1$ interval for f.*

(c) $\Big($ *The sine series (8.4) converges to f uniformly on $[0,L]$* $\Big) \iff \left(\sum_{n=1}^{\infty} |B_n| < \infty \right)$.

(d) *If $f \in \mathcal{C}^1[0,L]$, then:* $\Big($ *The sine series (8.4) converges to f uniformly on $[0,L]$* $\Big)$

$\iff \Big($ *f satisfies homogeneous Dirichlet boundary conditions (ie. $f(0) = f(L) = 0$)* $\Big)$.

(e) *If f is piecewise $\mathcal{C}^1$, and $\mathbf{K} \subset (j_m, j_{m+1})$ is any closed subset of a $\mathcal{C}^1$ interval of f, then the series (8.4) converges uniformly to f on $\mathbf{K}$.*

Proof: **Exercise 8.11** Hint: Deduce each statement from the corresponding statement of Theorem 8.1 on page 145. Use the change-of-variables $y = \frac{\pi}{L}x$ to pass from $y \in [0,L]$ to $x \in [0,\pi]$. □

Example 8.8:

(a) If $f(x) = \sin\left(\frac{5\pi}{L}x\right)$, then the Fourier sine series of f is just "$\sin\left(\frac{5\pi}{L}x\right)$". In other words, the Fourier coefficients B_n are all zero, except that $B_5 = 1$.

(b) Suppose $f(x) \equiv 1$. For all $n \in \mathbb{N}$,

$$B_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) dx = \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{x=0}^{x=L} = \frac{2}{n\pi}\Big[1 - (-1)^n\Big]$$
$$= \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is } \textbf{odd} \\ 0 & \text{if } n \text{ is } \textbf{even} \end{cases}.$$

Thus, the Fourier sine series is given: $\displaystyle\frac{4}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi}{L}x\right)$. Figure 8.1 displays some partial sums of this series. The **Gibbs phenomenon** is clearly evident just as in Example 8.2(b) on page 146.

(c) If $f(x) = \cos\left(\frac{m\pi}{L}x\right)$, then the Fourier sine series of f is: $\displaystyle\frac{4}{\pi}\sum_{\substack{n=1\\ n+m \text{ odd}}}^{\infty} \frac{n}{n^2 - m^2}\sin\left(\frac{n\pi}{L}x\right)$.

(**Exercise 8.12** Hint: Use Theorem 7.7 on page 117).

(d) If $\alpha > 0$, and $f(x) = \sinh\left(\frac{\alpha\pi x}{L}\right)$, then its Fourier sine coefficients are computed:

$$B_n = \frac{2}{L}\int_0^L \sinh\left(\frac{\alpha\pi x}{L}\right)\cdot\sin\left(\frac{n\pi x}{L}\right) dx = \frac{2\sinh(\alpha\pi)}{\pi}\frac{n(-1)^{n+1}}{\alpha^2 + n^2}.$$

(**Exercise 8.13**). ♢

8.2(b) Cosine Series on $[0, L]$

Recommended: §6.5(b), §8.1(b)

If $f \in \mathbf{L}^2[0, L]$, we define the **Fourier cosine coefficients** of f:

$$A_0 = \langle f, \mathbb{1} \rangle = \boxed{\frac{1}{L}\int_0^L f(x)\, dx} \text{ and } A_n = \frac{\langle f, \mathbf{C}_n \rangle}{\|\mathbf{C}_n\|_2^2} = \boxed{\frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx} \text{ for all } n > 0.$$

The **Fourier cosine series** of f is then the infinite summation of functions:

$$\boxed{\sum_{n=0}^{\infty} A_n \mathbf{C}_n(x)} \tag{8.5}$$

Theorem 8.9: Fourier Cosine Series Convergence on $[0, L]$

(a) *The set $\{\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \ldots\}$ is an orthogonal basis for $\mathbf{L}^2[0, L]$. Thus, if $f \in \mathbf{L}^2[0, L]$, then the cosine series (8.5) converges to f in $\mathbf{L}^2$-norm.*

Furthermore, $\{A_n\}_{n=0}^{\infty}$ is the unique sequence of coefficients with this property.

(b) *If f is piecewise $\mathcal{C}^1$, then the cosine series (8.5) converges to f pointwise on each $\mathcal{C}^1$ interval for f.*

(c) $\left(\text{The cosine series (8.5) converges to } f \text{ uniformly on } [0, L]\right) \iff \left(\sum_{n=0}^{\infty} |A_n| < \infty\right).$

(d) *If $f \in \mathcal{C}^1[0, L]$, then the cosine series (8.5) converges uniformly on $[0, L]$.*

(e) *More generally, if f is piecewise $\mathcal{C}^1$, and $\mathbf{K} \subset (j_m, j_{m+1})$ is any closed subset of a $\mathcal{C}^1$ interval of f, then the cosine series (8.5) converges uniformly to f on $\mathbf{K}$.*

(f) *If $f \in \mathcal{C}^1[0, L]$, then:*

$$\left(f \text{ satisfies homogeneous Neumann boundary conditions (ie. } f'(0) = f'(L) = 0) \right)$$

$$\iff \left(\sum_{n=0}^{\infty} n\, |A_n| < \infty. \right).$$

....and in this case, the cosine series (8.5) also converges to f uniformly on $[0, L]$.

Proof: **Exercise 8.14** Hint: Deduce each statement from the corresponding statement of Theorem 8.4 on page 149. Use the change-of-variables $y = \frac{\pi}{L}x$ to pass from $y \in [0, L]$ to $x \in [0, \pi]$. □

Jean Baptiste Joseph Fourier
Born: March 21, 1768 in Auxerre, France
Died: May 16, 1830 in Paris

Example 8.10:

(a) If $f(x) = \cos\left(\frac{13\pi}{L}x\right)$, then the Fourier cosine series of f is just "$\cos\left(\frac{13\pi}{L}x\right)$". In other words, the Fourier coefficients A_n are all zero, except that $A_{13} = 1$.

(b) Suppose $f(x) \equiv 1$. Then $f = \mathbf{C}_0$, so the Fourier cosine coefficients are: $A_0 = 1$, while $A_1 = A_2 = A_3 = \ldots 0$.

(c) Let $f(x) = \sin\left(\frac{m\pi}{L}x\right)$. If m is *even*, then the Fourier cosine series of f is: $\displaystyle\frac{4}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{n}{n^2 - m^2}\cos\left(\frac{n\pi}{L}x\right)$.

If m is *odd*, then the Fourier cosine series of f is: $\displaystyle\frac{2}{\pi m} + \frac{4}{\pi}\sum_{\substack{n=2\\ n \text{ even}}}^{\infty} \frac{n}{n^2 - m^2}\cos(nx)$.

(**Exercise 8.15** Hint: Use Theorem 7.7 on page 117). $\diamondsuit$

8.3 Computing Fourier (co)sine coefficients

Prerequisites: §8.2

When computing the Fourier sine coefficient $B_n = \frac{2}{L}\int_0^L f(x)\cdot\sin\left(\frac{n\pi}{L}x\right)\,dx$, it is simpler to first compute the integral $\int_0^L f(x)\cdot\sin\left(\frac{n\pi}{L}x\right)\,dx$, and then multiply the result by $\frac{2}{L}$. Likewise, to compute a Fourier cosine coefficients, first compute the integral $\int_0^L f(x)\cdot\cos\left(\frac{n\pi}{L}x\right)\,dx$, and then multiply the result by $\frac{2}{L}$. In this section, we review some useful techniques to compute these integrals.

8.3(a) Integration by Parts

Computing Fourier coefficients almost always involves integration by parts. Generally, if you can't compute it with integration by parts, you can't compute it. When evaluating a Fourier integral by parts, one almost always ends up with boundary terms of the form "$\cos(n\pi)$" or "$\sin\left(\frac{n}{2}\pi\right)$", etc. The following formulae are useful in this regard:

$$\boxed{\sin(n\pi) = 0 \quad \text{for any } n \in \mathbb{Z}.} \tag{8.6}$$

For example, $\sin(-\pi) = \sin(0) = \sin(\pi) = \sin(2\pi) = \sin(3\pi) = 0$.

$$\boxed{\cos(n\pi) = (-1)^n \quad \text{for any } n \in \mathbb{Z}.} \tag{8.7}$$

For example, $\cos(-\pi) = -1$, $\cos(0) = 1$, $\cos(\pi) = -1$, $\cos(2\pi) = 1$, $\cos(3\pi) = -1$, etc.

$$\boxed{\sin\left(\frac{n}{2}\pi\right) = \begin{cases} 0 & \text{if } n \text{ is } \textit{even} \\ (-1)^k & \text{if } n \text{ is } \textit{odd}, \text{ and } n = 2k+1 \end{cases}} \tag{8.8}$$

For example, $\sin(0) = 0$, $\sin\left(\frac{1}{2}\pi\right) = 1$, $\sin(\pi) = 0$, $\sin\left(\frac{3}{2}\pi\right) = -1$, etc.

$$\boxed{\cos\left(\frac{n}{2}\pi\right) = \begin{cases} 0 & \text{if } n \text{ is } \textit{odd} \\ (-1)^k & \text{if } n \text{ is } \textit{even}, \text{ and } n = 2k \end{cases}} \tag{8.9}$$

For example, $\cos(0) = 1$, $\cos\left(\frac{1}{2}\pi\right) = 0$, $\cos(\pi) = -1$, $\cos\left(\frac{3}{2}\pi\right) = 0$, etc.

Exercise 8.16 Verify these formulae.

8.3(b) Polynomials

Theorem 8.11: *Let $n \in \mathbb{N}$. Then*

$$\textbf{(a)} \quad \int_0^L \sin\left(\frac{n\pi}{L}x\right) dx = \begin{cases} \dfrac{2L}{n\pi} & \textit{if } n \textit{ is odd;} \\ 0 & \textit{if } n \textit{ is even.} \end{cases} \tag{8.10}$$

$$\textbf{(b)} \quad \int_0^L \cos\left(\frac{n\pi}{L}x\right) dx = \begin{cases} L & \textit{if } n = 0 \\ 0 & \textit{if } n > 0. \end{cases} \tag{8.11}$$

For any $k \in \{1, 2, 3, \ldots\}$, we have the following recurrence relations:

$$\textbf{(c)} \quad \int_0^L x^k \cdot \sin\left(\frac{n\pi}{L}x\right) dx = \frac{(-1)^{n+1}}{n} \cdot \frac{L^{k+1}}{\pi} + \frac{k}{n} \cdot \frac{L}{\pi} \int_0^L x^{k-1} \cdot \cos\left(\frac{n\pi}{L}x\right), \tag{8.12}$$

$$\textbf{(d)} \quad \int_0^L x^k \cdot \cos\left(\frac{n\pi}{L}x\right) dx = \frac{-k}{n} \cdot \frac{L}{\pi} \int_0^L x^{k-1} \cdot \sin\left(\frac{n\pi}{L}x\right). \tag{8.13}$$

Proof: **Exercise 8.17** Hint: for **(c)** and **(d)**, use integration by parts. □

Example 8.12: In all of the following examples, let $L = \pi$.

(a) $\frac{2}{\pi}\int_0^\pi \sin(nx)\ dx \quad = \quad \frac{2}{\pi}\frac{1-(-1)^n}{n}.$

(b) $\frac{2}{\pi}\int_0^\pi x\cdot\sin(nx)\ dx \quad = \quad (-1)^{n+1}\frac{2}{n}.$

(c) $\frac{2}{\pi}\int_0^\pi x^2\cdot\sin(nx)\ dx \quad = \quad (-1)^{n+1}\frac{2\pi}{n} \ + \ \frac{4}{\pi n^3}\Big((-1)^n - 1\Big).$

(d) $\frac{2}{\pi}\int_0^\pi x^3\cdot\sin(nx)\ dx \quad = \quad (-1)^n\left(\frac{12}{n^3} - \frac{2\pi^2}{n}\right).$

(e) $\frac{2}{\pi}\int_0^\pi \cos(nx)\ dx \quad = \quad \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$

(f) $\frac{2}{\pi}\int_0^\pi x\cdot\cos(nx)\ dx \quad = \quad \frac{2}{\pi n^2}\Big((-1)^n - 1\Big)$, if $n > 0$.

(g) $\frac{2}{\pi}\int_0^\pi x^2\cdot\cos(nx)\ dx \quad = \quad (-1)^n\frac{4}{n^2}$, if $n > 0$.

(h) $\frac{2}{\pi}\int_0^\pi x^3\cdot\cos(nx)\ dx \quad = \quad (-1)^n\frac{6\pi}{n^2} - \frac{12}{\pi n^4}\Big((-1)^n - 1\Big)$, if $n > 0$. ♢

Proof: **(b)**: We will show this in two ways. First, by direct computation:

$$\begin{aligned}\int_0^\pi x\cdot\sin(nx)\ dx &= \frac{-1}{n}\left(x\cdot\cos(nx)\ \Big|_{x=0}^{x=\pi} - \int_0^\pi \cos(nx)\ dx\right) \\ &= \frac{-1}{n}\left(\pi\cdot\cos(n\pi) - \frac{1}{n}\sin(nx)\ \Big|_{x=0}^{x=\pi}\right) \\ &= \frac{-1}{n}(-1)^n\pi \quad = \quad \frac{(-1)^{n+1}\pi}{n}\end{aligned}$$

Thus, $\frac{2}{\pi}\int_0^\pi x\cdot\sin(nx)\ dx \ = \ \frac{2(-1)^{n+1}}{n}$, as desired.

Next, we verify **(b)** using Theorem 8.11. Setting $L = \pi$ and $k = 1$ in (8.12), we have:

$$\begin{aligned}\int_0^\pi x\cdot\sin(nx)\ dx &= \frac{(-1)^{n+1}}{n}\cdot\frac{\pi^{1+1}}{\pi} \quad + \quad \frac{1}{n}\cdot\frac{\pi}{\pi}\int_0^\pi x^{k-1}\cdot\cos(nx)\ dx \\ &= \frac{(-1)^{n+1}}{n}\cdot\pi \quad + \quad \frac{1}{n}\int_0^\pi \cos(nx)\ dx \quad = \quad \frac{(-1)^{n+1}}{n}\cdot\pi.\end{aligned}$$

Because $\int_0^\pi \cos(nx)\ dx \ = \ 0$ by (8.11). Thus, $\frac{2}{\pi}\int_0^\pi x\cdot\sin(nx)\ dx \ = \ \frac{2(-1)^{n+1}}{n}$, as desired.

Proof of **(c)**:

$$\begin{aligned}
\int_0^\pi x^2 \cdot \sin(nx)\, dx &= \frac{-1}{n}\left(x^2 \cdot \cos(nx) \Big|_{x=0}^{x=\pi} - 2\int_0^\pi x\cos(nx)\, dx \right) \\
&= \frac{-1}{n}\left[\pi^2 \cdot \cos(n\pi) - \frac{2}{n}\left(x \cdot \sin(nx) \Big|_{x=0}^{x=\pi} - \int_0^\pi \sin(nx)\, dx \right)\right] \\
&= \frac{-1}{n}\left[\pi^2 \cdot (-1)^n + \frac{2}{n}\left(\frac{-1}{n}\cos(nx) \Big|_{x=0}^{x=\pi} \right)\right] \\
&= \frac{-1}{n}\left[\pi^2 \cdot (-1)^n - \frac{2}{n^2}\Big((-1)^n - 1 \Big)\right] \\
&= \frac{2}{n^3}\Big((-1)^n - 1 \Big) \quad + \quad \frac{(-1)^{n+1}\pi^2}{n}
\end{aligned}$$

The result follows.

Exercise 8.18 Verify **(c)** using Theorem 8.11.

(g) We will show this in two ways. First, by direct computation:

$$\begin{aligned}
\int_0^\pi x^2 \cdot \cos(nx)\, dx &= \frac{1}{n}\left[x^2 \cdot \sin(nx) \Big|_{x=0}^{x=\pi} \quad - \quad 2\int_0^\pi x \cdot \sin(nx)\, dx \right] \\
&= \frac{-2}{n}\int_0^\pi x \cdot \sin(nx)\, dx \qquad \text{(because } \sin(nx) = \sin(0) = 0\text{)} \\
&= \frac{2}{n^2}\left[x \cdot \cos(nx) \Big|_{x=0}^{x=\pi} \quad - \quad \int_0^\pi \cos(nx)\, dx \right] \\
&= \frac{2}{n^2}\left[\pi \cdot (-1)^n - \frac{1}{n}\sin(nx) \Big|_{x=0}^{x=\pi} \right] \\
&= \frac{2\pi \cdot (-1)^n}{n^2}
\end{aligned}$$

Thus, $\frac{2}{\pi}\int_0^\pi x^2 \cdot \cos(nx)\, dx \quad = \quad \frac{4 \cdot (-1)^n}{n^2}$, as desired.

Next, we verify **(g)** using Theorem 8.11. Setting $L = \pi$ and $k = 2$ in (8.13), we have:

$$\int_0^\pi x^2 \cdot \cos(nx)\, dx \quad = \quad \frac{-k}{n} \cdot \frac{L}{\pi}\int_0^p ix^{k-1} \cdot \sin(nx) \quad = \quad \frac{-2}{n} \cdot \int_0^\pi x \cdot \sin(nx), \tag{8.14}$$

Next, applying (8.12) with $k = 1$, we get:

$$\int_0^\pi x \cdot \sin(nx) \quad = \quad \frac{(-1)^{n+1}}{n} \cdot \frac{\pi^2}{\pi} + \frac{1}{n} \cdot \frac{\pi}{\pi}\int_0^\pi \cos(nx) \quad = \quad \frac{(-1)^{n+1}\pi}{n} + \frac{1}{n}\int_0^\pi \cos(nx),$$

Substituting this into (8.14), we get

$$\int_0^\pi x^2 \cdot \cos(nx)\, dx \quad = \quad \frac{-2}{n} \cdot \left[\frac{(-1)^{n+1}\pi}{n} + \frac{1}{n}\int_0^\pi \cos(nx) \right] \tag{8.15}$$

We're assuming $n > 0$. But then according to (8.11),

$$\int_0^\pi \cos(nx) \quad = \quad 0$$

Hence, we can simplify (8.15) to conclude:

$$\frac{2}{\pi}\int_0^\pi x^2 \cdot \cos(nx)\ dx \quad = \quad \frac{2}{\pi}\cdot\frac{-2}{n}\cdot\frac{(-1)^{n+1}\pi}{n} \quad = \quad \frac{4(-1)^n}{n^2},$$

as desired. □

Exercise 8.19 Verify all of the other parts of Example 8.12, both using Theorem 8.11, and through direct integration.

To compute the Fourier series of an arbitrary polynomial, we integrate one term at a time...

Example 8.13: Let $L = \pi$ and let $f(x) = x^2 - \pi \cdot x$. Then the Fourier sine series of f is:

$$\frac{-8}{\pi}\sum_{\substack{n=1\\ n\ \text{odd}}}^{\infty} \frac{1}{n^3}\sin(nx) \quad = \quad \frac{-8}{\pi}\left(\sin(x) + \frac{\sin(3x)}{27} + \frac{\sin(5x)}{125} + \frac{\sin(7x)}{343} + \cdots\cdots\right)$$

To see this, first, note that, by Example 8.12(b)

$$\int_0^\pi x \cdot \sin(nx)\ dx \ = \ \frac{-1}{n}(-1)^n\pi \ = \ \frac{(-1)^{n+1}\pi}{n}.$$

Next, by Example 8.12(c),

$$\int_0^\pi x^2 \cdot \sin(nx)\ dx \quad = \quad \frac{2}{n^3}\Big((-1)^n - 1\Big) \ + \ \frac{(-1)^{n+1}\pi^2}{n}.$$

Thus,

$$\begin{aligned}\int_0^\pi \Big(x^2 - \pi x\Big)\cdot\sin(nx)\ dx \quad &= \quad \int_0^\pi x^2\cdot\sin(nx)\ dx \ - \ \pi\cdot\int_0^\pi x\cdot\sin(nx)\ dx \\ &= \quad \frac{2}{n^3}\Big((-1)^n - 1\Big) \ + \ \frac{(-1)^{n+1}\pi^2}{n} \ - \ \pi\cdot\frac{(-1)^{n+1}\pi}{n} \\ &= \quad \frac{2}{n^3}\Big((-1)^n - 1\Big).\end{aligned}$$

Thus,

$$B_n \ = \ \frac{2}{\pi}\int_0^\pi \Big(x^2 - \pi x\Big)\cdot\sin(nx)\ dx \ = \ \frac{4}{\pi n^3}\Big((-1)^n - 1\Big) \ = \ \begin{cases} -8/\pi n^3 & \text{if} \quad n \text{ is odd} \\ 0 & \text{if} \quad n \text{ is even}\end{cases}.$$

♢

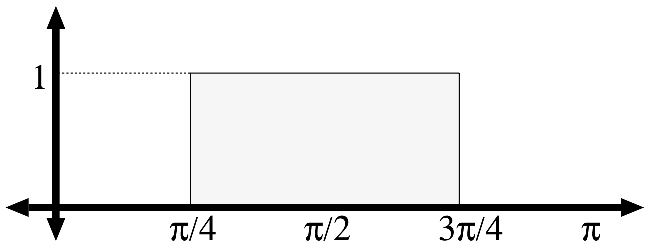

Figure 8.2: Example 8.14.

8.3(c) Step Functions

Example 8.14: Let $L = \pi$, and suppose $f(x) = \begin{cases} 1 & \text{if } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \\ 0 & \text{otherwise} \end{cases}$ (see Figure 8.2).
Then the Fourier sine coefficients of f are given:

$$B_n = \begin{cases} 0 & \text{if } n \text{ is } \textbf{even} \\ \frac{2\sqrt{2}(-1)^k}{n\pi} & \text{if } n \text{ is } \textbf{odd}\text{, and } n = 4k \pm 1 \text{ for some } k \in \mathbb{N} \end{cases}$$

To see this, observe that

$$\int_0^\pi f(x)\sin(nx)\,dx = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin(nx)\,dx = \frac{-1}{n}\cos(nx)\Big|_{x=\frac{\pi}{4}}^{x=\frac{3\pi}{4}} = \frac{-1}{n}\left(\cos\left(\frac{3n\pi}{4}\right) - \cos\left(\frac{n\pi}{4}\right)\right)$$
$$= \begin{cases} 0 & \text{if } n \text{ is } \textbf{even} \\ \frac{\sqrt{2}(-1)^{k+1}}{n} & \text{if } n \text{ is } \textbf{odd}\text{, and } n = 4k \pm 1 \text{ for some } k \in \mathbb{N} \end{cases}$$ (**Exercise 8.20**)

Thus, the Fourier sine series for f is:

$$\frac{2\sqrt{2}}{\pi}\left(\sin(x) + \sum_{k=1}^{N}(-1)^k\left(\frac{\sin\big((4k-1)x\big)}{4k-1} + \frac{\sin\big((4k+1)x\big)}{4k+1}\right)\right)$$ (**Exercise 8.21**)

Figure 8.3 shows some of the partial sums of this series. The series converges *pointwise* to $f(x)$ in the interior of the intervals $\left[0, \frac{\pi}{4}\right)$, $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$, and $\left(\frac{3\pi}{4}, \pi\right]$. However, it does not converge to f at the discontinuity points $\frac{\pi}{4}$ and $\frac{3\pi}{4}$. In the plots, this is betrayed by the violent oscillations of the partial sums near these discontinuity points –this is an example of the **Gibbs phenomenon**. ♢

Example 8.14 is an example of a *step function.* A function $F : [0, L] \longrightarrow \mathbb{R}$ is a **step function** (see Figure 8.4 on the facing page) if there are numbers $0 = x_0 < x_1 < x_2 < x_3 <$

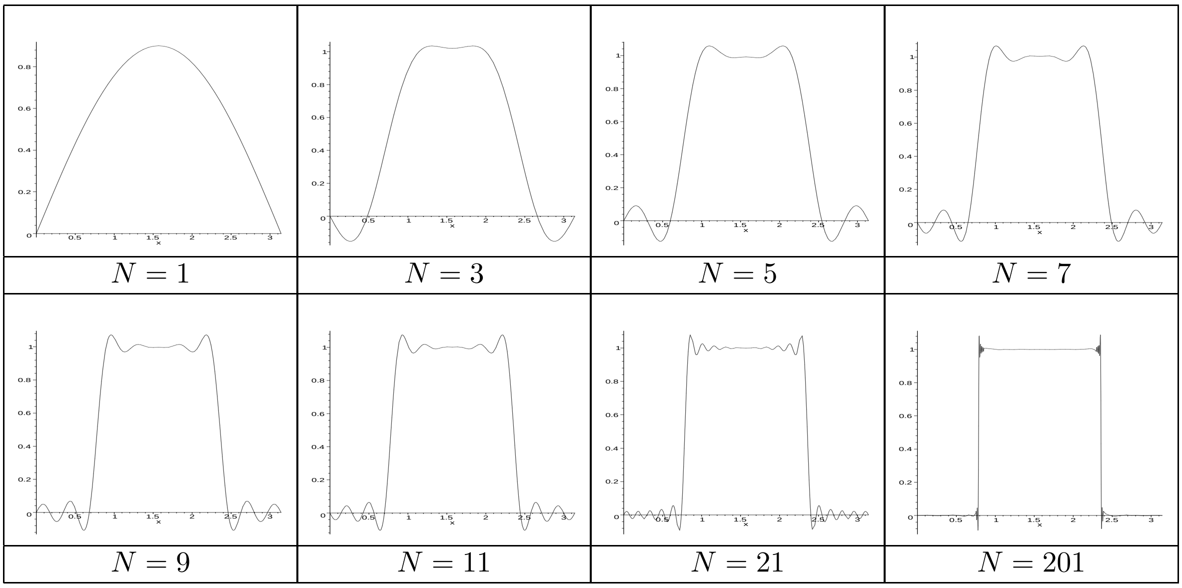

Figure 8.3: Partial Fourier sine series for Example 8.14, for $N = 0, 1, 2, 3, 4, 5, 10$ and 100. Notice the Gibbs phenomenon in the plots for large N.

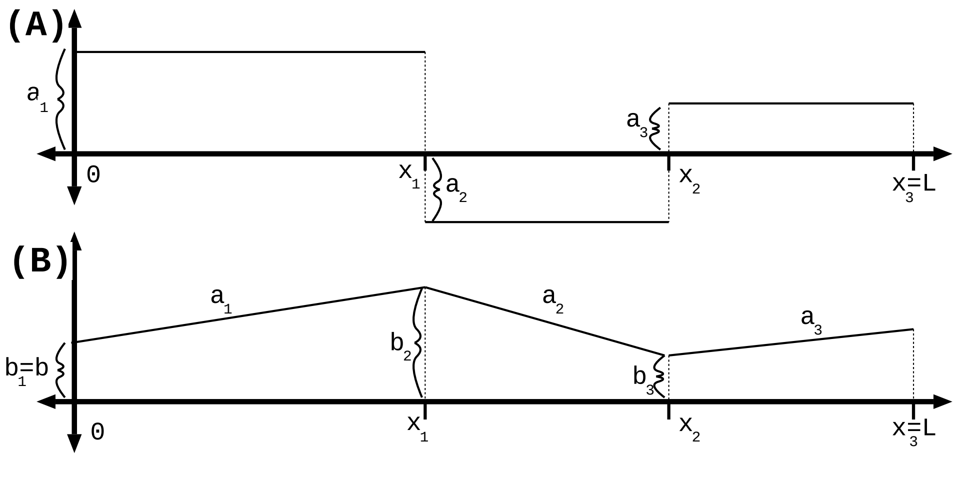

Figure 8.4: (A) A step function. (B) A piecewise linear function.

$\ldots < x_{M-1} < x_M = L$ and constants $a_1, a_2, \ldots, a_M \in \mathbb{R}$ so that

$$\left.\begin{array}{rcll} F(x) & = & a_1 & \text{if } 0 < x < x_1, \\ F(x) & = & a_2 & \text{if } x_1 < x < x_2, \\ & \vdots & & \\ F(x) & = & a_m & \text{if } x_{m-1} < x < x_m, \\ & \vdots & & \\ F(x) & = & a_M & \text{if } x_{M-1} < x < L \end{array}\right\} \tag{8.16}$$

For instance, in Example 8.14, $M = 3$; $x_0 = 0$, $x_1 = \frac{\pi}{4}$, $x_2 = \frac{3\pi}{4}$, and $x_3 = \pi$; $a_1 = 0 = a_3$, and $a_2 = 1$.

To compute the Fourier coefficients of a step function, we simply break the integral into 'pieces', as in Example 8.14. The general formula is given by the following theorem, but it is really not worth memorizing the formula. Instead, understand the idea.

Theorem 8.15: *Suppose $F : [0, L] \longrightarrow \mathbb{R}$ is a step function like (8.16). Then the Fourier coefficients of F are given:*

$$\begin{aligned} \frac{1}{L}\int_0^L F(x) &= \frac{1}{L}\sum_{m=1}^{M} a_m \cdot (x_m - x_{m-1}) \\ \frac{2}{L}\int_0^L F(x)\cdot\cos\left(\frac{n\pi}{L}x\right)\,dx &= \frac{-2}{\pi n}\sum_{m=1}^{M-1}\sin\left(\frac{n\pi}{L}\cdot x_m\right)\cdot\Big(a_{m+1} - a_m\Big) \\ \frac{2}{L}\int_0^L F(x)\cdot\sin\left(\frac{n\pi}{L}x\right)\,dx &= \frac{2}{\pi n}\Big(a_1 + (-1)^{n+1}a_M\Big) + \frac{2}{\pi n}\sum_{m=1}^{M-1}\cos\left(\frac{n\pi}{L}\cdot x_m\right)\cdot\Big(a_{m+1} - a_m\Big), \end{aligned}$$

Proof: **Exercise 8.22** Hint: Integrate the function piecewise. □

Remark: Note that the Fourier series of a step function f will converge uniformly to f on the *interior* of each "step", but will *not* converge to f at any of the step boundaries, because f is not continuous at these points.

Example 8.16: Suppose $L = \pi$, and $g(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \leq x \end{cases}$ (see Figure 8.5A).

Then the Fourier cosine series of $g(x)$ is:

$$\frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\Big((2k+1)x\Big)$$

In other words, $A_0 = \frac{1}{2}$, and, for all $n > 0$, $A_n = \begin{cases} \frac{2}{\pi}\frac{(-1)^k}{2k+1} & \text{if } n \text{ is odd and } n = 2k+1; \\ 0 & \text{if } n \text{ is even.} \end{cases}$.

Exercise 8.23 Show this in two ways: first by direct integration, and then by applying the formula from Theorem 8.15. ♢

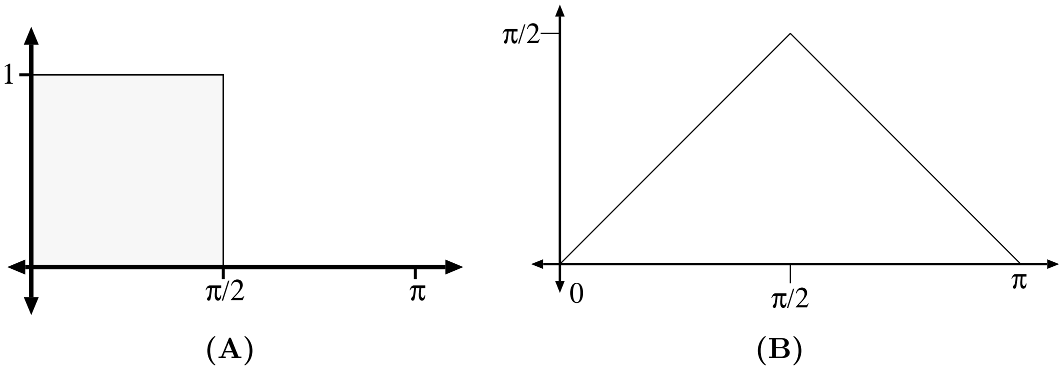

Figure 8.5: **(A)** The step function $g(x)$ in Example 8.16. **(B)** The tent function $f(x)$ in Example 8.17.

8.3(d) Piecewise Linear Functions

Example 8.17: (The Tent Function)

Let $\mathbb{X} = [0, \pi]$ and let $f(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} < x \le \pi. \end{cases}$ (see Figure 8.5B)

The Fourier sine series of f is: $\displaystyle \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}; \\ n=2k+1}}^{\infty} \frac{(-1)^k}{n^2} \sin(nx).$

To prove this, we must show that, for all $n > 0$,

$$B_n \quad = \quad \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\ dx \quad = \quad \begin{cases} \dfrac{4}{n^2 \pi}(-1)^k & \text{if } n \text{ is odd, } n = 2k+1; \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

To verify this, we observe that

$$\int_0^{\pi} f(x) \sin(nx)\ dx \quad = \quad \int_0^{\pi/2} x \sin(nx)\ dx \ + \ \int_{\pi/2}^{\pi} (\pi - x) \sin(nx)\ dx.$$

Exercise 8.24 Complete the computation of B_n. ◇

The tent function in Example 8.17 is *piecewise linear*. A function $F : [0, L] \longrightarrow \mathbb{R}$ is **piecewise linear** (see Figure 8.4 on page 161) if there are numbers $0 = x_0 < x_1 < x_1 < x_2 <$

$\ldots < x_{M-1} < x_M = L$ and constants $a_1, a_2, \ldots, a_M \in \mathbb{R}$ and $b \in \mathbb{R}$ so that

$$\left.\begin{array}{rcll} F(x) & = & a_1(x-L)+b_1 & \text{if } 0 < x < x_1, \\ F(x) & = & a_2(x-x_1)+b_2 & \text{if } x_1 < x < x_2, \\ & \vdots & & \\ F(x) & = & a_m(x-x_m)+b_{m+1} & \text{if } x_m < x < x_{m+1}, \\ & \vdots & & \\ F(x) & = & a_M(x-x_{M-1})+b_M & \text{if } x_{M-1} < x < L \end{array}\right\} \quad (8.17)$$

where $b_1 = b$, and, for all $m > 1$, $b_m = a_m(x_m - x_{m-1}) + b_{m-1}$.

For instance, in Example 8.17, $M = 2$, $x_1 = \frac{\pi}{2}$ and $x_2 = \pi$; $a_1 = 1$ and $a_2 = -1$.

To compute the Fourier coefficients of a piecewise linear function, we can break the integral into 'pieces', as in Example 8.17. The general formula is given by the following theorem, but it is really not worth memorizing the formula. Instead, understand the *idea*.

Theorem 8.18: *Suppose $F : [0, L] \longrightarrow \mathbb{R}$ is a piecewise-linear function like (8.17). Then the Fourier coefficients of F are given:*

$$\frac{1}{L}\int_0^L F(x) = \frac{1}{L}\sum_{m=1}^{M} \frac{a_m}{2}(x_m - x_{m-1})^2 + b_m \cdot (x_m - x_{m-1}).$$

$$\frac{2}{L}\int_0^L F(x)\cdot\cos\left(\frac{n\pi}{L}x\right)\,dx = \frac{2L}{(\pi n)^2}\sum_{m=1}^{M}\cos\left(\frac{n\pi}{L}\cdot x_m\right)\cdot\Big(a_m - a_{m+1}\Big)$$

$$\frac{2}{L}\int_0^L F(x)\cdot\sin\left(\frac{n\pi}{L}x\right)\,dx = \frac{2L}{(\pi n)^2}\sum_{m=1}^{M-1}\sin\left(\frac{n\pi}{L}\cdot x_m\right)\cdot\Big(a_m - a_{m+1}\Big)$$

(where we define $a_{M+1} := a_1$ for convenience).

Proof: <u>**Exercise 8.25**</u> Hint: invoke Theorem 8.15 and integration by parts. □

Note that the summands in this theorem read "$a_m - a_{m+1}$", not the other way around.

Example 8.19: (Cosine series of the tent function)

Let Let $\mathbb{X} = [0, \pi]$ and let $f(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} < x \le \pi. \end{cases}$ as in Example 8.17. The Fourier cosine series of f is:

$$\frac{\pi}{4} - \frac{8}{\pi}\sum_{\substack{n=1\\ n=4j+2,\\ \text{for some } j}}^{\infty} \frac{1}{n^2}\cos(nx)$$

In other words,

$$f(x) = \frac{\pi}{4} - \frac{8}{\pi}\left(\frac{\cos(2x)}{4} + \frac{\cos(6x)}{36} + \frac{\cos(10x)}{100} + \frac{\cos(14x)}{196} + \frac{\cos(18x)}{324} + \ldots\right)$$

To see this, first observe that

$$\begin{aligned} A_0 &= \frac{1}{\pi}\int_0^\pi f(x)\,dx = \frac{1}{\pi}\left(\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx\right) \\ &= \frac{1}{\pi}\left(\left[\frac{x^2}{2}\right]_0^{\pi/2} + \frac{\pi^2}{2} - \left[\frac{x^2}{2}\right]_{\pi/2}^{\pi}\right) = \frac{1}{\pi}\left(\frac{\pi^2}{8} + \frac{\pi^2}{2} - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right) \\ &= \frac{\pi^2}{4\pi} = \frac{\pi}{4}. \end{aligned}$$

Now let's compute A_n for $n > 0$.

$$\begin{aligned} \text{First, } \int_0^{\pi/2} x\cos(nx)\,dx &= \frac{1}{n}\left[x\sin(nx)\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin(nx)\,dx\right] \\ &= \frac{1}{n}\left[\frac{\pi}{2}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n}\cos(nx)\Big|_0^{\pi/2}\right] \\ &= \frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}. \end{aligned}$$

$$\begin{aligned} \text{Next, } \int_{\pi/2}^{\pi} x\cos(nx)\,dx &= \frac{1}{n}\left[x\sin(nx)\Big|_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\sin(nx)\,dx\right] \\ &= \frac{1}{n}\left[\frac{-\pi}{2}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n}\cos(nx)\Big|_{\pi/2}^{\pi}\right] \\ &= \frac{-\pi}{2n}\sin\left(\frac{n\pi}{2}\right) + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right). \end{aligned}$$

$$\begin{aligned} \text{Finally, } \int_{\pi/2}^{\pi} \pi\cos(nx)\,dx &= \frac{\pi}{n}\sin(nx)\Big|_{\pi/2}^{\pi} \\ &= \frac{-\pi}{n}\sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

Putting it all together, we have:

$$\begin{aligned} \int_0^\pi f(x)\cos(nx)\,dx &= \int_0^{\pi/2} x\cos(nx)\,dx + \int_{\pi/2}^{\pi}\pi\cos(nx)\,dx - \int_{\pi/2}^{\pi} x\cos(nx)\,dx \\ &= \frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2} - \frac{\pi}{n}\sin\left(\frac{n\pi}{2}\right) \\ &\qquad + \frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) - \frac{(-1)^n}{n^2} + \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right) \\ &= \frac{2}{n^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1+(-1)^n}{n^2} \end{aligned}$$

Now,

$$\cos\left(\frac{n\pi}{2}\right) = \begin{cases} (-1)^k & \text{if } n \text{ is even and } n = 2k; \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

$$\text{while} \quad 1 + (-1)^n = \begin{cases} 2 & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Thus,

$$2\cos\left(\frac{n\pi}{2}\right) - \left(1 + (-1)^n\right) = \begin{cases} -4 & \text{if } n \text{ is even, } n = 2k \text{ and } k = 2j+1 \text{ for some } j; \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} -4 & \text{if } n = 4j+2 \text{ for some } j; \\ 0 & \text{otherwise.} \end{cases}$$

(for example, $n = 2, 6, 10, 14, 18, \ldots$). Thus $A_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\, dx$

$$= \begin{cases} \frac{-8}{n^2\pi} & \text{if } n = 4j+2 \text{ for some } j; \\ 0 & \text{otherwise.} \end{cases}.$$ ♢

8.3(e) Differentiating Fourier (co)sine Series

Prerequisites: §8.2, §1.7

Suppose $f(x) = 3\sin(x) - 5\sin(2x) + 7\sin(3x)$. Then $f'(x) = 3\cos(x) - 10\cos(2x) + 21\cos(3x)$. Likewise, if $f(x) = 3 + 2\cos(x) - 6\cos(2x) + 11\cos(3x)$, then $f'(x) = -2\sin(x) + 12\sin(2x) - 33\sin(3x)$. This illustrates a general pattern.

Theorem 8.20: *Suppose $f \in \mathcal{C}^\infty[0, L]$*

- *Suppose f has Fourier sine series $\sum_{n=1}^\infty B_n \mathbf{S}_n(x)$. If $\sum_{n=1}^\infty n|B_n| < \infty$, then f' has Fourier cosine series: $f'(x) = \frac{\pi}{L}\sum_{n=1}^\infty nB_n\mathbf{C}_n(x)$, and this series converges uniformly.*

- *Suppose f has Fourier cosine series $\sum_{n=0}^\infty A_n \mathbf{C}_n(x)$. If $\sum_{n=1}^\infty n|A_n| < \infty$, then f' has Fourier sine series: $f'(x) = \frac{-\pi}{L}\sum_{n=1}^\infty nA_n\mathbf{S}_n(x)$, and this series converges uniformly.*

Proof: **Exercise 8.26** Hint: Apply Proposition 1.7 on page 16 □

Consequence: If $f(x) = \cos\left(\frac{n\pi x}{L}\right) + \sin\left(\frac{n\pi x}{L}\right)$, then $f'(x) = \frac{-n\pi}{L} f(x)$. In other words, f is an **eigenfunction**[1] for the differentation operator ∂_x, with eigenvalue $\lambda = \frac{-n\pi}{L}$. Hence, for any $k \in \mathbb{N}$, $\partial_x^k f \ = \ \lambda^k \cdot f$.

8.4 Practice Problems

In all of these problems, the domain is $\mathbb{X} = [0, \pi]$.

1. Let $\alpha > 0$ be a constant. Compute the Fourier *sine* series of $f(x) = \exp(\alpha \cdot x)$. At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

2. Compute the Fourier *cosine* series of $f(x) = \sinh(x)$. At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

3. Let $\alpha > 0$ be a constant. Compute the Fourier *sine* series of $f(x) = \cosh(\alpha x)$. At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

4. Compute the Fourier *cosine* series of $f(x) = x$. At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

5. Let $g(x) \ = \ \begin{cases} 1 & \text{if } 0 \leq x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \leq x \end{cases}$ (Fig. 8.5A on p. 163)

   (a) Compute the Fourier *cosine* series of $g(x)$. At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

   (b) Compute the Fourier *sine* series of $g(x)$. At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

6. Compute the Fourier *cosine* series of $g(x) \ = \ \begin{cases} 3 & \text{if } 0 \leq x < \frac{\pi}{2} \\ 1 & \text{if } \frac{\pi}{2} \leq x \end{cases}$

   At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

7. Compute the Fourier *sine* series of $f(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} < x \leq \pi. \end{cases}$

   (Fig. 8.5B on p.163) At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

   Hint: Note that $\displaystyle\int_0^\pi f(x)\sin(nx)\ dx \ = \ \int_0^{\pi/2} x\sin(nx)\ dx \ + \ \int_{\pi/2}^\pi (\pi - x)\sin(nx)\ dx.$

[1] See § 5.2(d) on page 81

8. Let $f : [0, \pi] \longrightarrow \mathbb{R}$ be defined: $f(x) = \begin{cases} x & \text{if} \quad 0 < x < \frac{\pi}{2} \\ 0 & \text{if} \quad \frac{\pi}{2} < x < \pi \end{cases}$.

   Compute the Fourier **sine** series for $f(x)$. At which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

Notes: ....

9 Real Fourier Series and Complex Fourier Series

9.1 Real Fourier Series on $[-\pi, \pi]$

Prerequisites: §7.4, §7.5 **Recommended:** §8.1, §6.5(d)

Throughout this section, let $\mathbf{S}_n(x) = \sin(nx)$, for all natural numbers $n \geq 1$, and and let $\mathbf{C}_n(x) = \cos(nx)$, for all natural numbers $n \geq 0$.

If $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ is any function with $\|f\|_2 < \infty$, we define the **(real) Fourier Coefficients:**

$$A_0 = \langle f, \mathbf{C}_0 \rangle = \langle f, \mathbb{1} \rangle = \boxed{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx},$$

$$A_n = \frac{\langle f, \mathbf{C}_n \rangle}{\|\mathbf{C}_n\|_2^2} = \boxed{\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\ dx} \quad \text{and} \quad B_n = \frac{\langle f, \mathbf{S}_n \rangle}{\|\mathbf{S}_n\|_2^2} = \boxed{\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\ dx} \quad \text{for } n > 0$$

The **(real) Fourier Series** of f is then the infinite summation of functions:

$$\boxed{A_0 + \sum_{n=1}^{\infty} \Big(A_n \mathbf{C}_n(x) + B_n \mathbf{S}_n(x) \Big)} \tag{9.1}$$

Let $\mathcal{C}^1[-\pi, \pi]$ be the set of all functions $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ which are continuously differentiable on $[-\pi, \pi]$. The real Fourier series eqn.(9.1) usually converges to f in $\mathbf{L}^2$. If $f \in \mathcal{C}^1[-\pi, \pi]$, then the Fourier series eqn.(9.1) also converges uniformly to f.

(**Exercise 9.1** Show that $\mathcal{C}^1[-\pi, \pi] \subset \mathbf{L}^2[-\pi, \pi]$.)

Theorem 9.1: Fourier Convergence on $[-\pi, \pi]$

(a) *The set $\{\mathbb{1}, \mathbf{S}_1, \mathbf{C}_1, \mathbf{S}_2, \mathbf{C}_2, \ldots\}$ is an orthogonal basis for $\mathbf{L}^2[-\pi, \pi]$. Thus, if $f \in \mathbf{L}^2[-\pi, \pi]$, then the Fourier series (9.1) converges to f in $\mathbf{L}^2$-norm.*

Furthermore, the coefficient sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ are the unique sequences of coefficients with this property. In other words, if $\{A'_n\}_{n=0}^{\infty}$ and $\{B'_n\}_{n=1}^{\infty}$ are two other sequences of coefficients such that $f \underset{L2}{\approx} \sum_{n=0}^{\infty} A'_n \mathbf{C}_n + \sum_{n=1}^{\infty} B'_n \mathbf{S}_n$, then we must have $A'_n = A_n$ and $B'_n = B_n$ for all $n \in \mathbb{N}$.

(b) *If $f \in \mathcal{C}^1[-\pi, \pi]$ then the Fourier series (9.1) converges pointwise on $(-\pi, \pi)$. In other words, if $-\pi < x < \pi$, then $f(x) = A_0 + \lim_{N\to\infty} \sum_{n=1}^{N} \Big(A_n \cos(nx) + B_n \sin(nx) \Big)$.*

(c) *If $f \in \mathcal{C}^1_{\text{per}}[-\pi, \pi]$ (i.e. f is $\mathcal{C}^1$ and satisfies periodic boundary conditions[1]), then the series (9.1) converges to f uniformly on $[-\pi, \pi]$.*

[1] i.e. $f(-\pi) = f(\pi)$ and $f'(-\pi) = f'(\pi)$; see § 6.5(d) on page 102.

(d) $\Big($ *The series* (9.1) *converges to* f *uniformly on* $[-\pi,\pi]$ $\Big) \Leftrightarrow \left(\sum_{n=0}^{\infty} |A_n| \; + \; \sum_{n=1}^{\infty} |B_n| \; < \; \infty \right)$.

Proof: For **(a)**, see [Kat76, p.29 of §1.5], or [Bro89, Theorem 2.3.10], or [CB87, §38]. For **(b)** see [Bro89, Theorem 1.4.4, p.15] or [CB87, §31]. For **(c)** see [Fol84, Theorem 8.43, p.256], or [WZ77, Theorem 12.20, p.219], or [CB87, §35]..

(d) is **Exercise 9.2** (Hint: Use the Weierstrass M-test, Proposition 7.18 on page 131.). □

Remark: There is nothing special about the interval $[-\pi,\pi]$. Real Fourier series can be defined for functions on an interval $[-L,L]$ for any $L>0$. We chose $L=\pi$ because it makes the computations simpler. For $L\neq\pi$, the previous results can be reformulated using the functions $\mathbf{S}_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ and $\mathbf{C}_n(x) = \cos\left(\frac{n\pi x}{L}\right)$.

9.2 Computing Real Fourier Coefficients

Prerequisites: §9.1 **Recommended:** §8.3

When computing the real Fourier coefficient $A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\;dx$ (or $B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin(nx)\;dx$), it is simpler to first compute the integral $\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\;dx$ (or $\int_{-\pi}^{\pi} f(x)\cdot\sin(nx)\;dx$), and then multiply the result by $\frac{1}{\pi}$. In this section, we review some useful techniques to compute this integral.

9.2(a) Polynomials

Recommended: §8.3(b)

Theorem 9.2: $\int_{-\pi}^{\pi} \sin(nx)\;dx \;=\; 0 \;=\; \int_{-\pi}^{\pi}\cos(nx)\;dx.$

For any $k\in\{1,2,3,\ldots\}$, *we have the following recurrence relations:*

- *If* k *is* **even**, *then:*

$$\int_{-\pi}^{\pi} x^k\cdot\sin(nx)\,dx \;=\; 0, \qquad \textit{and} \qquad \int_{-\pi}^{\pi} x^k\cdot\cos(nx)\,dx \;=\; \frac{-k}{n}\int_{-\pi}^{\pi} x^{k-1}\cdot\sin(nx)\,dx.$$

- *If* $k>0$ *is* **odd:**, *then:*

$$\int_{-\pi}^{\pi} x^k\cdot\sin(nx)\;dx \;=\; \frac{2(-1)^{n+1}\pi^k}{n} \;+\; \frac{k}{n}\int_{-\pi}^{\pi} x^{k-1}\cdot\cos(nx)\;dx$$

$$\textit{and} \qquad \int_{-\pi}^{\pi} x^k\cdot\cos(nx)\;dx \;=\; 0.$$

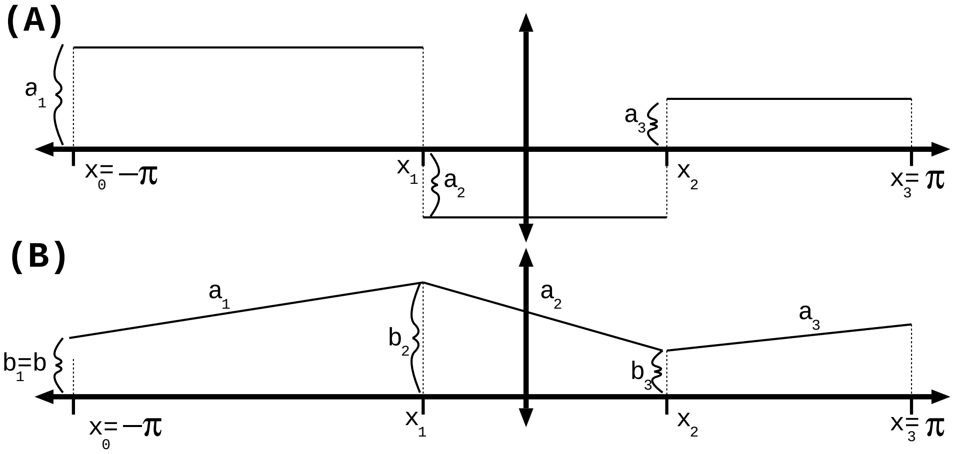

Figure 9.1: (A) A step function. (B) A piecewise linear function.

Proof: **Exercise 9.3** Hint: use integration by parts. □

Example 9.3:

(a) $p(x) = x$. Since $k = 1$ is *odd*, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \cos(nx)\, dx = 0,$$

and
$$\frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \sin(nx)\, dx = \frac{2(-1)^{n+1}\pi^0}{n} + \frac{1}{n\pi}\int_{-\pi}^{\pi} \cos(nx)\, dx =_{[*]} \frac{2(-1)^{n+1}}{n}.$$

where equality [∗] follows from case $k = 0$ in the theorem.

(b) $p(x) = x^2$. Since $k = 2$ is *even*, we have, for all n,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin(nx)\, dx = 0,$$
$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\, dx = \frac{-2}{n\pi}\int_{-\pi}^{\pi} x^1 \cdot \sin(nx)\, dx =_{[*]} \frac{-2}{n}\left(\frac{2(-1)^{n+1}}{n}\right) = \frac{4(-1)^n}{n^2}.$$

where equality [∗] follows from the previous example. ♢

9.2(b) Step Functions

Recommended: §8.3(c)

A function $F : [-\pi, \pi] \longrightarrow \mathbb{R}$ is a **step function** (see Figure 9.1(A)) if there are numbers $-\pi = x_0 < x_1 < x_1 < x_2 < \ldots < x_{M-1} < x_M = \pi$ and constants $a_1, a_2, \ldots, a_M \in \mathbb{R}$ so that

$$\left.\begin{array}{rcll} F(x) & = & a_1 & \text{if } -\pi < x < x_1, \\ F(x) & = & a_2 & \text{if } x_1 < x < x_2, \\ & \vdots & & \\ F(x) & = & a_m & \text{if } x_{m-1} < x < x_m, \\ & \vdots & & \\ F(x) & = & a_M & \text{if } x_{M-1} < x < \pi \end{array}\right\} \quad (9.2)$$

To compute the Fourier coefficients of a step function, we break the integral into 'pieces'. The general formula is given by the following theorem, but it is really not worth memorizing the formula. Instead, understand the *idea*.

Theorem 9.4: *Suppose $F : [-\pi, \pi] \longrightarrow \mathbb{R}$ is a step function like (9.2). Then the Fourier coefficients of F are given:*

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\, dx &= \frac{1}{2\pi}\sum_{m=1}^{M} a_m \cdot (x_m - x_{m-1}) \\ \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cdot\cos(nx)\, dx &= \frac{-1}{\pi n}\sum_{m=1}^{M-1} \sin(n\cdot x_m)\cdot\Big(a_{m+1} - a_m\Big) \\ \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cdot\sin(nx)\, dx &= \frac{(-1)^n}{\pi n}\Big(a_1 - a_M\Big) + \frac{1}{\pi n}\sum_{m=1}^{M-1} \cos(n\cdot x_m)\cdot\Big(a_{m+1} - a_m\Big), \end{aligned}$$

Proof: **Exercise 9.4** Hint: Integrate the function piecewise. Use the fact that $\int_{x_{m-1}}^{x_m} f(x)\sin(nx) = \frac{a_m}{n}\Big(\cos(n\cdot x_{m-1}) - \cos(n\cdot x_m)\Big)$, and $\int_{x_{m-1}}^{x_m} f(x)\cos(nx) = \frac{a_m}{n}\Big(\cos(n\cdot x_m) - \cos(n\cdot x_{m-1})\Big)$.
□

Remark: Note that the Fourier series of a step function f will converge uniformly to f on the *interior* of each "step", but will *not* converge to f at any of the step boundaries, because f is not continuous at these points.

Example 9.5:

Suppose $f(x) = \begin{cases} -3 & \text{if} \quad -\pi \le x < \frac{-\pi}{2}; \\ 5 & \text{if} \quad \frac{-\pi}{2} \le x < \frac{\pi}{2}; \\ 2 & \text{if} \quad \frac{\pi}{2} \le x \le \pi. \end{cases}$ (see Figure 9.2).

In the notation of Theorem 9.4, we have $M = 3$, and

$$\begin{array}{lllllllll} x_0 = -\pi; & x_1 = \frac{-\pi}{2}; & x_2 = \frac{\pi}{2}; & x_3 = \pi; \\ & a_1 = -3; & a_2 = 5; & a_3 = 2. \end{array}$$

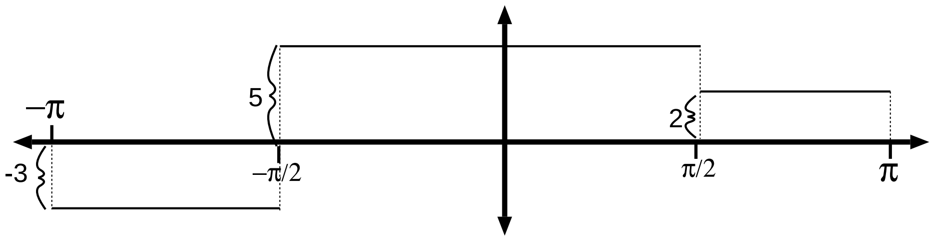

Figure 9.2: (A) A step function. (B) A piecewise linear function.

Thus, $$A_n = \frac{-1}{\pi n}\left[8\cdot\sin\left(n\cdot\frac{-\pi}{2}\right) - 3\cdot\sin\left(n\cdot\frac{\pi}{2}\right)\right] = \begin{cases} 0 & \text{if} \quad n \text{ is even;} \\ (-1)^k\cdot\frac{11}{\pi n} & \text{if} \quad n = 2k+1 \text{ is odd.}\end{cases}$$

and $$\begin{aligned} B_n &= \frac{1}{\pi n}\left[8\cdot\cos\left(n\cdot\frac{-\pi}{2}\right) - 3\cdot\cos\left(n\cdot\frac{\pi}{2}\right) - 5\cdot\cos(n\cdot\pi)\right] \\ &= \begin{cases} \frac{5}{\pi n} & \text{if} \quad n \text{ is odd;} \\ \frac{5}{\pi n}\left((-1)^k - 1\right) & \text{if} \quad n = 2k \text{ is even.}\end{cases} \\ &= \begin{cases} \frac{5}{\pi n} & \text{if} \quad n \text{ is odd;} \\ 0 & \text{if} \quad n \text{ is divisible by 4;} \\ \frac{-10}{\pi n} & \text{if} \quad n \text{ is even but } \textit{not} \text{ divisible by 4.}\end{cases} \end{aligned}$$ ♢

9.2(c) Piecewise Linear Functions

Recommended: §8.3(d)

A continuous function $F : [-\pi, \pi] \longrightarrow \mathbb{R}$ is **piecewise linear** (see Figure 9.1(B)) if there are numbers $-\pi = x_0 < x_1 < x_1 < x_2 < \ldots < x_{M-1} < x_M = \pi$ and constants $a_1, a_2, \ldots, a_M \in \mathbb{R}$ and $b \in \mathbb{R}$ so that

$$\left.\begin{array}{rcll} F(x) &=& a_1(x-\pi) + b_1 & \text{if } -\pi < x < x_1, \\ F(x) &=& a_2(x-x_1) + b_2 & \text{if } x_1 < x < x_2, \\ &\vdots& & \\ F(x) &=& a_m(x-x_m) + b_{m+1} & \text{if } x_m < x < x_{m+1}, \\ &\vdots& & \\ F(x) &=& a_M(x-x_{M-1}) + b_M & \text{if } x_{M-1} < x < \pi \end{array}\right\} \quad (9.3)$$

where $b_1 = b$, and, for all $m > 1$, $b_m = a_m(x_m - x_{m-1}) + b_{m-1}$.

Example 9.6: If $f(x) = |x|$, then f is piecewise linear, with: $x_0 = -\pi$, $x_1 = 0$, and $x_2 = \pi$; $a_1 = -1$ and $a_2 = 1$; $b_1 = \pi$, and $b_2 = 0$. ♢

To compute the Fourier coefficients of a piecewise linear function, we break the integral into 'pieces'. The general formula is given by the following theorem, but it is really not worth memorizing the formula. Instead, understand the *idea*.

Theorem 9.7: *Suppose $F : [-\pi, \pi] \longrightarrow \mathbb{R}$ is a piecewise-linear function like (9.3). Then the Fourier coefficients of F are given:*

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\,dx &= \frac{1}{2\pi}\sum_{m=1}^{M} \frac{a_m}{2}(x_m - x_{m-1})^2 \;+\; b_m \cdot (x_m - x_{m-1}).\\
\frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cdot\cos(nx)\,dx &= \frac{1}{\pi n^2}\sum_{m=1}^{M} \cos(nx_m)\cdot\Big(a_m - a_{m+1}\Big)\\
\frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cdot\sin(nx)\,dx &= \frac{1}{\pi n^2}\sum_{m=1}^{M-1} \sin(nx_m)\cdot\Big(a_m - a_{m+1}\Big)
\end{aligned}$$

(where we define $a_{M+1} := a_1$ for convenience).

Proof: **Exercise 9.5** Hint: invoke Theorem 9.4 and integration by parts. □

Note that the summands in this theorem read "$a_m - a_{m+1}$", not the other way around.

Example 9.8: Recall $f(x) = |x|$, from Example 9.6. Applying Theorem 9.7, we have

$$\begin{aligned}
A_0 &= \frac{1}{2\pi}\left[\frac{-1}{2}(0+\pi)^2 + \pi\cdot(0+\pi) \;+\; \frac{1}{2}(\pi-0)^2 + 0\cdot(\pi-0)\right] \;=\; \frac{\pi}{2}\\
A_n &= \frac{\pi}{\pi n^2}\left[(-1-1)\cdot\cos(n0)\,(1+1)\cdot\cos(n\pi)\right]\\
&= \frac{1}{\pi n^2}\left[-2+2(-1)^n\right] \;=\; \frac{-2}{\pi n^2}\left[1-(-1)^n\right],
\end{aligned}$$

while $B_n = 0$ for all n because f is even. ♢

9.2(d) Differentiating Real Fourier Series

Prerequisites: §9.1, §1.7

Suppose $f(x) = 3 + 2\cos(x) - 6\cos(2x) + 11\cos(3x) + 3\sin(x) - 5\sin(2x) + 7\sin(3x)$. Then $f'(x) = -2\sin(x) + 12\sin(2x) - 33\sin(3x) + 3\cos(x) - 10\cos(2x) + 21\cos(3x)$. This illustrates a general pattern.

Theorem 9.9: *Let $f \in \mathcal{C}^\infty[-\pi,\pi]$, and suppose f has Fourier series* $\sum_{n=0}^\infty A_n \mathbf{C}_n + \sum_{n=1}^\infty B_n \mathbf{S}_n$.
If $\sum_{n=1}^\infty n|A_n| < \infty$ *and* $\sum_{n=1}^\infty n|B_n| < \infty$, *then f' has Fourier Series:* $\sum_{n=1}^\infty n\left(B_n\mathbf{C}_n - A_n\mathbf{S}_n\right)$.

Proof: <u>**Exercise 9.6**</u> Hint: Apply Proposition 1.7 on page 16 □

Consequence: If $f(x) = \cos(nx) + \sin(nx)$, then $f'(x) = -nf(x)$. In other words, f is an **eigenfunction** for the differentation operator $\frac{d}{dx}$, with eigenvalue $-n$. Hence, for any k,

$$\partial^k f \;=\; (-n)^k \cdot f$$

9.3 (*)Relation between (Co)sine series and Real series

Prerequisites: §1.5, §9.1, §8.1

We have seen how the functions $\mathbf{C}_n$ and $\mathbf{S}_n$ form an orthogonal basis for $\mathbf{L}^2[-\pi,\pi]$. However, if we confine our attention to *half* this interval —that is, to $\mathbf{L}^2[0,\pi]$ —then it seems we only need half as many basis elements. Why is this?

Recall from § 1.5 on page 11 that a function $f : [-\pi,\pi] \longrightarrow \mathbb{R}$ is **even** if $f(-x) = f(x)$ for all $x \in [0,\pi]$, and f is **odd** if $f(-x) = -f(x)$ for all $x \in [0,\pi]$.

Define the vector spaces:

$$\begin{aligned} \mathbf{L}^2_{\text{even}}[-\pi,\pi] &= \text{all } \textit{even} \text{ elements in } \mathbf{L}^2[-\pi,\pi] \\ \mathbf{L}^2_{\text{odd}}[-\pi,\pi] &= \text{all } \textit{odd} \text{ elements in } \mathbf{L}^2[-\pi,\pi] \end{aligned}$$

Recall (Theorem 1.6 on page 12) that any function f has an **even-odd decomposition**. That is, there is a unique even function $\check{f}$ and a unique odd function $\acute{f}$ so that $f = \check{f} + \acute{f}$. We indicate this by writing: $\mathbf{L}^2[-\pi,\pi] = \mathbf{L}^2_{\text{even}}[-\pi,\pi] \oplus \mathbf{L}^2_{\text{odd}}[-\pi,\pi]$,

Lemma 9.10: *For any $n \in \mathbb{N}$:*

(a) *The function $\mathbf{C}_n(x) = \cos(nx)$ is <u>even</u>.*

(b) *The function $\mathbf{S}_n(x) = \sin(nx)$ is <u>odd</u>.*

Let $f : [-\pi,\pi] \longrightarrow \mathbb{R}$ be any function.

(c) *If $f(x) = \sum_{n=0}^\infty A_n \mathbf{C}_n(x)$, then f is <u>even</u>.*

(d) *If $f(x) = \sum_{n=1}^\infty B_n \mathbf{S}_n(x)$, then f is <u>odd</u>.*

Proof: Exercise 9.7 □

In other words, cosine series are even, and sine series are odd. The converse is also true. To be precise:

Proposition 9.11: *Let $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ be any function, and suppose f has real Fourier series $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \mathbf{C}_n(x) + \sum_{n=1}^{\infty} B_n \mathbf{S}_n(x)$. Then:*

(a) *If f is odd, then $A_n = 0$ for every $n \in \mathbb{N}$.*

(b) *If f is even, then $B_n = 0$ for every $n \in \mathbb{N}$.*

Proof: Exercise 9.8 □

From this, it follows immediately:

Proposition 9.12:

(a) *The set $\{\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \ldots\}$ is an orthogonal basis for $\mathbf{L}^2_{\mathsf{even}}[-\pi, \pi]$ (where $\mathbf{C}_0 = \mathbb{1}$).*

(b) *The set $\{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \ldots\}$ is an orthogonal basis for $\mathbf{L}^2_{\mathsf{odd}}[-\pi, \pi]$.*

In either case, the Fourier series will converge pointwise or uniformly if the relevant conditions from the earlier Fourier Convergence Theorems are satisfied. □

Exercise 9.9 Suppose f has even-odd decomposition $f = \check{f} + \acute{f}$, and f has real Fourier series $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \mathbf{C}_n(x) + \sum_{n=1}^{\infty} B_n \mathbf{S}_n(x)$. Show that:

(a) If $\check{f}(x) = \sum_{n=0}^{\infty} A_n \mathbf{C}_n(x)$, then f is even. (b) If $\acute{f}(x) = \sum_{n=1}^{\infty} B_n \mathbf{S}_n(x)$, then f is odd.

Suppose $f : [0, \pi] \longrightarrow \mathbb{R}$. Recall from §1.5 that we can "extend" f to an even function $f_{\mathsf{even}} : [-\pi, \pi] \longrightarrow \mathbb{R}$, or to an odd function $f_{\mathsf{odd}} : [-\pi, \pi] \longrightarrow \mathbb{R}$.

Proposition 9.13:

(a) *The Fourier **cosine** series for f is the same as the **real** Fourier series for f_{even}. In other words, the nth Fourier cosine coefficient is given: $A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_{\mathsf{even}}(x) \mathbf{C}_n(x)\, dx$.*

(b) *The Fourier **sine** series for f is the same as the **real** Fourier series for f_{odd}. In other words, the nth Fourier sine coefficient is given: $B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_{\mathsf{odd}}(x) \mathbf{S}_n(x)\, dx$.*

Proof: Exercise 9.10 □

9.4 (∗) Complex Fourier Series

Prerequisites: §7.4, §7.5, §1.3 **Recommended:** §9.1

If $f, g : \mathbb{X} \longrightarrow \mathbb{C}$ are complex-valued functions, then we define their **inner product**:

$$\langle f, g \rangle \;=\; \frac{1}{M} \int_{\mathbb{X}} f(\mathbf{x}) \cdot \overline{g(\mathbf{x})} \; d\mathbf{x}$$

where M is the length/area/volume of domain $\mathbb{X}$. Once again,

$$\|f\|_2 \;=\; \langle f, f \rangle^{1/2} \;=\; \left(\frac{1}{M} \int_{\mathbb{X}} f(\mathbf{x})\overline{f(\mathbf{x})} \; d\mathbf{x} \right)^{1/2} \;=\; \left(\frac{1}{M} \int_{\mathbb{X}} |f(\mathbf{x})|^2 \; d\mathbf{x} \right)^{1/2}.$$

The concepts of orthogonality, $\mathbf{L}^2$ distance, and $\mathbf{L}^2$ convergence are exactly the same as before.

Let $\mathbf{L}^2([-L, L]; \mathbb{C})$ be the set of all complex-valued functions $f : [-L, L] \longrightarrow \mathbb{C}$ with finite $\mathbf{L}^2$-norm. What is the Fourier series of such a function?

For all $n \in \mathbb{Z}$, let

$$\boxed{\mathbf{E}_n(x) = \exp\left(\frac{\pi \mathbf{i} n x}{L} \right)}$$

(thus, $\mathbf{E}_0 = 1\!\!1$ is the constant unit function). For all $n > 0$, notice that the *de Moivre Formulae* imply:

$$\begin{aligned} \mathbf{E}_n(x) &= \mathbf{C}_n(x) \;+\; \mathbf{i} \cdot \mathbf{S}_n(x) \\ \text{and} \;\; \mathbf{E}_{-n}(x) &= \mathbf{C}_n(x) \;-\; \mathbf{i} \cdot \mathbf{S}_n(x) \end{aligned} \qquad (9.4)$$

Also, note that $\langle \mathbf{E}_n, \mathbf{E}_m \rangle = 0$ if $n \neq m$, and $\|\mathbf{E}_n\|_2 = 1$ (**Exercise 9.11**), so these functions form an *orthonormal set*.

If $f : [-L, L] \longrightarrow \mathbb{C}$ is any function with $\|f\|_2 < \infty$, then we define the **(complex) Fourier coefficients** of f:

$$\widehat{f}_n \;=\; \langle f, \mathbf{E}_n \rangle \;=\; \boxed{\frac{1}{2L} \int_{-L}^{L} f(x) \cdot \exp\left(\frac{-\pi \mathbf{i} n x}{L} \right) \; dx} \qquad (9.5)$$

The **(complex) Fourier Series** of f is then the infinite summation of functions:

$$\boxed{\sum_{n=-\infty}^{\infty} \widehat{f}_n \cdot \mathbf{E}_n} \qquad (9.6)$$

(note that in this sum, n ranges from $-\infty$ to ∞).

Theorem 9.14: Complex Fourier Convergence

(a) *The set* $\{\ldots, \mathbf{E}_{-1}, \mathbf{E}_0, \mathbf{E}_1, \ldots\}$ *is an* orthonormal basis *for* $\mathbf{L}^2([-L, L]; \mathbb{C})$. *Thus, if* $f \in \mathbf{L}^2([-L, L]; \mathbb{C})$, *then the complex Fourier series* (9.6) *converges to* f *in* $\mathbf{L}^2$*-norm.*

Furthermore, $\{\widehat{f}_n\}_{n=-\infty}^{\infty}$ *is the* unique *sequence of coefficients with this property.*

(b) *If* f *is continuously differentiable*[2] *on* $[-\pi, \pi]$, *then the Fourier series* (9.6) *converges* pointwise *on* $(-\pi, \pi)$. *In other words, if* $-\pi < x < \pi$, *then* $f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \widehat{f}_n \mathbf{E}_n(x)$.

(c) *If* f *is continuously differentiable on* $[-\pi, \pi]$, *and* f *satisfies* periodic *boundary conditions* [i.e. $f(-\pi) = f(\pi)$ and $f'(-\pi) = f'(\pi)$], *then the series* (9.6) *converges to* f uniformly *on* $[-\pi, \pi]$.

(d) $\Big($ *The series* (9.6) *converges to* f uniformly *on* $[-\pi, \pi]$ $\Big) \iff \left(\sum_{n=-\infty}^{\infty} \left|\widehat{f}_n\right| < \infty \right)$.

Proof: **Exercise 9.12** Hint: use Theorem 9.1 on page 169, along with the equations (9.4). □

9.5 (∗) Relation between Real and Complex Fourier Coefficients

Prerequisites: §9.1, §9.4

If $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ is a real-valued function, then f can also be regarded as a complex valued function, and we can evaluate its complex Fourier series. Suppose $n > 0$. Then:

(a) $\widehat{f}_n = \frac{1}{2}(A_n - \mathbf{i}B_n)$, and $\widehat{f}_{-n} = \overline{\widehat{f}_n} = \frac{1}{2}(A_n + \mathbf{i}B_n)$.

(b) Thus, $A_n = \widehat{f}_n + \widehat{f}_{-n}$, and $B_n = \mathbf{i}(\widehat{f}_{-n} - \widehat{f}_n)$.

(c) $\widehat{f}_0 = A_0$.

Proof: **Exercise 9.13** Hint: use the equations (9.4). □

[2] This means that $f(x) = f_r(x) + \mathbf{i}f_i(x)$, where $f_r : [-L, L] \longrightarrow \mathbb{R}$ and $f_i : [-L, L] \longrightarrow \mathbb{R}$ are both continuously differentiable, real-valued functions.

Notes: ....

10 Multidimensional Fourier Series

10.1 ...in two dimensions

Prerequisites: §7.4, §7.5 **Recommended:** §8.2

Let $X, Y > 0$, and let $\mathbb{X} := [0, X] \times [0, Y]$ be an $X \times Y$ rectangle in the plane. Suppose $f : [0, X] \times [0, Y] \longrightarrow \mathbb{R}$ is a real-valued function of two variables. For all $n, m \in \mathbb{N}$ (both nonzero), we define the **two-dimensional Fourier sine coefficients:**

$$B_{n,m} = \boxed{\frac{4}{XY}\int_0^X\int_0^Y f(x,y)\sin\left(\frac{\pi n x}{X}\right)\sin\left(\frac{\pi m y}{Y}\right)\,dx\,dy}$$

The **two-dimensional Fourier sine series** of f is the doubly infinite summation:

$$\boxed{\sum_{n,m=1}^{\infty} B_{n,m}\sin\left(\frac{\pi n x}{X}\right)\sin\left(\frac{\pi m y}{Y}\right)} \qquad (10.1)$$

Notice that we are now summing over *two* independent indices, n and m.

Example 10.1: Let $X = \pi = Y$, so that $\mathbb{X} = [0, \pi] \times [0, \pi]$, and let $f(x, y) = x \cdot y$. Then f has two-dimensional Fourier sine series:

$$4\sum_{n,m=1}^{\infty}\frac{(-1)^{n+m}}{nm}\sin(nx)\sin(my).$$

To see this, recall from By Example 8.12(c) on page 157, we know that

$$\frac{2}{\pi}\int_0^{\pi} x\sin(x)\ dx = \frac{2(-1)^{n+1}}{n}$$

$$\text{Thus,}\quad B_{n,m} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} xy\cdot\sin(nx)\sin(my)\ dx\ dy$$

$$= \left(\frac{2}{\pi}\int_0^{\pi} x\sin(nx)\ dx\right)\cdot\left(\frac{2}{\pi}\int_0^{\pi} y\sin(my)\ dy\right)$$

$$= \left(\frac{2(-1)^{n+1}}{n}\right)\cdot\left(\frac{2(-1)^{m+1}}{m}\right) = \frac{4(-1)^{m+n}}{nm}. \qquad \diamondsuit$$

Example 10.2:

Let $X = \pi = Y$, so that $\mathbb{X} = [0, \pi] \times [0, \pi]$, and let $f(x, y) = 1$ be the constant 1 function. Then f has two-dimensional Fourier sine series:

$$\frac{4}{\pi^2}\sum_{n,m=1}^{\infty}\frac{[1-(-1)^n]}{n}\frac{[1-(-1)^m]}{m}\sin(nx)\sin(my) = \frac{16}{\pi^2}\sum_{\substack{n,m=1\\ \text{both odd}}}^{\infty}\frac{1}{n\cdot m}\sin(nx)\sin(my)$$

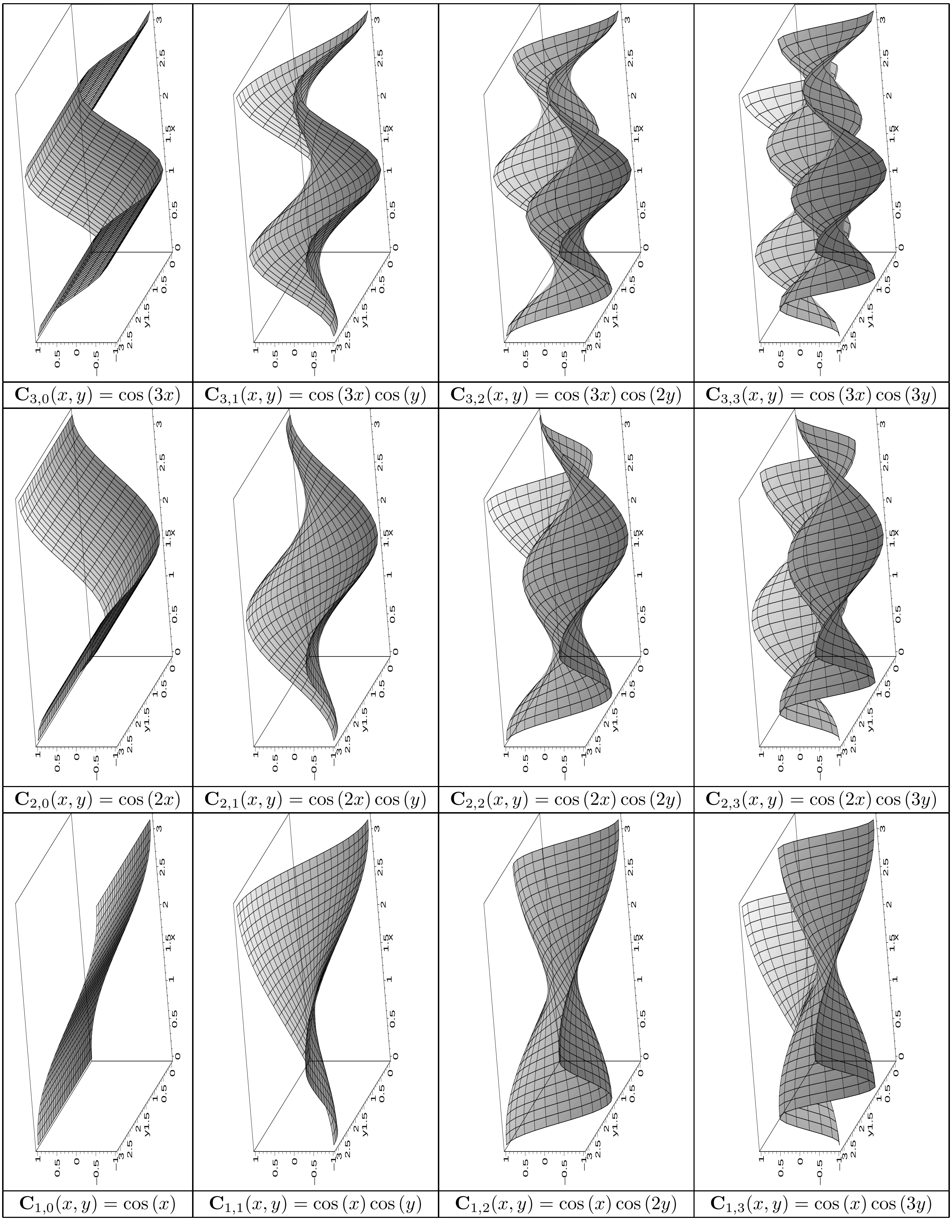

Figure 10.1: $\mathbf{C}_{n,m}$ for $n = 1..3$ and $m = 0..3$ (rotate page).

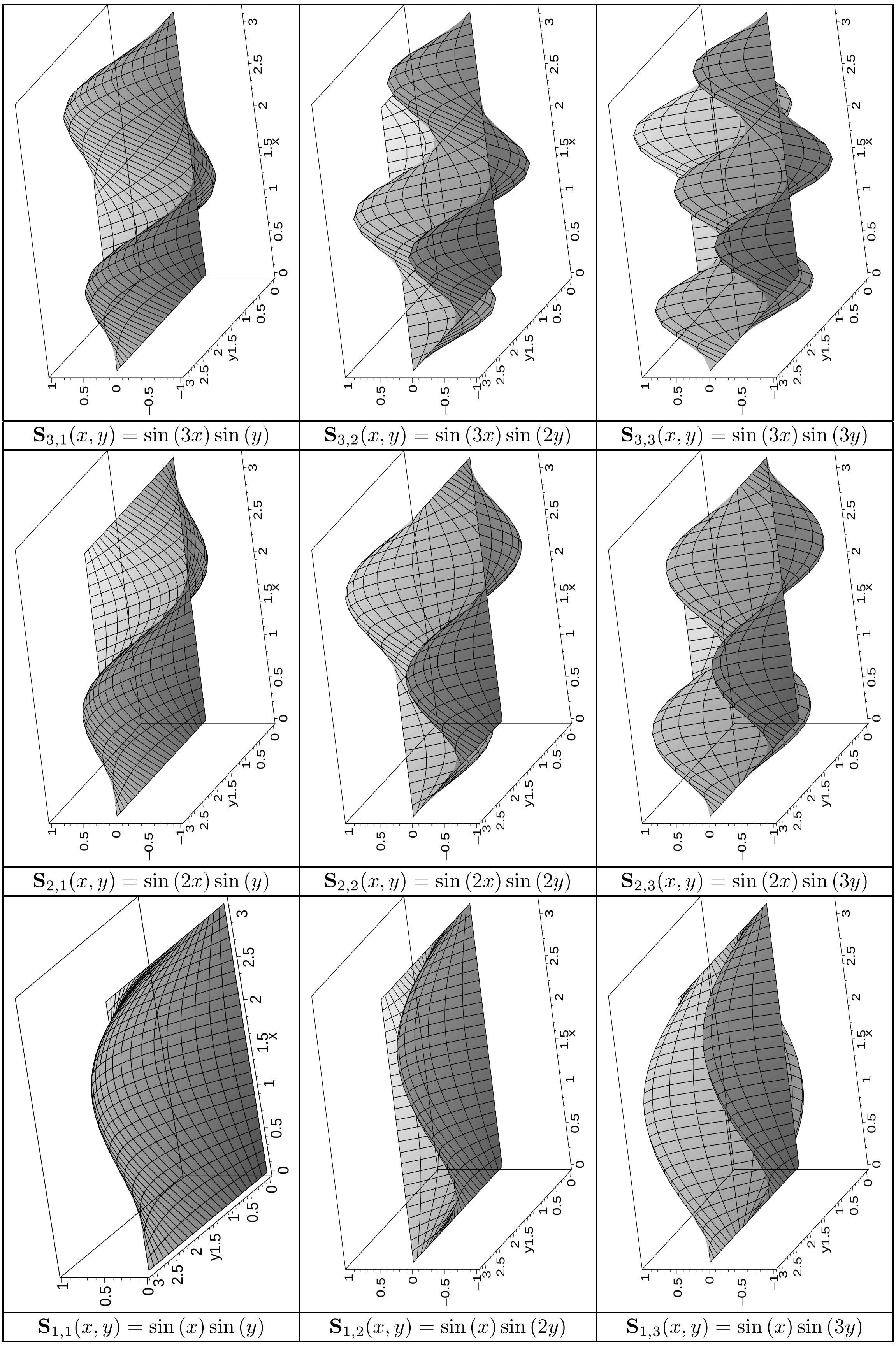

Figure 10.2: $\mathbf{S}_{n,m}$ for $n = 1..3$ and $m = 1..3$ (rotate page).

Exercise 10.1 Verify this. ♢

For all $n, m \in \mathbb{N}$ (possibly zero), we define the **two-dimensional Fourier cosine coefficients** of f:

$$A_0 = \boxed{\frac{1}{XY}\int_0^X\int_0^Y f(x,y)\,dx\,dy}$$

$$A_{n,0} = \boxed{\frac{2}{XY}\int_0^X\int_0^Y f(x,y)\cos\left(\frac{\pi n x}{X}\right)\,dx\,dy} \quad \text{for } n>0;$$

$$A_{0,m} = \boxed{\frac{2}{XY}\int_0^X\int_0^Y f(x,y)\cos\left(\frac{\pi m y}{X}\right)\,dx\,dy} \quad \text{for } m>0; \text{ and}$$

$$A_{n,m} = \boxed{\frac{4}{XY}\int_0^X\int_0^Y f(x,y)\cos\left(\frac{\pi n x}{X}\right)\cos\left(\frac{\pi m y}{Y}\right)\,dx\,dy} \quad \text{for } n,m>0.$$

The **two-dimensional Fourier cosine series** of f is the doubly infinite summation:

$$\boxed{\sum_{n,m=0}^{\infty} A_{n,m}\cos\left(\frac{\pi n x}{X}\right)\cos\left(\frac{\pi m y}{Y}\right)} \tag{10.2}$$

In what sense do these series converge to f? For any $n, m \in \mathbb{N}$, define:

$$\mathbf{C}_{n,m}(x,y) = \boxed{\cos\left(\frac{\pi n x}{X}\right)\cdot\cos\left(\frac{\pi m y}{Y}\right)} \quad \text{and} \quad \mathbf{S}_{n,m}(x,y) = \boxed{\sin\left(\frac{\pi n x}{X}\right)\cdot\sin\left(\frac{\pi m y}{Y}\right)}$$

(see Figures 10.1 and 10.2)

Theorem 10.3: Two-dimensional Co/Sine Series Convergence on $[0,X]\times[0,Y]$

(a) *The set* $\{\mathbf{S}_{n,m}\ ;\ 0 \neq n,m \in \mathbb{N}\}$ *is an* _orthogonal basis_ *for* $\mathbf{L}^2([0,X]\times[0,Y])$.

(b) *The set* $\{\mathbf{C}_{n,m}\ ;\ n,m \in \mathbb{N}\}$ *is an* _orthogonal basis_ *for* $\mathbf{L}^2([0,X]\times[0,Y])$.

(c) *Thus, if* $f \in \mathbf{L}^2([0,X]\times[0,Y])$, *then the series* (10.1) *and* (10.2) *both converge to* f *in* $\mathbf{L}^2$*-norm. Furthermore, the coefficient sequences* $\{A_{n,m}\}_{n,m=0}^{\infty}$ *and* $\{B_{n,m}\}_{n,m=1}^{\infty}$ *are the* _unique_ *sequences of coefficients with this property.*

(d) *If* f *is continuously differentiable on* $[0,X]\times[0,Y]$, *then the two-dimensional Fourier cosine series* (10.2) *converges to to* f _uniformly_ (and hence, _pointwise_) *on* $[0,X]\times[0,Y]$.

(e) *If* f *is continuously differentiable on* $[0,X]\times[0,Y]$, *then the two-dimensional Fourier sine series* (10.1) *converges to to* f _pointwise_ *on* $(0,X)\times(0,Y)$. *Furthermore, in this case,*

$$\left(\begin{array}{l}\textit{The sine series (10.1) converges}\\ \textit{to } f \textit{ uniformly on } [0,X]\times[0,Y]\end{array}\right) \iff \left(\begin{array}{l} f \textit{ satisfies homogeneous Dirichlet boundary conditions:}\\ \quad f(x,0) = f(x,Y) = 0, \textit{ for all } x\in[0,X], \textit{ and}\\ \quad f(0,y) = f(X,y) = 0, \textit{ for all } y\in[0,Y].\end{array}\right)$$

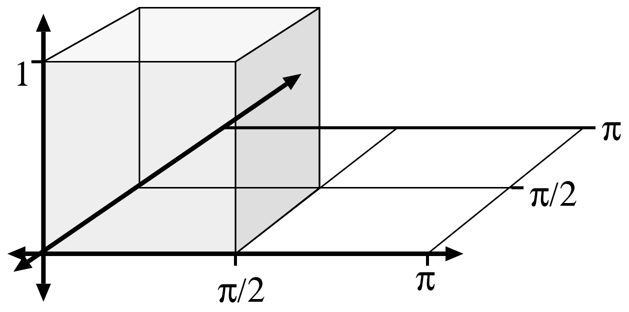

Figure 10.3: The box function $f(x,y)$ in Example 10.4.

(f) $\left(\text{ The cosine series (10.2) converges to } f \text{ \underline{uniformly} on } [0,X]\times[0,Y]\right) \iff \left(\sum_{n,m=0}^{\infty} |A_{n,m}| < \infty\right)$.

(g) $\left(\begin{array}{c} f \text{ satisfies \underline{homogeneous Neumann} boundary conditions:} \\ \partial_y f(x,0) = \partial_y f(x,Y) = 0, \text{ for all } x \in [0,X], \text{ and} \\ \partial_x f(0,y) = \partial_x f(X,y) = 0, \text{ for all } y \in [0,Y]. \end{array}\right)$

$$\iff \left(\sum_{n,m=0}^{\infty} n\cdot |A_{n,m}| < \infty \text{ and } \sum_{n,m=0}^{\infty} m\cdot |A_{n,m}| < \infty\right).$$

...and in this case, the cosine series (10.2) *converges uniformly to* f *on* $[0,X]\times[0,Y]$.

(h) $\left(\text{ The sine series (10.1) converges to } f \text{ \underline{uniformly} on } [0,X]\times[0,Y]\right) \iff \left(\sum_{n,m=1}^{\infty} |B_{n,m}| < \infty\right)$.

In this case, f *satisfies homogeneous Dirichlet boundary conditions.*

Proof: **(a,b,c)** See [Kat76, p.29 of §1.5]. **(d,e)** See [Fol84, Theorem 8.43, p.256]. **(f)** is **Exercise 10.2**. **(g)** is **Exercise 10.3**. **(h)** is **Exercise 10.4**. □

Example 10.4: Suppose $X = \pi = Y$, and $f(x,y) = \begin{cases} 1 & \text{if } 0 \le x < \frac{\pi}{2} \text{ and } 0 \le y < \frac{\pi}{2}; \\ 0 & \text{if } \frac{\pi}{2} \le x \text{ or } \frac{\pi}{2} \le y. \end{cases}$
(See Figure 10.3). Then the two-dimensional Fourier cosine series of f is:

$$\frac{1}{4} + \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\Big((2k+1)x\Big) + \frac{1}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\cos\Big((2j+1)y\Big)$$
$$+ \frac{4}{\pi^2}\sum_{k,j=0}^{\infty}\frac{(-1)^{k+j}}{(2k+1)(2j+1)}\cos\Big((2k+1)x\Big)\cdot\cos\Big((2j+1)y\Big)$$

To see this, note that $f(x,y) = g(x)\cdot g(y)$, where $g(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \le x \end{cases}$. Recall from Example 8.16 on page 162 that the (one-dimensional) Fourier cosine series of $g(x)$ is

$$g(x) \underset{L2}{\approx} \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\Big((2k+1)x\Big)$$

Thus, the cosine series for $f(x,y)$ is given:

$$\begin{aligned} f(x,y) &= g(x)\cdot g(y) \\ &\underset{L2}{\approx} \left[\frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\Big((2k+1)x\Big)\right]\cdot\left[\frac{1}{2} + \frac{2}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\cos\Big((2j+1)y\Big)\right]. \quad \diamondsuit \end{aligned}$$

(∗) Mixed series:

We can also define the **mixed Fourier sine/cosine coefficients:**

$$\begin{aligned} C_{n,0}^{[sc]} &= \frac{2}{XY}\int_0^X\int_0^Y f(x,y)\sin\left(\frac{\pi n x}{X}\right)\,dx\,dy, \quad \text{for } n>0. \\ C_{n,m}^{[sc]} &= \frac{4}{XY}\int_0^X\int_0^Y f(x,y)\sin\left(\frac{\pi n x}{X}\right)\cos\left(\frac{\pi m y}{Y}\right)\,dx\,dy, \quad \text{for } n,m>0. \\ C_{0,m}^{[cs]} &= \frac{2}{XY}\int_0^X\int_0^Y f(x,y)\sin\left(\frac{\pi m y}{Y}\right)\,dx\,dy, \quad \text{for } m>0. \\ C_{n,m}^{[cs]} &= \frac{4}{XY}\int_0^X\int_0^Y f(x,y)\cos\left(\frac{\pi n x}{X}\right)\sin\left(\frac{\pi m y}{Y}\right)\,dx\,dy, \quad \text{for } n,m>0. \end{aligned}$$

The **mixed Fourier sine/cosine series** of f are then:

$$\sum_{n=1,m=0}^{\infty} C_{n,m}^{[sc]}\sin\left(\frac{\pi n x}{X}\right)\cos\left(\frac{\pi m y}{Y}\right) \quad \text{and} \quad \sum_{n=0,m=1}^{\infty} C_{n,m}^{[cs]}\cos\left(\frac{\pi n x}{X}\right)\sin\left(\frac{\pi m y}{Y}\right) \tag{10.3}$$

Define

$$\mathbf{M}_{n,m}^{[sc]}(x,y) = \sin\left(\frac{\pi n_1 x}{X}\right)\cos\left(\frac{\pi n_2 y}{Y}\right) \quad \text{and} \quad \mathbf{M}_{n,m}^{[cs]}(x,y) = \cos\left(\frac{\pi n_1 x}{X}\right)\sin\left(\frac{\pi n_2 y}{Y}\right).$$

Proposition 10.5: Two-dimensional Mixed Co/Sine Series Convergence on $[0,X]\times[0,Y]$

The sets of "mixed" functions, $\left\{\mathbf{M}_{n,m}^{[sc]}\ ;\ n,m\in\mathbb{N}\right\}$ *and* $\left\{\mathbf{M}_{n,m}^{[cs]}\ ;\ n,m\in\mathbb{N}\right\}$ *are both* <u>*orthogonal basis*</u> *for* $\mathbf{L}^2([0,X]\times[0,Y])$. *In other words, if* $f\in\mathbf{L}^2([0,X]\times[0,Y])$, *then the series (10.3) both converge to* f *in* $\mathbf{L}^2$. □

<u>**Exercise 10.5**</u> Formulate conditions for pointwise and uniform convergence of the mixed series.

10.2 ...in many dimensions

Prerequisites: §7.4, §7.5 **Recommended:** §10.1

Let $X_1, \ldots, X_D > 0$, and let $\mathbb{X} := [0, X_1] \times \cdots \times [0, X_D]$ be an $X_1 \times \cdots \times X_D$ box in D-dimensional space. For any $\mathbf{n} \in \mathbb{N}$, and all $(x_1, \ldots, x_D) \in \mathbb{X}$, define:

$$\mathbf{C_n}(x_1, \ldots, x_D) = \boxed{\cos\left(\frac{\pi n_1 x_1}{X_1}\right) \cos\left(\frac{\pi n_2 x_2}{X_2}\right) \ldots \cos\left(\frac{\pi n_D x_D}{X_D}\right)}$$

$$\mathbf{S_n}(x_1, \ldots, x_D) = \boxed{\sin\left(\frac{\pi n_1 x_1}{X_1}\right) \sin\left(\frac{\pi n_2 x_2}{X_2}\right) \ldots \sin\left(\frac{\pi n_D x_D}{X_D}\right).}$$

Also, for any sequence $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_D)$ of D symbols "s" and "c", we can define the "mixed" functions, $\mathbf{M}_\mathbf{n}^{\boldsymbol{\omega}}$. For example, if $D = 3$, then define

$$\mathbf{M}_\mathbf{n}^{[scs]}(x, y, z) = \sin\left(\frac{\pi n_1 x}{X_x}\right) \cos\left(\frac{\pi n_2 y}{X_y}\right) \sin\left(\frac{\pi n_3 z}{X_z}\right)$$

If $f : \mathbb{X} \longrightarrow \mathbb{R}$ is any function with $\|f\|_2 < \infty$, then, for all $\mathbf{n} \in \mathbb{N}$, we define the **multiple Fourier sine coefficients:**

$$B_\mathbf{n} = \frac{\langle f, \mathbf{S_n} \rangle}{\|\mathbf{S_n}\|_2^2} = \boxed{\frac{2^D}{X_1 \cdots X_D} \int_\mathbb{X} f(\mathbf{x}) \cdot \mathbf{S}_n(\mathbf{x})\, d\mathbf{x}}$$

The **multiple Fourier sine series** of f is then:

$$\boxed{\sum_{\mathbf{n} \in \mathbb{N}^D} B_\mathbf{n} \mathbf{S_n}} \tag{10.4}$$

For all $\mathbf{n} \in \mathbb{N}$, we define the **multiple Fourier cosine coefficients:**

$$A_0 = \langle f, \mathbb{1} \rangle = \boxed{\frac{1}{X_1 \cdots X_D} \int_\mathbb{X} f(\mathbf{x})\, d\mathbf{x}} \quad \text{and} \quad A_\mathbf{n} = \frac{\langle f, \mathbf{C_n} \rangle}{\|\mathbf{C_n}\|_2^2} = \boxed{\frac{2^{d_\mathbf{n}}}{X_1 \cdots X_D} \int_\mathbb{X} f(\mathbf{x}) \cdot \mathbf{C}_n(\mathbf{x})\, d\mathbf{x}}$$

where, for each $\mathbf{n} \in \mathbb{N}$, the number $d_\mathbf{n}$ is the number of nonzero entries in $\mathbf{n} = (n_1, n_2, \ldots, n_D)$. The **multiple Fourier cosine series** of f is then:

$$\boxed{\sum_{\mathbf{n} \in \mathbb{N}^D} A_\mathbf{n} \mathbf{C_n}} \tag{10.5}$$

Finally, we define the **mixed Fourier Sine/Cosine coefficients:**

$$C_\mathbf{n}^{\boldsymbol{\omega}} = \frac{\langle f, \mathbf{M}_\mathbf{n}^{\boldsymbol{\omega}} \rangle}{\|\mathbf{M}_\mathbf{n}^{\omega}\|_2^2} = \frac{2^{d_\mathbf{n}}}{X_1 \cdots X_D} \int_\mathbb{X} f(\mathbf{x}) \cdot \mathbf{M}_n^{\boldsymbol{\omega}}(\mathbf{x})\, d\mathbf{x}$$

where , for each $\mathbf{n} \in \mathbb{N}$, the number $d_\mathbf{n}$ is the number of nonzero entries n_i in $\mathbf{n} = (n_1, \ldots, n_D)$ such that $\omega_i = c$. The **mixed Fourier Sine/Cosine series** of f is then:

$$\sum_{\mathbf{n} \in \mathbb{N}^D} C_\mathbf{n}^{\boldsymbol{\omega}} \mathbf{M}_\mathbf{n}^{\boldsymbol{\omega}} \tag{10.6}$$

Theorem 10.6: Multidimensional Co/Sine Series Convergence on $\mathbb{X}$

(a) *The set* $\{\mathbf{S_n} \; ; \; 0 \neq \mathbf{n} \in \mathbb{N}^D\}$ *is an orthogonal basis for* $\mathbf{L}^2(\mathbb{X})$.

(b) *The set* $\{\mathbf{C_n} \; ; \; \mathbf{n} \in \mathbb{N}^D\}$ *is an orthogonal basis for* $\mathbf{L}^2(\mathbb{X})$.

(c) *For any sequence* $\boldsymbol{\omega}$ *of* D *symbols "s" and "c", the set of "mixed" functions,* $\{\mathbf{M_n^{\boldsymbol{\omega}}} \; ; \; \mathbf{n} \in \mathbb{N}^D\}$ *is an orthogonal basis for* $\mathbf{L}^2(\mathbb{X})$.

(d) *In other words, if* $f \in \mathbf{L}^2(\mathbb{X})$, *then the series* (10.4), (10.5), *and* (10.6) *all converge to* f *in* $\mathbf{L}^2$*-norm. Furthermore, the coefficient sequences* $\{A_\mathbf{n}\}_{\mathbf{n}\in\mathbb{N}^D}$, $\{B_\mathbf{n}\}_{0\neq\mathbf{n}\in\mathbb{N}^D}$, *and* $\{C_\mathbf{n}^{\boldsymbol{\omega}}\}_{\mathbf{n}\in\mathbb{N}^D}$ *are the unique sequences of coefficients with these properties.*

(e) *If* f *is continuously differentiable on* $\mathbb{X}$, *then the cosine series* (10.5) *converges uniformly (and hence pointwise) to* f *on* $\mathbb{X}$.

(f) *If* f *is continuously differentiable on* $\mathbb{X}$, *then the sine series* (10.4) *converges pointwise to* f *on the interior of* $\mathbb{X}$. *Furthermore, in this case,*

$$\left(\begin{array}{l}\text{The sine series (10.4) converges}\\ \text{to } f \text{ uniformly on } \mathbb{X}\end{array}\right) \iff \left(\begin{array}{l} f \text{ satisfies homogeneous Dirichlet}\\ \text{boundary conditions on } \partial\mathbb{X}\end{array}\right).$$

(g) *If* f *is continuously differentiable on* $\mathbb{X}$, *then the mixed series* (10.6) *converges pointwise to* f *on the interior of* $\mathbb{X}$. *Furthermore, in this case,*

$$\left(\begin{array}{l}\text{The mixed series (10.6) converges}\\ \text{to } f \text{ uniformly on } \mathbb{X}\end{array}\right) \iff \left(\begin{array}{l} f \text{ satisfies homogeneous Dirichlet boundary}\\ \text{conditions on the } i\text{th face of } \partial\mathbb{X}, \text{ for any}\\ i \in [1...D] \text{ with } \omega_i = s.\end{array}\right).$$

(h) $\left(f \text{ satisfies homogeneous Neumann boundary conditions on } \partial\mathbb{X} \right)$

$$\iff \left(\text{For all } d \in [1...D], \text{ we have } \sum_{\mathbf{n}\in\mathbb{N}} n_d \cdot |A_\mathbf{n}| \; < \; \infty \right)$$

...and in this case, the cosine series (10.5) *converges uniformly to* f *on* $\mathbb{X}$.

Proof: **(a,b,c,d)** See [Kat76, p.29 of §1.5]. **(e,f,g)** follow from [Fol84, Theorem 8.43, p.256]. □

Proposition 10.7: Differentiating Multiple Fourier (co)sine series

Let $\mathbb{X} := [0, X_1] \times \cdots \times [0, X_D]$. *Let* $f : \mathbb{X} \longrightarrow \mathbb{R}$ *be differentiable, with Fourier series* $f \underset{\text{unif}}{\equiv} A_0 + \sum_{\mathbf{n}\in\mathbb{N}^D} A_\mathbf{n}\mathbf{C_n} + \sum_{\mathbf{n}\in\mathbb{N}^D} B_\mathbf{n}\mathbf{S_n}$.

(a) *Fix $i \in [1..D]$. Suppose that* $\sum_{\mathbf{n}\in\mathbb{N}^D} n_i^2 |A_\mathbf{n}| + \sum_{\mathbf{n}\in\mathbb{N}^D} n_i^2 |B_\mathbf{n}| < \infty$. *Then*

$$\partial_i^2 f \underset{L2}{\approx} \sum_{\mathbf{n}\in\mathbb{N}^D} -\left(\frac{\pi n_i}{X_i}\right)^2 \cdot \Big(A_\mathbf{n}\mathbf{C_n} + B_\mathbf{n}\mathbf{S_n}\Big).$$

(b) *Suppose that* $\sum_{\mathbf{n}\in\mathbb{N}^D} |\mathbf{n}|^2 |A_\mathbf{n}| + \sum_{\mathbf{n}\in\mathbb{N}^D} |\mathbf{n}|^2 |B_\mathbf{n}| < \infty$ (where we define $|\mathbf{n}|^2 := n_1^2 + ... + n_D^2$).

Then $$\triangle f \underset{L2}{\approx} -\pi^2 \sum_{\mathbf{n}\in\mathbb{N}^D} \left[\left(\frac{n_1}{X_1}\right)^2 + \cdots + \left(\frac{n_D}{X_D}\right)^2\right] \cdot \Big(A_\mathbf{n}\mathbf{C_n} + B_\mathbf{n}\mathbf{S_n}\Big).$$

Proof: **Exercise 10.6** Hint: Apply Proposition 1.7 on page 16 □

Example 10.8: Fix $\mathbf{n} \in \mathbb{N}^D$. If $f = A \cdot \mathbf{C_n} + B \cdot \mathbf{S_n}$, then

$$\triangle f = -\pi^2 \left[\left(\frac{n_1}{X_1}\right)^2 + \cdots + \left(\frac{n_D}{X_D}\right)^2\right] \cdot f.$$

In particular, if $X_1 = \cdots = X_D = \pi$, then this simplifies to:

$$\triangle f = -\pi^2 |\mathbf{n}|^2 \cdot f.$$

In other words, any pure wave function with wave vector $\mathbf{n} = (n_1, \ldots, n_D)$ is an eigenfunction of the Laplacian operator, with eigenvalue $\lambda = -\pi^2|\mathbf{n}|^2$. ♢

10.3 Practice Problems

Compute the two-dimensional Fourier sine transforms of the following functions. For each question, also determine: at which points does the series converge pointwise? Why? Does the series converge uniformly? Why or why not?

1. $f(x,y) = x^2 \cdot y$.

2. $g(x,y) = x + y$.

3. $f(x,y) = \cos(Nx) \cdot \cos(My)$, for some integers $M, N > 0$.

4. $f(x,y) = \sin(Nx) \cdot \sinh(Ny)$, for some integer $N > 0$.

Notes: ....

IV BVPs in Cartesian Coordinates

Fourier theory is relevant to boundary value problems because the orthogonal trigonometric functions $\mathbf{S}_n$ and $\mathbf{C}_n$ in a Fourier series are *eigenfunctions* of the Laplacian operator $\triangle$. Thus, we can use these functions as 'building blocks' to construct a solution to a given partial differential equation —a solution which also satisfies specified initial conditions and/or boundary conditions. In particular, we will use Fourier sine series to obtain homogeneous *Dirichlet* boundary conditions [by Theorems 8.1(d), 8.7(d), 10.3(e) and 10.6(f)] , and Fourier cosine series to obtain homogeneous *Neumann* boundary conditions [by Theorems 8.4(f), 8.9(f), 10.3(g) and 10.6(h)]. This basic strategy underlies all the solution methods developed in Chapters 11 to 13, and many of the methods of Chapter 14.

11 Boundary Value Problems on a Line Segment

Prerequisites: §8.1, §6.5

Fourier series can be used to find solutions to boundary value problems on the line segment $[0, L]$. The key idea is that the functions $\mathbf{S}_n(x) = \sin\left(\frac{n\pi}{L}x\right)$ and $\mathbf{C}_n(x) = \cos\left(\frac{n\pi}{L}x\right)$ are *eigenfunctions* of the Laplacian operator. Furthermore, $\mathbf{S}_n$ satisfies *Dirichlet* boundary conditions, so any (uniformly convergent) Fourier sine series will also. Likewise, $\mathbf{C}_n$ satisfies *Neumann* boundary conditions, so any (uniformly convergent) Fourier cosine series will also.

For simplicity, we will assume throughout that $L = \pi$. Thus $\mathbf{S}_n(x) = \sin(nx)$ and $\mathbf{C}_n(x) = \cos(nx)$. We will also assume that the physical constants in the various equations are all set to one. Thus, the Heat Equation becomes "$\partial_t u = \triangle u$", the Wave Equation is "$\partial_t^2 u = \triangle u$", etc.

This does not limit the generality of our results. For example, faced with a general heat equation of the form "$\partial_t u(x,t) \;=\; \kappa \cdot \triangle u$" for $x \in [0, L]$, (with $\kappa \neq 1$ and $L \neq \pi$) you can simply replace the coordinate x with a new space coordinate $y = \frac{\pi}{L}x$ and replace t with a new time coordinate $s = \kappa t$, to reformulate the problem in a way compatible with the following methods.

11.1 The Heat Equation on a Line Segment

Prerequisites: §8.2, §6.4, §6.5, §2.2(a),§1.7 **Recommended:** §8.3(e)

Proposition 11.1: (Heat Equation; homogeneous Dirichlet boundary)

Let $\mathbb{X} = [0, \pi]$, and let $f \in \mathbf{L}^2[0, \pi]$ be some function describing an initial heat distribution. Suppose f has Fourier Sine Series $f(x) \underset{\text{L2}}{\widetilde{\approx}} \sum_{n=1}^{\infty} B_n \sin(nx)$, *and define:*

$$u(x;t) \quad \underset{\text{L2}}{\widetilde{\approx}} \quad \sum_{n=1}^{\infty} B_n \sin(nx) \cdot \exp\Big(-n^2 \cdot t\Big), \qquad \text{for all } x \in [0, \pi] \text{ and } t \geq 0.$$

Then $u(x;t)$ is the unique solution to the one-dimensional Heat Equation "$\partial_t u = \partial_x^2 u$", with homogeneous **Dirichlet** *boundary conditions*

$$u(0;t) = u(\pi;t) \;=\; 0, \qquad \text{for all } t > 0.$$

and initial conditions: $u(x;0) = f(x)$, for all $x \in [0, \pi]$.

Furthermore, the series defining u converges semiuniformly on $\mathbb{X} \times (0, \infty)$.

Proof: **Exercise 11.1** Hint:

(a) Show that, when $t = 0$, the Fourier series of $u(x;0)$ agrees with that of $f(x)$; hence $u(x;0) = f(x)$.

(b) Show that, for all $t > 0$, $\displaystyle\sum_{n=1}^{\infty} \left| n^2 \cdot B_n \cdot e^{-n^2 t} \right| \;<\; \infty.$

(c) For any $T > 0$, apply Proposition 1.7 on page 16 to conclude that $\partial_t u(x;t) \underset{\text{unif}}{\equiv} \sum_{n=1}^{\infty} -n^2 B_n \sin(nx) \cdot \exp\left(-n^2 \cdot t\right) \underset{\text{unif}}{\equiv} \triangle u(x;t)$ on $[T, \infty)$.

(d) Observe that for all $t > 0$, $\sum_{n=1}^{\infty} \left| B_n e^{-n^2 t} \right| < \infty$.

(e) Apply part (c) of Theorem 8.1 on page 145 to show that the Fourier series of $u(x;t)$ converges uniformly for all $t > 0$.

(f) Apply part (d) of Theorem 8.1 on page 145 to conclude that $u(0;t) = 0 = u(\pi, t)$ for all $t > 0$.

(g) Apply Theorem 6.16(a) on page 107 to show that this solution is unique. □

Example 11.2: Consider a metal rod of length π, with initial temperature distribution $f(x) = \tau \cdot \sinh(\alpha x)$ (where $\tau, \alpha > 0$ are constants), and homogeneous Dirichlet boundary condition. Proposition 11.1 tells us to get the Fourier sine series for $f(x)$. In Example 8.3 on page 148, we computed this to be $\frac{2\tau \sinh(\alpha\pi)}{\pi} \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{\alpha^2 + n^2} \cdot \sin(nx)$. The evolving temperature distribution is therefore given:

$$u(x;t) = \frac{2\tau \sinh(\alpha\pi)}{\pi} \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{\alpha^2 + n^2} \cdot \sin(nx) \cdot e^{-n^2 t}.$$ ♢

Proposition 11.3: (Heat Equation; homogeneous Neumann boundary)

Let $\mathbb{X} = [0, \pi]$, and let $f \in \mathbf{L}^2[0, \pi]$ be some function describing an initial heat distribution. Suppose f has Fourier Cosine Series $f(x) \underset{L2}{\approx} \sum_{n=0}^{\infty} A_n \cos(nx)$, and define:

$$u(x;t) \underset{L2}{\approx} \sum_{n=0}^{\infty} A_n \cos(nx) \cdot \exp\left(-n^2 \cdot t\right), \qquad \text{for all } x \in [0, \pi] \text{ and } t \geq 0.$$

Then $u(x;t)$ is the unique solution to the one-dimensional Heat Equation "$\partial_t u = \partial_x^2 u$", with homogeneous **Neumann** *boundary conditions*

$$\partial_x u(0;t) = \partial_x u(\pi;t) = 0, \qquad \text{for all } t > 0.$$

and initial conditions: $u(x;0) = f(x)$, for all $x \in [0, \pi]$.

Furthermore, the series defining u converges semiuniformly on $\mathbb{X} \times (0, \infty)$.

Proof: Setting $t = 0$, we get:

$$u(x;t) = \sum_{n=1}^{\infty} A_n \cos(nx) \cdot \exp\left(-n^2 \cdot 0\right). = \sum_{n=1}^{\infty} A_n \cos(nx) \cdot \exp(0)$$
$$= \sum_{n=1}^{\infty} A_n \cos(nx) \cdot 1 = \sum_{n=1}^{\infty} A_n \cos(nx) = f(x),$$

so we have the desired initial conditions.

Let $M := \max_{n\in\mathbb{N}} |A_n|$. Then $M < \infty$ (because $f \in \mathbf{L}^2$).

Claim 1: *For all $t > 0$,* $\sum_{n=0}^{\infty} \left| n^2 \cdot A_n \cdot e^{-n^2 t} \right| < \infty.$

Proof: Since $M = \max_{n\in\mathbb{N}} |A_n|$, we know that $|A_n| < M$ for all $n \in \mathbb{N}$. Thus,

$$\sum_{n=0}^{\infty} \left| n^2 \cdot A_n \cdot e^{-n^2 t} \right| \leq \sum_{n=0}^{\infty} \left| n^2 \right| \cdot M \cdot \left| e^{-n^2 t} \right| = M \cdot \sum_{n=0}^{\infty} n^2 \cdot e^{-n^2 t}$$

Hence,it suffices to show that $\sum_{n=0}^{\infty} n^2 \cdot e^{-n^2 t} < \infty$. To see this, let $E = e^t$. Then $E > 1$ (because $t > 0$). Also, $n^2 \cdot e^{-n^2 t} = \frac{n^2}{E^{n^2}}$, for each $n \in \mathbb{N}$. Thus,

$$\sum_{n=1}^{\infty} n^2 e^{-n^2 t} = \sum_{n=1}^{\infty} \frac{n^2}{E^{n^2}} \leq \sum_{m=1}^{\infty} \frac{m}{E^m} \tag{11.1}$$

We must show that right-hand series in (11.1) converges. We apply the Ratio Test:

$$\lim_{m\to\infty} \frac{\frac{m+1}{E^{m+1}}}{\frac{m}{E^m}} = \lim_{m\to\infty} \frac{m+1}{m} \frac{E^m}{E^{m+1}} = \lim_{m\to\infty} \frac{1}{E} < 1.$$

Hence the right-hand series in (11.1) converges. $\diamond_{\text{Claim 1}}$

Claim 2: *For any $T > 0$, we have* $\partial_x u(x;t) \underset{\text{unif}}{\equiv} -\sum_{n=1}^{\infty} nA_n \sin(nx) \cdot \exp\left(-n^2 \cdot t\right)$ *on* $[T, \infty)$, *and also* $\partial_x^2 u(x;t) \underset{\text{unif}}{\equiv} -\sum_{n=1}^{\infty} n^2 A_n \cos(nx) \cdot \exp\left(-n^2 \cdot t\right)$ *on* $[T, \infty)$.

Proof: This follows from Claim 1 and two applications of Proposition 1.7 on page 16. $\diamond_{\text{Claim 2}}$

Claim 3: *For any $T > 0$, we have* $\partial_t u(x;t) \underset{\text{unif}}{\equiv} -\sum_{n=1}^{\infty} n^2 A_n \cos(nx) \cdot \exp\left(-n^2 \cdot t\right)$ *on* $[T, \infty)$.

Proof:

$$\begin{aligned} \partial_t u(x;t) &= \partial_t \sum_{n=1}^{\infty} A_n \cos(nx) \cdot \exp\left(-n^2 \cdot t\right) \underset{(*)}{=} \sum_{n=1}^{\infty} A_n \cos(nx) \cdot \partial_t \exp\left(-n^2 \cdot t\right) \\ &= \sum_{n=1}^{\infty} A_n \cos(nx) \cdot (-n^2) \exp\left(-n^2 \cdot t\right), \end{aligned}$$

where $(*)$ is by Prop. 1.7 on page 16. $\diamond_{\text{Claim 3}}$

Combining Claims 2 and 3, we conclude that $\partial_t u(x;t) = \triangle u(x;t)$.

Finally Claim 1 also implies that, for any $t > 0$,

$$\sum_{n=0}^{\infty} \left| n \cdot A_n \cdot e^{-n^2 t} \right| \quad < \quad \sum_{n=0}^{\infty} \left| n^2 \cdot A_n \cdot e^{-n^2 t} \right| \quad < \quad \infty.$$

Hence, Theorem 8.4(d) on p.149 implies that $u(x;t)$ satisfies homogeneous Neumann boundary conditions for any $t > 0$.

Finally, Theorem 6.16(b) on page 107 says this solution is unique. □

Example 11.4: Consider a metal rod of length π, with initial temperature distribution $f(x) = \cosh(x)$ and homogeneous Neumann boundary condition. Proposition 11.3 tells us to get the Fourier cosine series for $f(x)$. In Example 8.6 on page 150, we computed this to be $\dfrac{\sinh(\pi)}{\pi} + \dfrac{2\sinh(\pi)}{\pi} \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n \cdot \cos(nx)}{n^2+1}$. The evolving temperature distribution is therefore given:

$$u(x;t) \underset{L2}{\approx} \frac{\sinh(\pi)}{\pi} + \frac{2\sinh(\pi)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \cdot \cos(nx)}{n^2+1} \cdot e^{-n^2 t}. \qquad \diamondsuit$$

Exercise 11.2 Let $L > 0$ and let $\mathbb{X} := [0, L]$. Let $\kappa > 0$ be a diffusion constant, and consider the general one-dimensional Heat Equation

$$\partial_t u \quad = \quad \kappa \, \partial_x^2 u. \tag{11.2}$$

(a) Generalize Proposition 11.1 to find the solution to eqn.(11.2) on $\mathbb{X}$ satisfying prescribed initial conditions and homogeneous Dirichlet boundary conditions.

(b) Generalize Proposition 11.3 to find the solution to eqn.(11.2) on $\mathbb{X}$ satisfying prescribed initial conditions and homogeneous Neumann boundary conditions.

In both cases, prove that your solution converges, satisfies the desired initial conditions and boundary conditions, and satisfies eqn.(11.2) (Hint: imitate the strategy suggested in Exercise 11.1)

Exercise 11.3 Let $\mathbb{X} = [0, \pi]$, and let $f \in \mathbf{L}^2(\mathbb{X})$ be a function whose Fourier sine series satisfies $\displaystyle\sum_{n=1}^{\infty} n^2 |B_n| < \infty$. Imitate Proposition 11.1, to find a 'Fourier series' solution to the initial value problem for the one-dimensional *free Schrödinger equation*

$$\mathbf{i}\partial_t \,\omega \quad = \quad \frac{-1}{2} \partial_x^2 \,\omega, \tag{11.3}$$

on $\mathbb{X}$, with initial conditions $\omega_0 = f$, and satisfying homogeneous Dirichlet boundary conditions. Prove that your solution converges, satisfies the desired initial conditions and boundary conditions, and satisfies eqn.(11.3). (Hint: imitate the strategy suggested in Exercise 11.1).

11.2 The Wave Equation on a Line (The Vibrating String)

Prerequisites: §8.2(a), §6.4, §6.5, §3.2(a) **Recommended:** §16.6(b)

Imagine a violin string stretched tightly between two points. At equilibrium, the string is perfectly flat, but if we pluck or strike the string, it will vibrate, meaning there will be a vertical displacement from equilibrium. Let $\mathbb{X} = [0, \pi]$ represent the string, and for any point $x \in \mathbb{X}$ on the string and time $t > 0$, let $u(x;t)$ be the vertical displacement of the drum. Then u will obey the two-dimensional Wave Equation:

$$\partial_t^2 u(x;t) \;=\; \triangle u(x;t). \tag{11.4}$$

However, since the string is fixed at its endpoints, the function u will also exhibit homogeneous **Dirichlet** boundary conditions

$$u(0;t) = u(\pi;t) \;=\; 0 \qquad \text{(for all } t > 0). \tag{11.5}$$

Proposition 11.5: (Initial Position Problem for Vibrating String with fixed endpoints)

$f_0 : \mathbb{X} \longrightarrow \mathbb{R}$ be a function describing the initial displacement of the string. Suppose f_0 has Fourier Sine Series $f_0(x) \underset{L2}{\approx} \sum_{n=1}^{\infty} B_n \sin(nx)$, *and define:*

$$w(x;t) \underset{L2}{\approx} \sum_{n=1}^{\infty} B_n \sin(nx) \cdot \cos(nt), \qquad \text{for all } x \in [0,\pi] \text{ and } t \geq 0. \tag{11.6}$$

Then $w(x;t)$ is the unique solution to the Wave Equation (11.4), *satisfying the Dirichlet boundary conditions* (11.5), *as well as*

$$\left.\begin{array}{lrcl} \textbf{Initial Position:} & w(x,0) & = & f_0(x), \\ \textbf{Initial Velocity:} & \partial_t w(x,0) & = & 0, \end{array}\right\} \quad \text{for all } x \in [0,\pi].$$

Proof: <u>**Exercise 11.4**</u> Hint:

(a) Prove the trigonometric identity $\sin(nx)\cos(nt) \;=\; \frac{1}{2}\Big(\sin(n(x-t)) + \sin(n(x+t))\Big)$.

(b) Use this identity to show that the Fourier sine series (11.6) converges in $\mathbf{L}^2$ to the d'Alembert solution from Theorem 16.28(a) on page 329.

(c) Apply Theorem 6.18 on page 108 to show that this solution is unique. □

Example 11.6: Let $f_0(x) = \sin(5x)$. Thus, $B_5 = 1$ and $B_n = 0$ for all $n \neq 5$. Proposition 11.5 tells us that the corresponding solution to the Wave Equation is $w(x,t) = \cos(5t)\sin(5x)$. To see that w satisfies the wave equation, note that, for any $x \in [0,\pi]$ and $t > 0$,

$$\begin{array}{rrcrcrcl} & \partial_t w(x,t) & = & -5\sin(5t)\sin(5x) & \text{and} & 5\cos(5t)\cos(5x) & = & \partial_x w(x,t); \\ \text{Thus} & \partial_t^2 w(x,t) & = & -25\cos(5t)\sin(5x) & = & -25\cos(5t)\cos(5x) & = & \partial_x^2 w(x,t). \end{array}$$

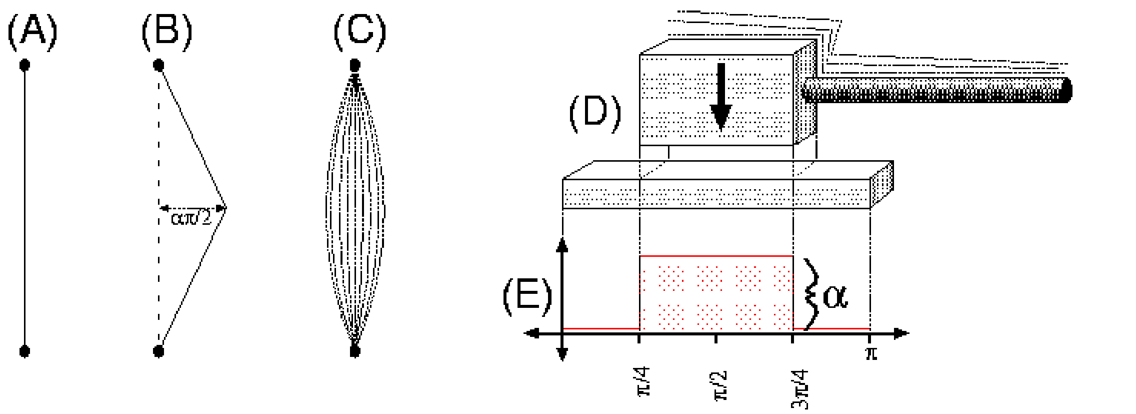

Figure 11.1: **(A)** A harpstring at rest. **(B)** A harpstring being plucked. **(C)** The harpstring vibrating. **(D)** A big hammer striking a xylophone. **(E)** The initial velocity of the xylophone when struck.

Also w has the desired initial position because, for any $x \in [0, \pi]$, we have $w(0,t) = \cos(0)\sin(5x) = \sin(5x) = f_0(x)$, because $\cos(0) = 1$.

Next, w has the desired initial velocity because for any $x \in [0, \pi]$, we have $\partial_t w(0,t) = 5\sin(0)\sin(5x) = 0$, because $\sin(0) = 0$.

Finally w satisfies homogeneous Dirichlet BC because, for any $t > 0$, we have $w(0,t) = \cos(5t)\sin(0) = 0$ and $w(\pi, t) = \cos(5t)\sin(5\pi) = 0$, because $\sin(0) = 0 = \sin(5\pi)$. ♢

Example 11.7: (The plucked harp string)

A harpist places her fingers at the midpoint of a harp string and plucks it. What is the formula describing the vibration of the string?

Solution: For simplicity, we imagine the string has length π. The taught string forms a straight line when at rest (Figure 11.1A); the harpist plucks the string by pulling it away from this resting position and then releasing it. At the moment she releases it, the string's *initial velocity* is zero, and its *initial position* is described by a **tent function** like the one in Example 8.17 on page 163

$$f_0(x) \quad = \quad \begin{cases} \alpha x & \text{if } 0 \le x \le \frac{\pi}{2} \\ \alpha(\pi - x) & \text{if } \frac{\pi}{2} < x \le \pi. \end{cases} \qquad \text{(Figure 11.1B)}$$

where $\alpha > 0$ is a constant describing the force with which she plucks the string (and its resulting amplitude).

The endpoints of the harp string are fixed, so it vibrates with *homogeneous Dirichlet* boundary conditions. Thus, Proposition 11.5 tells us to find the Fourier sine series for f_0. In Example 8.17, we computed this to be:

$$f_0 \quad \underset{L2}{\approx} \quad \frac{4 \cdot \alpha}{\pi} \sum_{\substack{n=1 \\ n \text{ odd;} \\ n=2k+1}}^{\infty} \frac{(-1)^k}{n^2} \sin(nx).$$

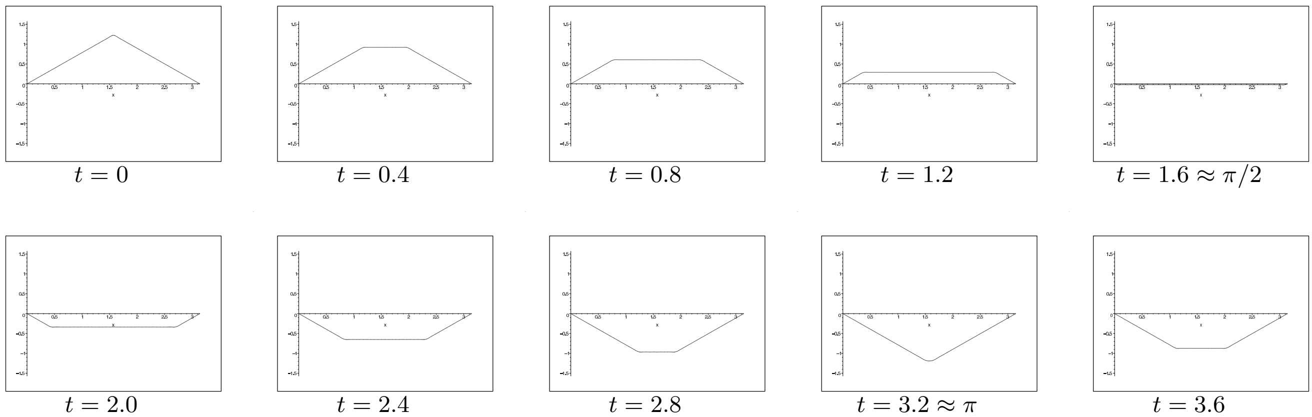

Figure 11.2: The plucked harpstring of Example 11.7. From $t = 0$ to $t = \pi/2$, the initially triangular shape is blunted; at $t = \pi/2$ it is totally flat. From $t = \pi/2$ to $t = \pi$, the process happens in reverse, only the triangle grows back upside down. At $t = \pi$, the original triangle reappears, upside down. Then the entire process happens in reverse, until the original triangle reappears at $t = 2\pi$.

Thus, the resulting solution is: $u(x;t) \underset{L2}{\approx} \frac{4 \cdot \alpha}{\pi} \sum_{\substack{n=1 \\ n \text{ odd;} \\ n=2k+1}}^{\infty} \frac{(-1)^k}{n^2} \sin(nx)\cos(nt);$

(See Figure 11.2). This is not a very accurate model because we have not accounted for energy loss due to friction. In a real harpstring, these 'perfectly triangular' waveforms rapidly decay into gently curving waves depicted in Figure 11.1(C); these slowly settle down to a stationary state. ♢

Proposition 11.8: (Initial Velocity Problem for Vibrating String with fixed endpoints)

$f_1 : \mathbb{X} \longrightarrow \mathbb{R}$ be a function describing the initial velocity of the string. Suppose f_1 has Fourier Sine Series $f_1(x) \underset{L2}{\approx} \sum_{n=1}^{\infty} B_n \sin(nx)$, *and define:*

$$v(x;t) \underset{L2}{\approx} \sum_{n=1}^{\infty} \frac{B_n}{n} \sin(nx) \cdot \sin(nt), \qquad \textit{for all } x \in [0,\pi] \textit{ and } t \geq 0. \tag{11.7}$$

Then $v(x;t)$ is the unique solution to the Wave Equation (11.4)*, satisfying the Dirichlet boundary conditions* (11.5)*, as well as*

$$\left.\begin{array}{lrcl} \textbf{Initial Position:} & v(x,0) & = & 0; \\ \textbf{Initial Velocity:} & \partial_t\, v(x,0) & = & f_1(x), \end{array}\right\} \quad \textit{for all } x \in [0,\pi].$$

Proof: **Exercise 11.5** Hint:

(a) Prove the trigonometric identity $-\sin(nx)\sin(nt) = \frac{1}{2}\Big(\cos\big(n(x+t)\big) - \cos\big(n(x-t)\big)\Big)$.

(b) Use this identity to show that the Fourier sine series (11.7) converges in $\mathbf{L}^2$ to the d'Alembert solution from Theorem 16.28(b) on page 329.

(c) Apply Theorem 6.18 on page 108 to show that this solution is unique. □

Example 11.9: Let $f_1(x) = 3\sin(8x)$. Thus, $B_8 = 3$ and $B_n = 0$ for all $n \neq 8$. Proposition 11.8 tells us that the corresponding solution to the Wave Equation is $w(x,t) = \frac{3}{8}\sin(8t)\sin(8x)$. To see that w satisfies the wave equation, note that, for any $x \in [0,\pi]$ and $t > 0$,

$$\begin{array}{rrclcrcl} & \partial_t\, w(x,t) & = & 3\sin(8t)\cos(8x) & \text{and} & 3\cos(8t)\sin(8x) & = & \partial_x\, w(x,t); \\ \text{Thus} & \partial_t^2\, w(x,t) & = & -24\cos(8t)\cos(8x) & = & -24\cos(8t)\cos(8x) & = & \partial_x^2\, w(x,t). \end{array}$$

Also w has the desired initial position because, for any $x \in [0,\pi]$, we have $w(0,t) = \frac{3}{8}\sin(0)\sin(8x) = 0$, because $\sin(0) = 0$.

Next, w has the desired initial velocity because for any $x \in [0,\pi]$, we have $\partial_t w(0,t) = \frac{3}{8}8\cos(0)\sin(8x) = 3\sin(8x) = f_1(x)$, because $\cos(0) = 1$.

Finally w satisfies homogeneous Dirichlet BC because, for any $t > 0$, we have $w(0,t) = \frac{3}{8}\sin(8t)\sin(0) = 0$ and $w(\pi,t) = \frac{3}{8}\sin(8t)\sin(8\pi) = 0$, because $\sin(0) = 0 = \sin(8\pi)$. ♢

Example 11.10: (The Xylophone)

A musician strikes the midpoint of a xylophone bar with a broad, flat hammer. What is the formula describing the vibration of the string?

Solution: For simplicity, we imagine the bar has length π and is fixed at its endpoints (actually most xylophones satisfy neither requirement). At the moment when the hammer strikes it, the string's *initial position* is zero, and its *initial velocity* is determined by the distribution of force imparted by the hammer head. For simplicity, we will assume the hammer head has width $\pi/2$, and hits the bar squarely at its midpoint (Figure 11.1D). Thus, the initial velocity is given by the function:

$$f_1(x) \quad = \quad \begin{cases} \alpha & \text{if } \frac{\pi}{4} \le x \le \frac{3\pi}{4} \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Figure 11.1E)}$$

where $\alpha > 0$ is a constant describing the force of the impact. Proposition 11.8 tells us to find the Fourier sine series for $f_1(x)$. From Example 8.14 on page 160, we know this to be

$$f_1(x) \underset{\text{L2}}{\approx} \frac{2\alpha\sqrt{2}}{\pi}\left(\sin(x) + \sum_{k=1}^{\infty}(-1)^k \frac{\sin\Big((4k-1)x\Big)}{4k-1} + \sum_{k=1}^{\infty}(-1)^k \frac{\sin\Big((4k+1)x\Big)}{4k+1}\right).$$

The resulting vibrational motion is therefore described by:

$$v(x,t) \underset{\text{L2}}{\approx} \frac{2\alpha\sqrt{2}}{\pi}\left(\sin(x)\sin(t) + \sum_{k=1}^{\infty}(-1)^k \frac{\sin\Big((4k-1)x\Big)\sin\Big((4k-1)t\Big)}{(4k-1)^2} + \sum_{k=1}^{\infty}(-1)^k \frac{\sin\Big((4k+1)x\Big)\sin\Big((4k+1)t\Big)}{(4k+1)^2}\right). \quad \diamondsuit$$

Exercise 11.6 Let $L > 0$ and let $\mathbb{X} := [0, L]$. Let $\lambda > 0$ be a parameter describing wave velocity (determined by the string's tension, elasticity, density, etc.), and consider the general one-dimensional Wave Equation

$$\partial_t^2 u = \lambda^2 \partial_x^2 u. \tag{11.8}$$

(a) Generalize Proposition 11.5 to find the solution to eqn.(11.8) on $\mathbb{X}$ having zero initial velocity and a prescribed initial position, and homogeneous Dirichlet boundary conditions.

(b) Generalize Proposition 11.8 to find the solution to eqn.(11.8) on $\mathbb{X}$ having zero initial position and a prescribed initial velocity, and homogeneous Dirichlet boundary conditions.

In both cases, prove that your solution converges, satisfies the desired initial conditions and boundary conditions, and satisfies eqn.(11.8) (Hint: imitate the strategy suggested in Exercises 11.4 and 11.5.)

11.3 The Poisson Problem on a Line Segment

Prerequisites: §8.2, §6.5, §2.4 **Recommended:** §8.3(e)

We can also use Fourier series to solve the one-dimensional Poisson problem on a line segment. This is not usually a practical solution method, because we already have a simple, complete solution to this problem using a double integral (see Example 2.7 on page 29). However, we include this result anyways, as a simple illustration of Fourier techniques.

Proposition 11.11: *Let $\mathbb{X} = [0, \pi]$, and let $q : \mathbb{X} \longrightarrow \mathbb{R}$ be some function, with* uniformly *convergent Fourier sine series: $q(x) \underset{\text{L2}}{\approx} \sum_{n=1}^{\infty} Q_n \sin(nx)$. Define the function $u(x)$ by*

$$u(x) \underset{\text{unif}}{\equiv} \sum_{n=1}^{\infty} \frac{-Q_n}{n^2}\sin(nx), \qquad \textit{for all } x \in [0,\pi].$$

Then $u(x)$ is the unique solution to the <u>Poisson equation</u> *"$\triangle u(x) = q(x)$" satisfying homogeneous Dirichlet boundary conditions: $u(0) = u(\pi) = 0$.*

Proof: **Exercise 11.7** Hint: (a) Apply Theorem 8.1(c) (p.145) to show that $\sum_{n=1}^{\infty} |Q_n| < \infty$.

(b) Apply Theorem 8.20 on page 166 to conclude that $\triangle u(x) \underset{\text{unif}}{\equiv} \sum_{n=1}^{\infty} Q_n \sin(nx) = q(x)$.

(c) Observe that $\sum_{n=1}^{\infty}\left|\frac{Q_n}{n^2}\right| < \infty$.

(d) Apply Theorem 8.1(c) (p.145) to show that the given Fourier sine series for $u(x)$ converges uniformly.

(e) Apply Theorem 8.1(d) (p.145) to conclude that $u(0) = 0 = u(\pi)$.

(f) Apply Theorem 6.14(a) on page 106 to conclude that this solution is unique. □

Proposition 11.12: *Let $\mathbb{X} = [0,\pi]$, and let $q : \mathbb{X} \longrightarrow \mathbb{R}$ be some some function, with* uniformly *convergent Fourier cosine series: $q(x) \underset{L2}{\approx} \sum_{n=1}^{\infty} Q_n \cos(nx)$, and suppose that $Q_0 = 0$.*

Fix any constant $K \in \mathbb{R}$, and define the function $u(x)$ by

$$u(x) \underset{\text{unif}}{\equiv} \sum_{n=1}^{\infty} \frac{-Q_n}{n^2}\cos(nx) \quad + \quad K, \qquad \text{for all } x \in [0,\pi]. \tag{11.9}$$

Then $u(x)$ is a solution to the Poisson equation "$\triangle u(x) = q(x)$", satisfying homogeneous Neumann boundary conditions $u'(0) = u'(\pi) = 0$.

Furthermore, all solutions to this Poisson equation with these boundary conditions have the form (11.9) for some choice of K.

If $Q_0 \neq 0$, however, the problem has no solution.

Proof: **Exercise 11.8** Hint: (a) Apply Theorem 8.4(c) (p.149) to $\sum_{n=1}^{\infty}|Q_n| < \infty$.

(b) Apply Theorem 8.20 on page 166 to conclude that $\triangle u(x) \underset{\text{unif}}{\equiv} \sum_{n=1}^{\infty} Q_n \cos(nx) = q(x)$.

(c) Observe that $\sum_{n=1}^{\infty}\left|\frac{Q_n}{n}\right| < \infty$.

(d) Apply Theorem 8.4(d) (p.149) to conclude that $u'(0) = 0 = u'(\pi)$.

(e) Apply Theorem 6.14(c) on page 106 to conclude that this solution is unique up to addition of a constant. □

Exercise 11.9 Mathematically, it is clear that the solution of Proposition 11.12 cannot be well-defined if $Q_0 \neq 0$. Provide a physical explanation for why this is to be expected.

11.4 Practice Problems

1. Let $g(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \le x \end{cases}$. (see problem #5 of §8.4)

(a) Find the solution to the one-dimensional Heat Equation $\partial_t u(x,t) = \triangle u(x,t)$ on the interval $[0,\pi]$, with initial conditions $u(x,0) = g(x)$ and homogeneous **Dirichlet** Boundary conditions.

(b) Find the solution to the one-dimensional Heat Equation $\partial_t u(x,t) = \triangle u(x,t)$ on the interval $[0,\pi]$, with initial conditions $u(x,0) = g(x)$ and homogeneous **Neumann** Boundary conditions.

(c) Find the solution to the one-dimensional Wave Equation $\partial_t^2 w(x,t) = \triangle w(x,t)$ on the interval $[0,\pi]$, satisfying homogeneous **Dirichlet** Boundary conditions, with initial **position** $w(x,0) = 0$ and initial **velocity** $\partial_t w(x,0) = g(x)$.

2. Let $f(x) = \sin(3x)$, for $x \in [0,\pi]$.

(a) Compute the Fourier **Sine** Series of $f(x)$ as an element of $\mathbf{L}^2[0,\pi]$.

(b) Compute the Fourier **Cosine** Series of $f(x)$ as an element of $\mathbf{L}^2[0,\pi]$.

(c) Solve the one-dimensional **Heat Equation** ($\partial_t u = \triangle u$) on the domain $\mathbb{X} = [0,\pi]$, with **initial conditions** $u(x;0) = f(x)$, and the following boundary conditions:

i. Homogeneous **Dirichlet** boundary conditions.

ii. Homogeneous **Neumann** boundary conditions.

(d) Solve the the one-dimensional **Wave Equation** ($\partial_t^2 v = \triangle v$) on the domain $\mathbb{X} = [0,\pi]$, with homogeneous **Dirichlet** boundary conditions, and with

$$\begin{array}{rrcl} \textbf{Initial position:} & v(x;0) & = & 0, \\ \textbf{Initial velocity:} & \partial_t v(x;0) & = & f(x). \end{array}$$

3. Let $f : [0,\pi] \longrightarrow \mathbb{R}$, and suppose f has

$$\begin{array}{rrcl} \text{Fourier cosine series:} & f(x) & = & \displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}\cos(nx) \\ \text{Fourier sine series:} & f(x) & = & \displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}\sin(nx) \end{array}$$

(a) Find the solution to the one-dimensional Heat Equation $\partial_t u = \triangle u$, with homogeneous **Neumann** boundary conditions, and initial conditions $u(x;0) = f(x)$ for all $x \in [0,\pi]$.

(b) **Verify** your solution in part (a). Check the Heat equation, the initial conditions, and boundary conditions. [Hint: Use Proposition 1.7 on page 16]

(c) Find the solution to the one-dimensional **wave equation** $\partial_t^2 u(x;t) = \triangle u(x;t)$ with homogeneous **Dirichlet** boundary conditions, and

$$\begin{array}{rrcll} \text{Initial position} & u(x;0) & = & f(x), & \text{for all } x \in [0,\pi]. \\ \text{Initial velocity} & \partial_t u(x;0) & = & 0, & \text{for all } x \in [0,\pi]. \end{array}$$

4. Let $f : [0,\pi] \longrightarrow \mathbb{R}$ be defined by $f(x) = x$.

(a) Compute the Fourier *sine* series for f.

(b) Does the Fourier sine series converge *pointwise* to f on $(0, \pi)$? Justify your answer.

(c) Does the Fourier sine series converge *uniformly* to f on $[0, \pi]$? Justify your answer in *two different* ways.

(d) Compute the Fourier *cosine* series for f.

(e) Solve the one-dimensional **Heat Equation** ($\partial_t u = \triangle u$) on the domain $\mathbb{X} := [0, \pi]$, with initial conditions $u(x, 0) := f(x)$, and with the following boundary conditions:

[i] Homogeneous **Dirichlet** boundary conditions.

[ii] Homogeneous **Neumann** boundary conditions.

(f) **Verify** your solution to question (e) part [i]. That is: check that your solution satisfies the heat equation, the desired initial conditions, and homogeneous Dirichlet BC. [You may assume that the relevent series converge uniformly, if necessary. You may differentiate Fourier series termwise, if necessary.]

(g) Find the solution to the one-dimensional **Wave Equation** on the domain $\mathbb{X} := [0, \pi]$, with homogeneous Dirichlet boundary conditions, and with

$$\begin{array}{rrcl} \text{Initial position} & u(x; 0) & = & f(x), \quad \text{for all } x \in [0, \pi]. \\ \text{Initial velocity} & \partial_t\, u(x; 0) & = & 0, \quad \text{for all } x \in [0, \pi]. \end{array}$$

Notes: ....

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12 Boundary Value Problems on a Square

Prerequisites: §10.1, §6.5 **Recommended:** §11

Multiple Fourier series can be used to find solutions to boundary value problems on a box $[0, X] \times [0, Y]$. The key idea is that the functions $\mathbf{S}_{n,m}(x, y) = \sin\left(\frac{n\pi}{X}x\right)\sin\left(\frac{m\pi}{Y}y\right)$ and $\mathbf{C}_{n,m}(x, y) = \cos\left(\frac{n\pi}{X}x\right)\cos\left(\frac{m\pi}{Y}y\right)$ are *eigenfunctions* of the Laplacian operator. Furthermore, $\mathbf{S}_{n,m}$ satisfies *Dirichlet* boundary conditions, so any (uniformly convergent) Fourier sine series will also. Likewise, $\mathbf{C}_{n,m}$ satisfies *Neumann* boundary conditions, so any (uniformly convergent) Fourier cosine series will also.

For simplicity, we will assume throughout that $X = Y = \pi$. Thus $\mathbf{S}_{n,m}(x) = \sin(nx)\sin(my)$ and $\mathbf{C}_{n,m}(x) = \cos(nx)\cos(my)$. We will also assume that the physical constants in the various equations are all set to one. Thus, the Heat Equation becomes "$\partial_t u = \triangle u$", the Wave Equation is "$\partial_t^2 u = \triangle u$", etc. This will allow us to develop the solution methods in the simplest possible scenario, without a lot of distracting technicalities.

The extension of these solution methods to equations with arbitrary physical constants on an arbitrary rectangular domain $[0, X] \times [0, Y]$ (for some $X, Y > 0$) are left as exercises. These exercises are quite straightforward, but are an effective test of your understanding of the solution techniques.

Remark on Notation: Throughout this chapter (and the following ones) we will often write a function $u(x, y; t)$ in the form $u_t(x, y)$. This emphasizes the distinguished role of the 'time' coordinate t, and makes it natural to think of fixing t at some value and applying the 2-dimensional Laplacian $\triangle = \partial_x^2 + \partial_y^2$ to the resulting 2-dimensional function u_t.

Some authors use the subscript notation "u_t" to denote the partial derivative $\partial_t u$. We *never* use this notation. In this book, partial derivatives are always denoted by "$\partial_t u$", etc.

12.1 The (nonhomogeneous) Dirichlet problem on a Square

Prerequisites: §10.1, §6.5(a), §2.3, §1.7 **Recommended:** §8.3(e)

In this section we will learn to solve the **Dirichlet problem** on a square domain $\mathbb{X}$: that is, to find a function which is harmonic on the interior of $\mathbb{X}$ and which satisfies specified Dirichlet boundary conditions on the boundary $\mathbb{X}$. Solutions to the Dirichlet problem have several physical interpretations.

Heat: Imagine that the boundaries of $\mathbb{X}$ are perfect heat conductors, which are in contact with external 'heat reservoirs' with fixed temperatures. For example, one boundary might be in contact with a heat source, and another, in contact with a coolant liquid. The solution to the Dirichlet problem is then the equilibrium temperature distribution on the interior of the box, given these constraints.

Electrostatic: Imagine that the boundaries of $\mathbb{X}$ are electrical conductors which are held at some fixed voltage by the application of an external electric potential (different boundaries, or different parts of the same boundary, may be held at different voltages). The

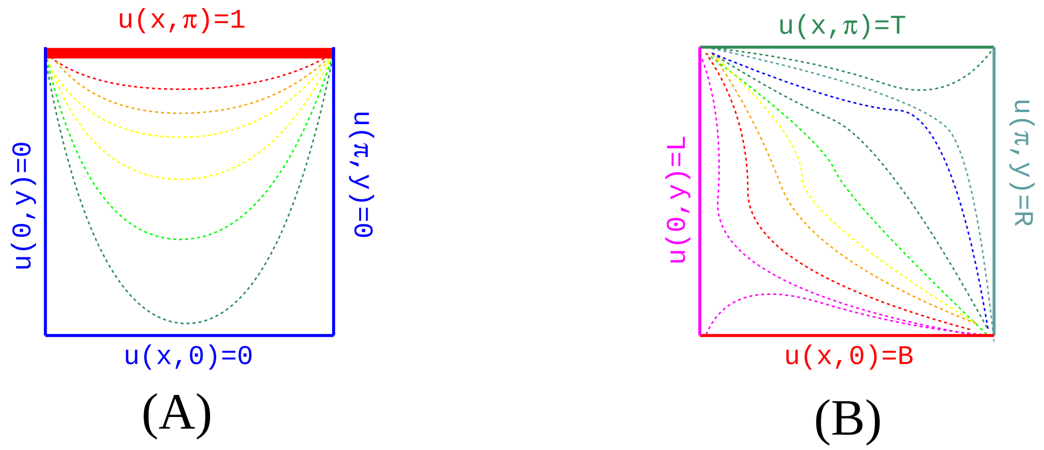


Figure 12.1: The Dirichlet problem on a square. **(A)** Proposition 12.1; **(B)** Propositions 12.2 and 12.4.

solution to the Dirichlet problem is then the electric potential field on the interior of the box, given these constraints.

Minimal surface: Imagine a squarish frame of wire, which we have bent in the vertical direction to have some shape. If we dip this wire frame in a soap solution, we can form a soap bubble (i.e. minimal-energy surface) which must obey the 'boundary conditions' imposed by the shape of the wire. The differential equation describing a minimal surface is not *exactly* the same as the Laplace equation; however, when the surface is not too steeply slanted (i.e. when the wire frame is not too bent), the Laplace equation is a good approximation; hence the solution to the Dirichlet problem is a good approximation of the shape of the soap bubble.

We will begin with the simplest problem: a constant, nonzero Dirichlet boundary condition on one side of the box, and zero boundary conditions on the other three sides.

Proposition 12.1: (Dirichlet problem; one constant nonhomogeneous boundary)

Let $\mathbb{X} = [0, \pi] \times [0, \pi]$*, and consider the Laplace equation "*$\triangle u = 0$*", with nonhomogeneous* **Dirichlet** *boundary conditions* [see Figure 12.1(A)]:

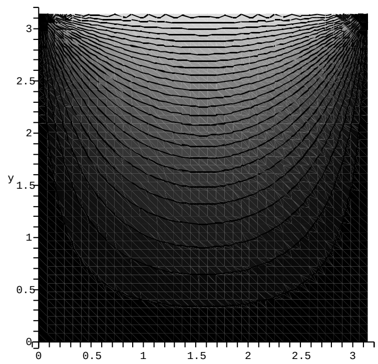
$$u(0, y) \;=\; u(\pi, y) \;=\; 0, \quad \textit{for all } y \in [0, \pi). \tag{12.1}$$

$$u(x, 0) \;=\; 0 \quad \textit{and} \quad u(x, \pi) \;=\; 1, \quad \textit{for all } x \in [0, \pi]. \tag{12.2}$$

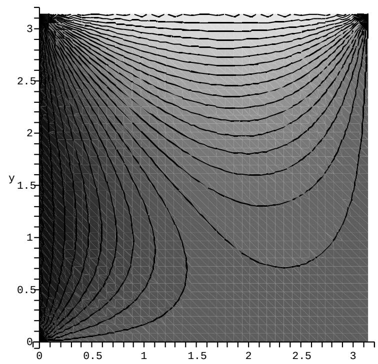
The unique solution to this problem is the function

$$u(x, y) \;\underset{L2}{\approx}\; \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n \sinh(n\pi)} \sin(nx) \cdot \sinh(ny), \qquad \textit{for all } (x, y) \in \mathbb{X}.$$

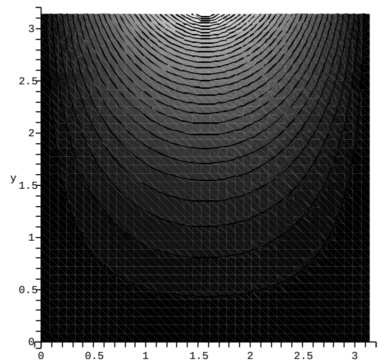
[See Figures 12.2(a) and 12.3(a).] *Furthermore, this series converges semiuniformly on* $\mathsf{int}(\mathbb{X})$.



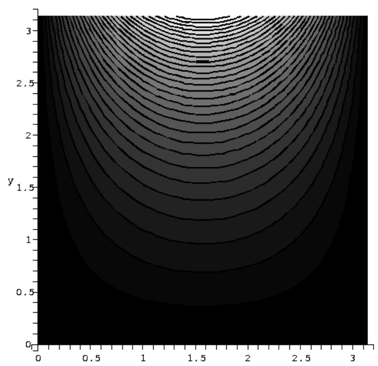
Proposition 12.1
$T = 1$, $R = L = B = 0$



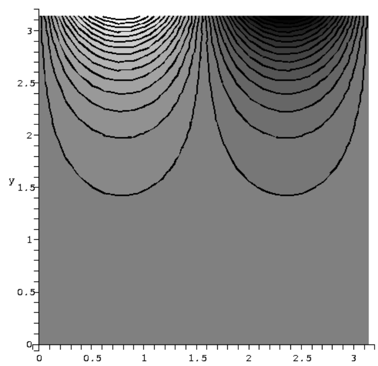
Example 12.3
$T = -3$, $L = 5$, $R = B = 0$



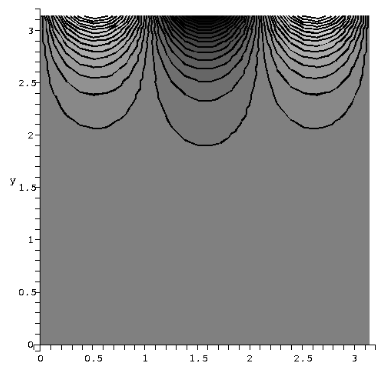
Example 12.6
$T =$ tent function , $R = L = B = 0$



$\sin(x)\sinh(y)$
$T(x) = \sin(x)$, $R = L = B = 0$



$\sin(2x)\sinh(2y)$
$T(x) = \sin(2x)$, $R = L = B = 0$



$\sin(3x)\sinh(3y)$
$T(x) = \sin(3x)$, $R = L = B = 0$

Figure 12.2: The Dirichlet problem on a box. The curves represent isothermal contours (of a temperature distribution) or equipotential lines (of an electric voltage field).

Proof: **Exercise 12.1**

(a) Check that, for all $n \in \mathbb{N}$, the function $u_n(x, y) = \sin(nx) \cdot \sinh(ny)$ satisfies the Laplace equation and the first boundary condition (12.1). See Figures 12.2(d,e,f) and 12.3(d,e,f).

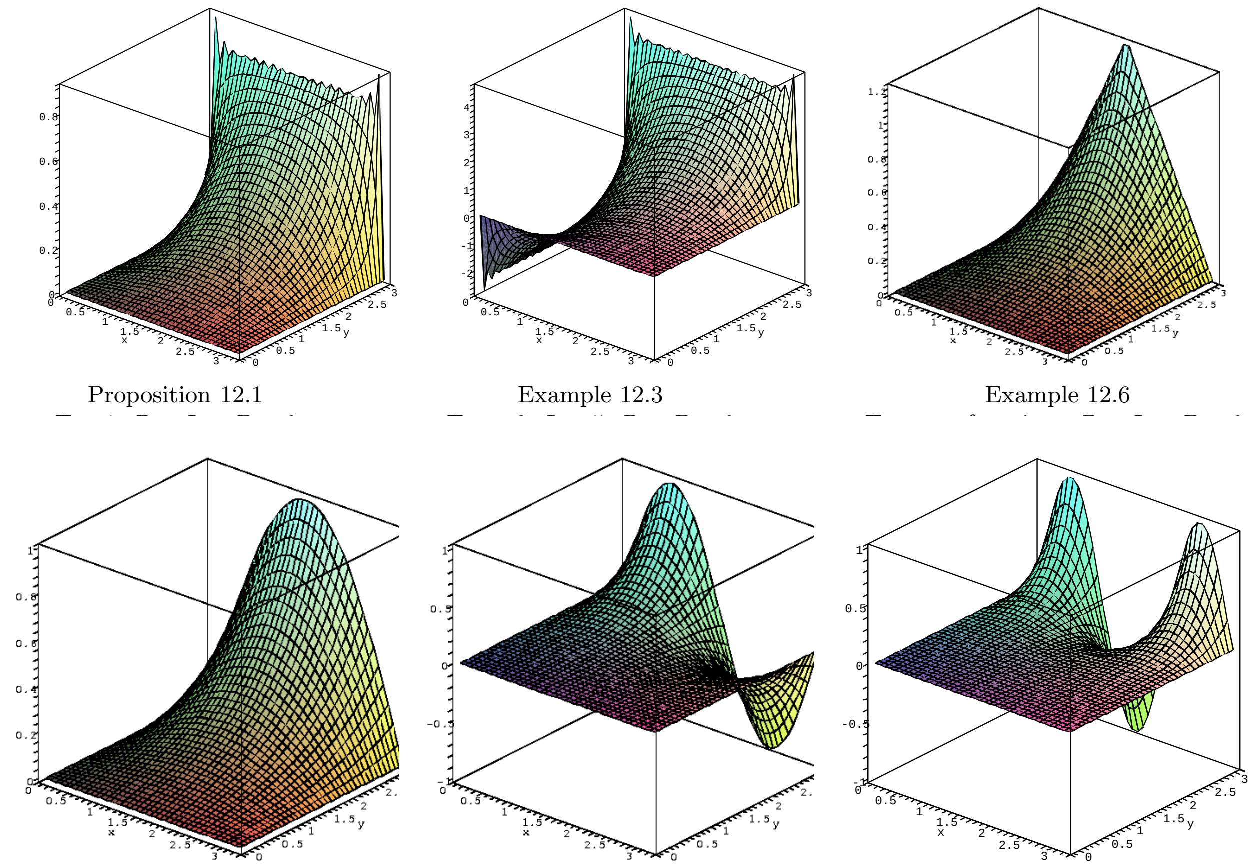
(b) Show that $\displaystyle\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} n^2 \left| \frac{\sinh(ny)}{n \sinh(n\pi)} \right| < \infty$, for any fixed $y < \pi$.

(c) Apply Proposition 1.7 on page 16 to conclude that $\triangle u(x, y) = 0$.

(d) Observe that $\displaystyle\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left| \frac{\sinh(ny)}{n \sinh(n\pi)} \right| < \infty$, for any fixed $y < \pi$.

(e) Apply part (c) of Theorem 8.1 on page 145 to show that the series given for $u(x, y)$ converges uniformly for any fixed $y < \pi$.

(f) Apply part (d) of Theorem 8.1 on page 145 to conclude that $u(0, y) = 0 = u(\pi, y)$ for all $y < \pi$.



Proposition 12.1 | Example 12.3 | Example 12.6

$\sin(x)\sinh(y)$
$T(x) = \sin(x)$, $R = L = B = 0$

$\sin(2x)\sinh(2y)$
$T(x) = \sin(2x)$, $R = L = B = 0$

$\sin(3x)\sinh(3y)$
$T(x) = \sin(3x)$, $R = L = B = 0$

Figure 12.3: The Dirichlet problem on a box: 3-dimensional plots. You can imagine these as soap films.

(g) Observe that $\sin(nx) \cdot \sinh(n \cdot 0) = 0$ for all $n \in \mathbb{N}$ and all $x \in [0, \pi]$. Conclude that $u(x, 0) = 0$ for all $x \in [0, \pi]$.

(h) To check that the solution also satisfies the boundary condition (12.2), subsititute $y = \pi$ to get:

$$u(x,\pi) \quad = \quad \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n \sinh(n\pi)} \sin(nx) \cdot \sinh(n\pi) \quad = \quad \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n} \sin(nx) \quad \underset{L2}{\approx} \quad 1.$$

because $\frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n} \sin(nx)$ is the (one-dimensional) Fourier sine series for the function $b(x) = 1$ (see Example 8.2(b) on page 146).

(i) Apply Theorem 6.14(a) on page 106 to conclude that this solution is unique. □

Proposition 12.2: (Dirichlet Problem; four constant nonhomogeneous boundaries)

Let $\mathbb{X} = [0, \pi] \times [0, \pi]$, *and consider the Laplace equation* "$\triangle u = 0$", *with nonhomogeneous Dirichlet boundary conditions* [see Figure 12.1(B)]:

$$\begin{array}{rclcrcll} u(0,y) & = & L & \text{and} & u(\pi,y) & = & R, & \text{for all } y \in (0,\pi); \\ u(x,\pi) & = & T & \text{and} & u(x,0) & = & B, & \text{for all } x \in (0,\pi). \end{array}$$

where L, R, T, *and* B *are four constants. The unique solution to this problem is the function:*

$$u(x,y) \;=\; l(x,y) + r(x,y) + t(x,y) + b(x,y), \quad \text{for all } (x,y) \in \mathbb{X}.$$

where, for all $(x,y) \in \mathbb{X}$,

$$l(x,y) \underset{\text{L2}}{\approx} L \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} c_n \sinh\Big(n(\pi - x)\Big) \cdot \sin(ny), \qquad r(x,y) \underset{\text{L2}}{\approx} R \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} c_n \sinh(nx) \cdot \sin(ny),$$

$$t(x,y) \underset{\text{L2}}{\approx} T \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} c_n \sin(nx) \cdot \sinh(ny), \qquad b(x,y) \underset{\text{L2}}{\approx} B \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} c_n \sin(nx) \cdot \sinh\Big(n(\pi - y)\Big).$$

where $c_n := \dfrac{4}{n\pi \sinh(n\pi)}$, *for all* $n \in \mathbb{N}$.

Furthermore, these four series converge semiuniformly on $\mathsf{int}(\mathbb{X})$.

Proof: <u>**Exercise 12.2**</u>

(a) Apply Proposition 12.1 to show that each of the functions $l(x,y)$, $r(x,y)$, $t(x,y)$, $b(x,y)$ satisfies a Dirichlet problem where one side has nonzero temperature and the other three sides have zero temperature.

(b) Add these four together to get a solution to the original problem.

(c) Apply Theorem 6.14(a) on page 106 to conclude that this solution is unique. □

<u>**Exercise 12.3**</u> What happens to the solution at the four corners $(0,0)$, $(0,\pi)$, $(\pi,0)$ and (π,π)?

Example 12.3: Suppose $R = 0 = B$, $T = -3$, and $L = 5$. Then the solution is:

$$\begin{aligned} u(x,y) &\underset{\text{L2}}{\approx} L \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} c_n \sinh\Big(n(\pi - x)\Big) \cdot \sin(ny) \;+\; T \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} c_n \sin(nx) \cdot \sinh(ny) \\ &= \frac{20}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sinh\Big(n(\pi - x)\Big) \cdot \sin(ny)}{n \sinh(n\pi)} \;-\; \frac{12}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin(nx) \cdot \sinh(ny)}{n \sinh(n\pi)}. \end{aligned}$$

See Figures 12.2(b) and 12.3(b). ◇

Proposition 12.4: (Dirichlet Problem; arbitrary nonhomogeneous boundaries)

Let $\mathbb{X} = [0,\pi]\times[0,\pi]$, and consider the Laplace equation "$\triangle u = 0$", with nonhomogeneous Dirichlet boundary conditions [see Figure 12.1(B)]:

$$\begin{array}{rclcrclr} u(0,y) &=& L(y) & \text{and} & u(\pi,y) &=& R(y), & \text{for all } y\in(0,\pi);\\ u(x,\pi) &=& T(x) & \text{and} & u(x,0) &=& B(x), & \text{for all } x\in(0,\pi). \end{array}$$

where $L(y)$, $R(y)$, $T(x)$, and $B(x)$ are four arbitrary functions. Suppose these functions have (one-dimensional) Fourier sine series:

$$L(y) \underset{L2}{\approx} \sum_{n=1}^{\infty} L_n \sin(ny), \qquad R(y) \underset{L2}{\approx} \sum_{n=1}^{\infty} R_n \sin(ny), \quad \text{for all } y\in[0,\pi];$$

$$T(x) \underset{L2}{\approx} \sum_{n=1}^{\infty} T_n \sin(nx), \quad \text{and} \quad B(x) \underset{L2}{\approx} \sum_{n=1}^{\infty} B_n \sin(nx), \quad \text{for all } x\in[0,\pi].$$

The unique solution to this problem is the function:

$$u(x,y) \;=\; l(x,y) + r(x,y) + t(x,y) + b(x,y), \quad \text{for all } (x,y)\in\mathbb{X}.$$

where, for all $(x,y)\in\mathbb{X}$,

$$l(x,y) \underset{L2}{\approx} \sum_{n=1}^{\infty} \frac{L_n}{\sinh(n\pi)} \sinh\Big(n(\pi-x)\Big)\cdot\sin(ny), \qquad r(x,y) \underset{L2}{\approx} \sum_{n=1}^{\infty} \frac{R_n}{\sinh(n\pi)} \sinh(nx)\cdot\sin(ny),$$

$$t(x,y) \underset{L2}{\approx} \sum_{n=1}^{\infty} \frac{T_n}{\sinh(n\pi)} \sin(nx)\cdot\sinh(ny), \qquad b(x,y) \underset{L2}{\approx} \sum_{n=1}^{\infty} \frac{B_n}{\sinh(n\pi)} \sin(nx)\cdot\sinh\Big(n(\pi-y)\Big).$$

Furthermore, these four series converge semiuniformly on int $(\mathbb{X})$.

Proof: **Exercise 12.4** First we consider the function $t(x,y)$.

(a,b) Same as Exercise 12.1(a,b)

(c) Apply Proposition 1.7 on page 16 to conclude that $t(x,y)$ is harmonic —i.e. $\triangle t(x,y) = 0$.

Through symmetric reasoning, conclude that the functions $\ell(x,y)$, $r(x,y)$ and $b(x,y)$ are also harmonic.

(d) Same as Exercise 12.1(d)

(e) Apply part (c) of Theorem 8.1 on page 145 to show that the series given for $t(x,y)$ converges uniformly for any fixed $y<\pi$.

(f) Apply part (d) of Theorem 8.1 on page 145 to conclude that $t(0,y) = 0 = t(\pi,y)$ for all $y<\pi$.

(g) Observe that $\sin(nx)\cdot\sinh(n\cdot 0) = 0$ for all $n\in\mathbb{N}$ and all $x\in[0,\pi]$. Conclude that $t(x,0) = 0$ for all $x\in[0,\pi]$.

(h) To check that the solution also satisfies the boundary condition (12.2), subsititute $y=\pi$ to get:

$$t(x,\pi) \;=\; \sum_{n=1}^{\infty} \frac{T_n}{\sinh(n\pi)} \sin(nx)\cdot\sinh(n\pi) \;=\; \frac{4}{\pi}\sum_{n=1}^{\infty} T_n \sin(nx) \;=\; T(x).$$

(j) At this point, we know that $t(x,\pi) = T(x)$ for all $x\in[0,\pi]$, and $t\equiv 0$ on the other three sides of the square. Through symmetric reasoning, show that:

- $\ell(0, y) = L(y)$ for all $y \in [0, \pi]$, and $\ell \equiv 0$ on the other three sides of the square.
- $r(\pi, y) = R(y)$ for all $y \in [0, \pi]$, and $r \equiv 0$ on the other three sides of the square.
- $b(x, 0) = B(x)$ for all $x \in [0, \pi]$, and $b \equiv 0$ on the other three sides of the square.

(k) Conclude that $u = t + b + r + \ell$ is harmonic and satisfies the desired boundary conditions.

(l) Apply Theorem 6.14(a) on page 106 to conclude that this solution is unique. □

Example 12.5: If $T(x) = \sin(3x)$, and $B \equiv L \equiv R \equiv 0$, then $u(x, y) = \dfrac{\sin(3x)\sinh(3y)}{\sinh(3\pi)}$. ♢

Example 12.6: *Let* $\mathbb{X} = [0, \pi] \times [0, \pi]$. *Solve the 2-dimensional Laplace Equation on* $\mathbb{X}$, *with* inhomogeneous Dirichlet *boundary conditions:*

$$u(0, y) \; = \; 0; \qquad u(\pi, y) \; = \; 0; \qquad u(x, 0) \; = \; 0;$$

$$u(x, \pi) \; = \; T(x) \; = \; \begin{cases} x & \textit{if } 0 \le x \le \frac{\pi}{2} \\ \pi - x & \textit{if } \frac{\pi}{2} < x \le \pi \end{cases} \qquad \text{(see Figure 8.5(B) on page 163)}$$

Solution: Recall from Example 8.17 on page 163 that $T(x)$ has Fourier series:

$$T(x) \;\underset{L2}{\approx}\; \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd;} \\ n=2k+1}}^{\infty} \frac{(-1)^k}{n^2} \sin(nx).$$

Thus, the solution is $u(x, y) \;\underset{L2}{\approx}\; \dfrac{4}{\pi} \displaystyle\sum_{\substack{n=1 \\ n \text{ odd;} \\ n=2k+1}}^{\infty} \frac{(-1)^k}{n^2 \sinh(n\pi)} \sin(nx)\sinh(ny).$

See Figures 12.2(c) and 12.3(c). ♢

<u>**Exercise 12.5**</u> Let $X, Y > 0$ and let $\mathbb{X} := [0, X] \times [0, Y]$. Generalize Proposition 12.4 to find the solution to the Laplace equation on $\mathbb{X}$, satisfying arbitrary nonhomogeneous Dirichlet boundary conditions on the four sides of $\partial\mathbb{X}$.

12.2 The Heat Equation on a Square

12.2(a) Homogeneous Boundary Conditions

Prerequisites: §10.1, §6.4, §6.5, §2.2(b), §1.7 **Recommended:** §11.1, §8.3(e)

Proposition 12.7: (Heat Equation; homogeneous Dirichlet boundary)

Consider the box $\mathbb{X} = [0,\pi] \times [0,\pi]$, and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be some function describing an initial heat distribution. Suppose f has Fourier Sine Series

$$f(x,y) \underset{L2}{\approx} \sum_{n,m=1}^{\infty} B_{n,m} \sin(nx)\sin(my)$$

and define:

$$u_t(x,y) \underset{L2}{\approx} \sum_{n,m=1}^{\infty} B_{n,m} \sin(nx)\cdot\sin(my)\cdot\exp\Big(-(n^2+m^2)\cdot t\Big), \qquad \text{for all } (x,y) \in \mathbb{X} \text{ and } t \geq 0.$$

Then $u_t(x,y)$ is the unique solution to the Heat Equation "$\partial_t u = \triangle u$", with homogeneous Dirichlet boundary conditions

$$u_t(x,0) = u_t(0,y) = u_t(\pi,y) = u_t(x,\pi) = 0, \qquad \text{for all } x,y \in [0,\pi] \text{ and } t > 0.$$

and initial conditions: $u_0(x,y) = f(x,y)$, for all $(x,y) \in \mathbb{X}$.

Furthermore, the series defining u converges semiuniformly on $\mathbb{X} \times (0,\infty)$.

Proof: <u>**Exercise 12.6**</u> Hint:

(a) Show that, when $t = 0$, the two-dimensional Fourier series of $u_0(x,y)$ agrees with that of $f(x,y)$; hence $u_0(x,y) = f(x,y)$.

(b) Show that, for all $t > 0$, $\displaystyle\sum_{n,m=1}^{\infty} \left|(n^2+m^2)\cdot B_{n,m}\cdot e^{-(n^2+m^2)t}\right| < \infty.$

(c) For any $T > 0$, apply Proposition 1.7 on page 16 to conclude that

$$\partial_t u_t(x,y) \underset{\text{unif}}{\equiv} \sum_{n,m=1}^{\infty} -(n^2+m^2)B_{n,m}\sin(nx)\cdot\sin(my)\cdot\exp\Big(-(n^2+m^2)\cdot t\Big) \underset{\text{unif}}{\equiv} \triangle u_t(x,y),$$

for all $(x,y;t) \in \mathbb{X} \times [T,\infty)$.

(d) Observe that for all $t > 0$, $\displaystyle\sum_{n,m=1}^{\infty} \left|B_{n,m}e^{-(n^2+m^2)t}\right| < \infty.$

(e) Apply part (e) of Theorem 10.3 on page 183 to show that the two-dimensional Fourier series of $u_t(x,y)$ converges uniformly for all $t > 0$.

(f) Apply part (e) of Theorem 10.3 on page 183 to conclude that u_t satisfies homogeneous Dirichlet boundary conditions, for all $t > 0$.

(g) Apply Theorem 6.16(a) on page 107 to show that this solution is unique. □

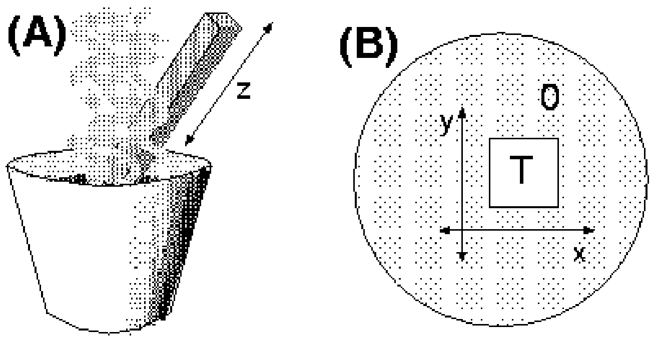


Figure 12.4: **(A)** A hot metal rod quenched in a cold bucket. **(B)** A cross section of the rod in the bucket.

Example 12.8: (The quenched rod)

On a cold January day, a blacksmith is tempering an iron rod. He pulls it out of the forge and plunges it, red-hot, into ice-cold water (Figure 12.4A). *The rod is very long and narrow, with a square cross section. We want to compute how the rod cooled.*

Answer: The rod is immersed in freezing cold water, and is a good conductor, so we can assume that its outer surface takes the the surrounding water temperature of 0 degrees. Hence, we assume homogeneous Dirichlet boundary conditions.

Endow the rod with coordinate system (x, y, z), where z runs along the length of the rod. Since the rod is extremely long relative to its cross-section, we can neglect the z coordinate, and reduce to a 2-dimensional equation (Figure 12.4B). Assume the rod was initially uniformly heated to a temperature of T. The initial temperature distribution is thus a constant function: $f(x, y) = T$. From Example 10.2 on page 180, we know that the constant function 1 has two-dimensional Fourier sine series:

$$1 \underset{\text{L2}}{\approx} \frac{16}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{1}{n \cdot m} \sin(nx)\sin(my)$$

Thus, $f(x, y) \underset{\text{L2}}{\approx} \frac{16T}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{1}{n \cdot m} \sin(nx)\sin(my)$. Thus, the time-varying thermal profile of the rod is given:

$$u_t(x, y) \underset{\text{L2}}{\approx} \frac{16T}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{1}{n \cdot m} \sin(nx)\sin(my)\exp\Big(-(n^2+m^2)\cdot t\Big). \qquad \diamondsuit$$

Proposition 12.9: (Heat Equation; homogeneous Neumann boundary)

Consider the box $\mathbb{X} = [0, \pi] \times [0, \pi]$, and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be some function describing an initial heat distribution. Suppose f has Fourier Cosine Series

$$f(x, y) \underset{\text{L2}}{\approx} \sum_{n,m=0}^{\infty} A_{n,m} \cos(nx) \cos(my)$$

and define:

$$u_t(x, y) \underset{\text{L2}}{\approx} \sum_{n,m=0}^{\infty} A_{n,m} \cos(nx) \cdot \cos(my) \cdot \exp\Big(-(n^2 + m^2) \cdot t \Big), \qquad \text{for all } (x, y) \in \mathbb{X} \text{ and } t \geq 0.$$

Then $u_t(x, y)$ is the unique solution to the Heat Equation "$\partial_t u = \triangle u$", with homogeneous Neumann boundary conditions

$$\partial_y u_t(x, 0) = \partial_y u_t(x, \pi) = \partial_x u_t(0, y) = \partial_x u_t(\pi, y) = 0, \qquad \text{for all } x, y \in [0, \pi] \text{ and } t > 0.$$

and initial conditions: $u_0(x, y) = f(x, y)$, for all $(x, y) \in \mathbb{X}$. □

Furthermore, the series defining u converges semiuniformly on $\mathbb{X} \times (0, \infty)$.

Proof: <u>**Exercise 12.7**</u> Hint:

(a) Show that, when $t = 0$, the two-dimensional Fourier cosine series of $u_0(x, y)$ agrees with that of $f(x, y)$; hence $u_0(x, y) = f(x, y)$.

(b) Show that, for all $t > 0$, $\displaystyle\sum_{n,m=0}^{\infty} \left|(n^2 + m^2) \cdot A_{n,m} \cdot e^{-(n^2+m^2)t}\right| < \infty$.

(c) Apply Proposition 1.7 on page 16 to conclude that

$$\partial_t u_t(x, y) \underset{\text{unif}}{\equiv} \sum_{n,m=0}^{\infty} -(n^2+m^2) A_{n,m} \cos(nx) \cdot \cos(my) \cdot \exp\Big(-(n^2 + m^2) \cdot t \Big) \underset{\text{unif}}{\equiv} \triangle u_t(x, y),$$

for all $(x, y) \in \mathbb{X}$ and $t > 0$.

(d) Observe that for all $t > 0$, $\displaystyle\sum_{n,m=0}^{\infty} n \cdot \left|A_{n,m} e^{-(n^2+m^2)t}\right| < \infty$ and $\displaystyle\sum_{n,m=0}^{\infty} m \cdot \left|A_{n,m} e^{-(n^2+m^2)t}\right| < \infty$.

(e) Apply part (g) of Theorem 10.3 on page 183 to conclude that u_t satisfies homogeneous Neumann boundary conditions, for all $t > 0$.

(f) Apply Theorem 6.16(b) on page 107 to show that this solution is unique. □

Example 12.10: Suppose $\mathbb{X} = [0, \pi] \times [0, \pi]$

(a) Let $f(x, y) = \cos(3x) \cos(4y) + 2 \cos(5x) \cos(6y)$. Then $A_{3,4} = 1$ and $A_{5,6} = 2$, and all other Fourier coefficients are zero. Thus, $u(x, y; t) = \cos(3x) \cos(4y) \cdot e^{-25t} + \cos(5x) \cos(6y) \cdot e^{-59t}$.

(b) Suppose $f(x,y) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{\pi}{2} \text{ and } 0 \leq y < \frac{\pi}{2}; \\ 0 & \text{if } \frac{\pi}{2} \leq x \text{ or } \frac{\pi}{2} \leq y. \end{cases}$ We know from Example 10.4 on page 184 that the two-dimensional Fourier cosine series of f is:

$$f(x,y) \underset{L2}{\approx} \frac{1}{4} + \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\cos\Big((2k+1)x\Big) + \frac{1}{\pi}\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}\cos\Big((2j+1)y\Big)$$
$$+ \frac{4}{\pi^2}\sum_{k,j=1}^{\infty} \frac{(-1)^{k+j}}{(2k+1)(2j+1)}\cos\Big((2k+1)x\Big)\cdot\cos\Big((2j+1)y\Big)$$

Thus, the solution to the heat equation, with initial conditions $u_0(x,y) = f(x,y)$ and homogeneous Neumann boundary conditions is given:

$$u_t(x,y) \underset{L2}{\approx}$$
$$\frac{1}{4} + \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\cos\Big((2k+1)x\Big)\cdot e^{-(2k+1)^2 t} + \frac{1}{\pi}\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}\cos\Big((2j+1)y\Big)\cdot e^{-(2j+1)^2 t}$$
$$+ \frac{4}{\pi^2}\sum_{k,j=1}^{\infty} \frac{(-1)^{k+j}}{(2k+1)(2j+1)}\cos\Big((2k+1)x\Big)\cdot\cos\Big((2j+1)y\Big)\cdot e^{-\left[(2k+1)^2+(2j+1)^2\right]\cdot t}$$

♢

Exercise 12.8 Let $X, Y > 0$ and let $\mathbb{X} := [0,X]\times[0,Y]$. Let $\kappa > 0$ be a diffusion constant, and consider the general two-dimensional Heat Equation

$$\partial_t u = \kappa \triangle u. \qquad (12.3)$$

(a) Generalize Proposition 12.7 to find the solution to eqn.(12.3) on $\mathbb{X}$ satisfying prescribed initial conditions and homogeneous Dirichlet boundary conditions.

(b) Generalize Proposition 12.9 to find the solution to eqn.(12.3) on $\mathbb{X}$ satisfying prescribed initial conditions and homogeneous Neumann boundary conditions.

In both cases, prove that your solution converges, satisfies the desired initial conditions and boundary conditions, and satisfies eqn.(12.3) (Hint: imitate the strategy suggested in Exercise 12.6)

Exercise 12.9 Let $f : \mathbb{X} \longrightarrow \mathbb{R}$ and suppose the Fourier sine series of f satisfies the constraint $\sum_{n,m=1}^{\infty}(n^2+m^2)|B_{nm}| < \infty$. Imitate Proposition 12.7 to find a Fourier series solution to the initial value problem for the two-dimensional *free Schrödinger equation*

$$\mathbf{i}\partial_t\,\omega = \frac{-1}{2}\triangle\,\omega \qquad (12.4)$$

on the box $\mathbb{X} = [0,\pi]^2$, with homogeneous Dirichlet boundary conditions. Prove that your solution converges, satisfies the desired initial conditions and boundary conditions, and satisfies eqn.(12.4). (Hint: imitate the strategy suggested in Exercise 12.6, and also Exercise 12.15 on page 222).

12.2(b) Nonhomogeneous Boundary Conditions

Prerequisites: §12.2(a), §12.1 **Recommended:** §12.3(b)

Proposition 12.11: (Heat Equation on Box; nonhomogeneous Dirichlet BC)

Let $\mathbb{X} = [0, \pi] \times [0, \pi]$. Let $f : \mathbb{X} \longrightarrow \mathbb{R}$ and let $L, R, T, B : [0, \pi] \longrightarrow \mathbb{R}$ be functions. Consider the heat equation

$$\partial_t u(x, y; t) = \triangle u(x, y; t),$$

with initial conditions

$$u(x, y; 0) = f(x, y), \qquad \text{for all } (x, y) \in \mathbb{X}, \tag{12.5}$$

and nonhomogeneous Dirichlet boundary conditions:

$$\left.\begin{array}{rclcrcll} u(x,\pi;t) & = & T(x) & \text{and} & u(x,0;t) & = & B(x), & \text{for all } x \in [0,\pi] \\ u(0,y;t) & = & L(y) & \text{and} & u(\pi,y;t) & = & R(y), & \text{for all } y \in [0,\pi] \end{array}\right\} \text{ for all } t > 0. \tag{12.6}$$

This problem is solved as follows:

1. *Let $w(x, y)$ be the solution[1] to the Laplace Equation "$\triangle w(x, y) = 0$", with the nonhomogeneous Dirichlet BC (12.6).*
2. *Define $g(x, y) = f(x, y) - w(x, y)$. Let $v(x, y; t)$ be the solution[2] to the heat equation "$\partial_t v(x, y; t) = \triangle v(x, y; t)$" with initial conditions $v(x, y; 0) = g(x, y)$, and homogeneous Dirichlet BC.*
3. *Define $u(x, y; t) = v(x, y; t) + w(x, y)$. Then $u(x, y; t)$ is a solution to the Heat Equation with initial conditions (12.5) and nonhomogeneous Dirichlet BC (12.6).*

Proof: **Exercise 12.10** □

Interpretation: In Proposition 12.11, the function $w(x, y)$ represents the *long-term thermal equilibrium* that the system is 'trying' to attain. The function $g(x, y) = f(x, y) - w(x, y)$ thus measures the *deviation* between the current state and this equilibrium, and the function $v(x, y; t)$ thus represents how this 'transient' deviation decays to zero over time.

Example 12.12: Suppose $T(x) = \sin(2x)$ and $R \equiv L \equiv 0$ and $B \equiv 0$. Then Proposition 12.4 on page 208 says

$$w(x, y) = \frac{\sin(2x)\sinh(2y)}{\sinh(2\pi)}.$$

[1] Obtained from Proposition 12.4 on page 208, for example.
[2] Obtained from Proposition 12.7 on page 210, for example.

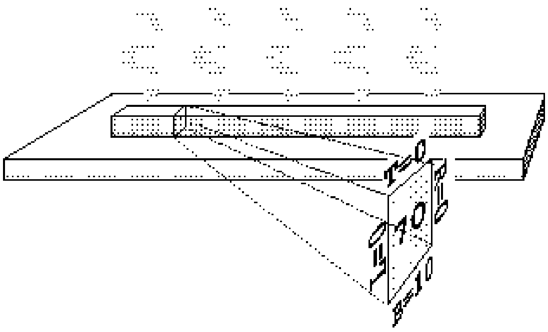


Figure 12.5: The temperature distribution of a baguette

Suppose $f(x, y) := \sin(2x)\sin(y)$. Then

$$\begin{aligned} g(x,y) &= f(x,y) - w(x,y) \quad = \quad \sin(2x)\sin(y) - \frac{\sin(2x)\sinh(2y)}{\sinh(2\pi)} \\ &\underset{(*)}{=} \sin(2x)\sin(y) \; - \; \left(\frac{\sin(2x)}{\sinh(2\pi)}\right)\left(\frac{2\sinh(2\pi)}{\pi}\right)\sum_{m=1}^{\infty}\frac{m(-1)^{m+1}}{2^2+m^2}\cdot\sin(my) \\ &= \sin(2x)\sin(y) \; - \; \frac{2\sin(2x)}{\pi}\sum_{m=1}^{\infty}\frac{m(-1)^{m+1}}{4+m^2}\cdot\sin(my). \end{aligned}$$

Here $(*)$ is because Example 8.3 on page 148 says $\sinh(2y) = \frac{2\sinh(2\pi)}{\pi}\sum_{m=1}^{\infty}\frac{m(-1)^{m+1}}{2^2+m^2}\cdot \sin(my)$. Thus, Proposition 12.7 on page 210 says that

$$v(x,y;t) \quad = \quad \sin(2x)\sin(y)e^{-5t} - \frac{2\sin(2x)}{\pi}\sum_{m=1}^{\infty}\frac{m(-1)^{m+1}}{4+m^2}\cdot\sin(mx)\exp(-(4+m^2)t).$$

Finally, Proposition 12.11 says the solution is $u(x,y;t) \;:=\; v(x,y;t) \;+\; \frac{\sin(2x)\sinh(2y)}{\sinh(2\pi)}$. ♢

Example 12.13: *A freshly baked baguette is removed from the oven and left on a wooden plank to cool near the window. The baguette is initially at a uniform temperature of* $90^o\,C$*; the air temperature is* $20^o\,C$*, and the temperature of the wooden plank (which was sitting in the sunlight) is* $30^o\,C$*.*

Mathematically model the cooling process near the center of the baguette. How long will it be before the baguette is cool enough to eat? (assuming 'cool enough' is below $40^o\,C$*.)*

Answer: For simplicity, we will assume the baguette has a square cross-section (and dimensions $\pi \times \pi$, of course). If we confine our attention to the middle of the baguette, we are far from the endpoints, so that we can neglect the longitudinal dimension and treat this as a two-dimensional problem.

Suppose the temperature distribution along a cross section through the center of the baguette is given by the function $u(x, y; t)$. To simplify the problem, we will subtract $20^o C$ off all temperatures. Thus, in the notation of Proposition 12.11 the boundary conditions are:

$$\begin{array}{rcl} L(y) \;=\; R(y) \;=\; T(x) &=& 0 \qquad \text{(the air)} \\ \text{and} \qquad B(x) &=& 10. \qquad \text{(the wooden plank)} \end{array}$$

and our initial temperature distribution is $f(x, y) = 70$ (see Figure 12.5).

From Proposition 12.1 on page 204, we know that the long-term equilibrium for these boundary conditions is given by:

$$w(x, y) \underset{L2}{\approx} \frac{40}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n \sinh(n\pi)} \sin(nx) \cdot \sinh(n(\pi - y)),$$

We want to represent this as a two-dimensional Fourier sine series. To do this, we need the (one-dimensional) Fourier sine series for $\sinh(nx)$. We set $\alpha = n$ in Example 8.3 on page 148, and get:

$$\sinh(nx) \underset{L2}{\approx} \frac{2\sinh(n\pi)}{\pi} \sum_{m=1}^{\infty} \frac{m(-1)^{m+1}}{n^2 + m^2} \cdot \sin(mx). \tag{12.7}$$

Thus,

$$\begin{aligned} \sinh\Big(n(\pi - y)\Big) &\underset{L2}{\approx} \frac{2\sinh(n\pi)}{\pi} \sum_{m=1}^{\infty} \frac{m(-1)^{m+1}}{n^2 + m^2} \cdot \sin(m\pi - my) \\ &= \frac{2\sinh(n\pi)}{\pi} \sum_{m=1}^{\infty} \frac{m}{n^2 + m^2} \cdot \sin(my), \end{aligned}$$

because $\sin(m\pi - ny) = \sin(m\pi)\cos(ny) - \cos(m\pi)\sin(ny) = (-1)^{m+1}\sin(ny)$. Substituting this into (12.7) yields:

$$\begin{aligned} w(x, y) &\underset{L2}{\approx} \frac{80}{\pi^2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sum_{m=1}^{\infty} \frac{m \cdot \sinh(n\pi)}{n \cdot \sinh(n\pi)(n^2 + m^2)} \sin(nx) \cdot \sin(my) \\ &= \frac{80}{\pi^2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sum_{m=1}^{\infty} \frac{m \cdot \sin(nx) \cdot \sin(my)}{n \cdot (n^2 + m^2)} \end{aligned} \tag{12.8}$$

Now, the initial temperature distribution is the constant function with value 70. Take the two-dimensional sine series from Example 10.2 on page 180, and multiply it by 70, to obtain:

$$f(x, y) = 70 \underset{L2}{\approx} \frac{1120}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{1}{n \cdot m} \sin(nx)\sin(my)$$

Thus,

$$g(x,y) = f(x,y) - w(x,y) \underset{\text{L2}}{\approx} \frac{1120}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{\sin(nx)\cdot\sin(my)}{n\cdot m} - \frac{80}{\pi^2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sum_{m=1}^{\infty} \frac{m\cdot\sin(nx)\cdot\sin(my)}{n\cdot(n^2+m^2)}$$

Thus,

$$v(x,y;t) \underset{\text{L2}}{\approx} \frac{1120}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{\sin(nx)\cdot\sin(my)}{n\cdot m} \exp\Big(-(n^2+m^2)t\Big) - \frac{80}{\pi^2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sum_{m=1}^{\infty} \frac{m\cdot\sin(nx)\cdot\sin(my)}{n\cdot(n^2+m^2)} \exp\Big(-(n^2+m^2)t\Big)$$

If we combine the second term in this expression with (12.8), we get the final answer:

$$u(x,y;t) = v(x,y;t) + w(x,y) \underset{\text{L2}}{\approx} \frac{1120}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{\sin(nx)\cdot\sin(my)}{n\cdot m} \exp\Big(-(n^2+m^2)t\Big) + \frac{80}{\pi^2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \sum_{m=1}^{\infty} \frac{m\cdot\sin(nx)\cdot\sin(my)}{n\cdot(n^2+m^2)} \Big[1-\exp\Big(-(n^2+m^2)t\Big)\Big]$$

◇

12.3 The Poisson Problem on a Square

12.3(a) Homogeneous Boundary Conditions

Prerequisites: §10.1, §6.5, §2.4 **Recommended:** §11.3, §8.3(e)

Proposition 12.14: *Let $\mathbb{X} = [0,\pi]\times[0,\pi]$, and let $q : \mathbb{X} \longrightarrow \mathbb{R}$ be some function. Suppose q has Fourier sine series: $q(x,y) \underset{\text{L2}}{\approx} \sum_{n,m=1}^{\infty} Q_{n,m}\sin(nx)\sin(my)$, and define the function $u(x,y)$ by $u(x,y) \underset{\text{unif}}{\equiv} \sum_{n,m=1}^{\infty} \frac{-Q_{n,m}}{n^2+m^2}\sin(nx)\sin(my)$, for all $(x,y)\in\mathbb{X}$.*

Then $u(x,y)$ is the unique solution to the Poisson equation "$\triangle u(x,y) = q(x,y)$", satisfying homogeneous Dirichlet boundary conditions $u(x,0) = u(0,y) = u(x,\pi) = u(\pi,y) = 0$.

Proof: **Exercise 12.11** (a) Use Proposition 1.7 on page 16 to show that u satisfies the Poisson equation.

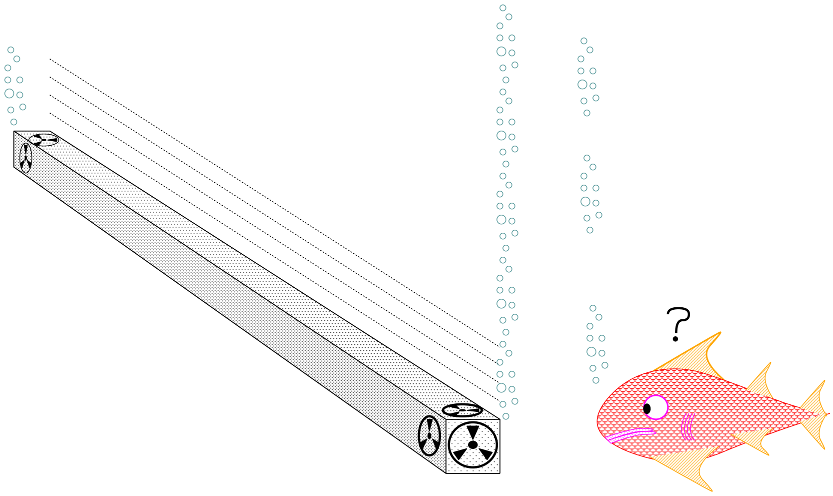


Figure 12.6: A jettisoned fuel rod in the Arctic Ocean

(b) Use Proposition 10.3(e) on page 183 to show that u satisfies homogeneous Dirichlet BC.

(c) Apply Theorem 6.14(a) on page 106 to conclude that this solution is unique. □

Example 12.15: *A nuclear submarine beneath the Arctic Ocean has jettisoned a fuel rod from its reactor core (Figure 12.6). The fuel rod is a very long, narrow, enriched uranium bar with square cross section. The radioactivity causes the fuel rod to be uniformly heated from within at a rate of Q, but the rod is immersed in freezing Arctic water. We want to compute its internal temperature distribution.*

Answer: The rod is immersed in freezing cold water, and is a good conductor, so we can assume that its outer surface takes the the surrounding water temperature of 0 degrees. Hence, we assume homogeneous Dirichlet boundary conditions.

Endow the rod with coordinate system (x, y, z), where z runs along the length of the rod. Since the rod is extremely long relative to its cross-section, we can neglect the z coordinate, and reduce to a 2-dimensional equation. The uniform heating is described by a constant function: $q(x, y) = Q$. From Example 10.2 on page 180, know that the constant function 1 has two-dimensional Fourier sine series:

$$1 \underset{\text{L2}}{\approx} \frac{16}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{1}{n \cdot m} \sin(nx)\sin(my)$$

Thus, $q(x, y) \underset{\text{L2}}{\approx} \frac{16Q}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{1}{n \cdot m} \sin(nx)\sin(my)$. The temperature distribution must satisfy Poisson's equation. Thus, the temperature distribution is:

$$u(x, y) \underset{\text{unif}}{\equiv} \frac{-16Q}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{1}{n \cdot m \cdot (n^2 + m^2)} \sin(nx)\sin(my).$$

◇

Example 12.16: Suppose $q(x,y) = x \cdot y$. Then the solution to the Poisson equation $\triangle u = q$ on the square, with homogeneous Dirichlet boundary conditions, is given by:

$$u(x,y) \underset{\text{unif}}{\equiv} 4 \sum_{n,m=1}^{\infty} \frac{(-1)^{n+m+1}}{nm \cdot (n^2+m^2)} \sin(nx)\sin(my)$$

To see this, recall from Example 10.1 on page 180 that the two-dimensional Fourier sine series for $q(x,y)$ is:

$$xy \underset{\text{L2}}{\approx} 4 \sum_{n,m=1}^{\infty} \frac{(-1)^{n+m}}{nm} \sin(nx)\sin(my).$$

Now apply Proposition 12.14. ♢

Proposition 12.17: *Let $\mathbb{X} = [0,\pi] \times [0,\pi]$, and let $q : \mathbb{X} \longrightarrow \mathbb{R}$ be some function. Suppose q has Fourier cosine series: $q(x,y) \underset{\text{L2}}{\approx} \sum_{n,m=0}^{\infty} Q_{n,m} \cos(nx)\cos(my)$, and suppose that $Q_{0,0} = 0$.*

Fix some constant $K \in \mathbb{R}$, and define the function $u(x,y)$ by

$$u(x,y) \underset{\text{unif}}{\equiv} \sum_{\substack{n,m=0 \\ \text{not both zero}}}^{\infty} \frac{-Q_{n,m}}{n^2+m^2} \cos(nx)\cos(my) \quad + \quad K, \qquad \text{for all } (x,y) \in \mathbb{X}. \quad (12.9)$$

Then $u(x,y)$ is a solution to the Poisson equation "$\triangle u(x,y) = q(x,y)$", satisfying homogeneous Neumann boundary conditions $\partial_y u(x,0) = \partial_x u(0,y) = \partial_y u(x,\pi) = \partial_x u(\pi,y) = 0$.

Furthermore, all solutions to this Poisson equation with these boundary conditions have the form (12.9).

If $Q_{0,0} \neq 0$, however, the problem has no solution.

Proof: **Exercise 12.12** (a) Use Proposition 1.7 on page 16 to show that u satisfies the Poisson equation.

(b) Use Proposition 10.3 on page 183 to show that u satisfies homogeneous Neumann BC.

(c) Apply Theorem 6.14(c) on page 106 to conclude that this solution is unique up to addition of a constant. □

Exercise 12.13 Mathematically, it is clear that the solution of Proposition 12.17 cannot be well-defined if $Q_{0,0} \neq 0$. Provide a physical explanation for why this is to be expected.

Example 12.18: Suppose $q(x,y) = \cos(2x)\cdot\cos(3y)$. Then the solution to the Poisson equation $\triangle u = q$ on the square, with homogeneous Neumann boundary conditions, is given by:

$$u(x,y) = \frac{-\cos(2x)\cdot\cos(3y)}{13}$$

To see this, note that the two-dimensional Fourier Cosine series of $q(x,y)$ is just $\cos(2x)\cdot\cos(3y)$. In other words, $A_{2,3} = 1$, and $A_{n,m} = 0$ for all other n and m. In particular, $A_{0,0} = 0$, so we can apply Proposition 12.17 to conclude: $u(x,y) = \frac{-\cos(2x)\cdot\cos(3y)}{2^2+3^2} = \frac{-\cos(2x)\cdot\cos(3y)}{13}$. ♢

12.3(b) Nonhomogeneous Boundary Conditions

Prerequisites: §12.3(a), §12.1 **Recommended:** §12.2(b)

Proposition 12.19: (Poisson Equation on Box; nonhomogeneous Dirichlet BC)

Let $\mathbb{X} = [0,\pi]\times[0,\pi]$. Let $q : \mathbb{X} \longrightarrow \mathbb{R}$ and $L, R, T, B : [0,\pi] \longrightarrow \mathbb{R}$ be functions. Consider the Poisson equation

$$\triangle\, u(x,y) = q(x,y), \qquad (12.10)$$

with nonhomogeneous Dirichlet boundary conditions:

$$\begin{array}{rclcrcll} u(x,\pi) &=& T(x) & \text{and} & u(x,0) &=& B(x), & \text{for all } x\in[0,\pi] \\ u(0,y) &=& L(y) & \text{and} & u(\pi,y) &=& R(y), & \text{for all } y\in[0,\pi] \end{array} \qquad (12.11)$$

(see Figure 12.1(B) on page 204). This problem is solved as follows:

1. *Let $v(x,y)$ be the solution[3] to the Poisson equation (12.10) with homogeneous Dirichlet BC: $v(x,0) = v(0,y) = v(x,\pi) = v(\pi,y) = 0$.*
2. *Let $w(x,y)$ be the solution[4] to the Laplace Eqation "$\triangle w(x,y) = 0$", with the nonhomogeneous Dirichlet BC (12.11).*
3. *Define $u(x,y) = v(x,y) + w(x,y)$; then $u(x,y)$ is a solution to the Poisson problem with the nonhomogeneous Dirichlet BC (12.11).*

Proof: **Exercise 12.14** □

Example 12.20: *Suppose $q(x,y) = x\cdot y$. Find the solution to the Poisson equation $\triangle u = q$ on the square, with* **nonhomogeneous** *Dirichlet boundary conditions:*

$$u(0,y) = 0; \quad u(\pi,y) = 0; \quad u(x,0) = 0; \qquad (12.12)$$

$$u(x,\pi) = T(x) = \begin{cases} x & \text{if } 0\le x\le \frac{\pi}{2} \\ \frac{\pi}{2} - x & \text{if } \frac{\pi}{2} < x \le \pi \end{cases} \quad \text{(see Figure 8.5(B) on page 163)} \qquad (12.13)$$

[3]Obtained from Proposition 12.14 on page 217, for example.
[4]Obtained from Proposition 12.4 on page 208, for example.

Solution: In Example 12.16, we found the solution to the Poisson equation $\triangle v = q$, with homogeneous Dirichlet boundary conditions; it was:

$$v(x,y) \underset{\text{unif}}{\equiv} 4 \sum_{n,m=1}^{\infty} \frac{(-1)^{n+m+1}}{nm \cdot (n^2+m^2)} \sin(nx)\sin(my).$$

In Example 12.6 on page 209, we found the solution to the Laplace equation $\triangle w = 0$, with nonhomogeneous Dirichlet boundary conditions (12.12) and (12.13); it was:

$$w(x,y) \underset{\text{L2}}{\approx} \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd;} \\ n=2k+1}}^{\infty} \frac{(-1)^k}{n^2 \sinh(n\pi)} \sin(nx)\sinh(ny).$$

Thus, according to Proposition 12.19 on the preceding page, the solution to the nonhomogeneous Poisson problem is:

$$\begin{aligned} u(x,y) &= v(x,y) + w(x,y) \\ &\underset{\text{L2}}{\approx} 4 \sum_{n,m=1}^{\infty} \frac{(-1)^{n+m+1}}{nm \cdot (n^2+m^2)} \sin(nx)\sin(my) + \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd;} \\ n=2k+1}}^{\infty} \frac{(-1)^k}{n^2 \sinh(n\pi)} \sin(nx)\sinh(ny). \end{aligned}$$

$\diamondsuit$

12.4 The Wave Equation on a Square (The Square Drum)

Prerequisites: §10.1, §6.4, §6.5, §3.2(b), §1.7 **Recommended:** §11.2, §8.3(e)

Imagine a drumskin stretched tightly over a square frame. At equilibrium, the drumskin is perfectly flat, but if we strike the skin, it will vibrate, meaning that the membrane will experience vertical displacements from equilibrium. Let $\mathbb{X} = [0,\pi] \times [0,\pi]$ represent the square skin, and for any point $(x,y) \in \mathbb{X}$ on the drumskin and time $t > 0$, let $u(x,y;t)$ be the vertical displacement of the drum. Then u will obey the two-dimensional Wave Equation:

$$\partial_t^2 u(x,y;t) = \triangle u(x,y;t). \tag{12.14}$$

However, since the skin is held down along the edges of the box, the function u will also exhibit homogeneous **Dirichlet** boundary conditions

$$\left.\begin{array}{rclcrcll} u(x,\pi;t) &=& 0 & \text{and} & u(x,0;t) &=& 0, & \text{for all } x \in [0,\pi] \\ u(0,y;t) &=& 0 & \text{and} & u(\pi,y;t) &=& 0, & \text{for all } y \in [0,\pi] \end{array}\right\} \text{ for all } t > 0. \tag{12.15}$$

Proposition 12.21: (Initial Position for Drumskin)

Let $\mathbb{X} = [0,\pi] \times [0,\pi]$, and let $f_0 : \mathbb{X} \longrightarrow \mathbb{R}$ be a function describing the initial displacement of the drumskin. Suppose f_0 has Fourier Sine Series $f_0(x,y) \underset{\text{unif}}{\equiv} \sum_{n,m=1}^{\infty} B_{n,m} \sin(nx)\sin(my)$, *suh that:*

$$\sum_{n,m=1}^{\infty} (n^2+m^2)|B_{n,m}| < \infty. \tag{12.16}$$

Define:

$$w(x,y;t) \underset{\text{unif}}{\equiv} \sum_{n,m=1}^{\infty} B_{n,m}\sin(nx)\cdot\sin(my)\cdot\cos\left(\sqrt{n^2+m^2}\ \cdot t\right), \quad \text{for all } (x,y)\in\mathbb{X} \text{ and } t\geq 0. \tag{12.17}$$

Then series (12.17) *converges uniformly, and* $w(x,y;t)$ *is the unique solution to the Wave Equation* (12.14), *satisfying the Dirichlet boundary conditions* (14.30), *as well as*

$$\left.\begin{array}{lrcl} \textbf{Initial Position:} & w(x,y,0) & = & f_0(x,y), \\ \textbf{Initial Velocity:} & \partial_t\, w(x,y,0) & = & 0, \end{array}\right\} \quad \textit{for all } (x,y)\in\mathbb{X}.$$

Proof: <u>**Exercise 12.15**</u> (a) Use the hypothesis (12.16) and Proposition 1.7 on page 16 to conclude that

$$\partial_t^2\, w(x,y;t) \underset{\text{unif}}{\equiv} -\sum_{n,m=1}^{\infty} (n^2+m^2)\cdot B_{n,m}\sin(nx)\cdot\sin(my)\cdot\cos\left(\sqrt{n^2+m^2}\ \cdot t\right) \underset{\text{unif}}{\equiv} \triangle w(x,y;t)$$

for all $(x,y)\in\mathbb{X}$ and $t>0$.

(b) Check that the Fourier series (12.17) converges uniformly.

(c) Use Theorem 10.3(e) on page 183 to conclude that $u(x,y;t)$ satisfies Dirichlet boundary conditions.

(d) Set $t=0$ to check the initial position.

(e) Set $t=0$ and use Proposition 1.7 on page 16 to check initial velocity.

(f) Apply Theorem 6.18 on page 108 to show that this solution is unique. □

Example 12.22: Suppose $f_0(x,y) = \sin(2x)\cdot\sin(3y)$. Then the solution to the wave equation on the square, with initial position f_0, and homogeneous Dirichlet boundary conditions, is given by:

$$w(x,y;t) \quad = \quad \sin(2x)\cdot\sin(3y)\cdot\cos(\sqrt{13}\ t)$$

To see this, note that the two-dimensional Fourier sine series of $f_0(x,y)$ is just $\sin(2x)\cdot\sin(3y)$. In other words, $B_{2,3}=1$, and $B_{n,m}=0$ for all other n and m. Apply Proposition 12.21 to conclude: $w(x,y;t) = \sin(2x)\cdot\sin(3y)\cdot\cos\left(\sqrt{2^2+3^2}\ t\right) = \sin(2x)\cdot\sin(3y)\cdot\cos(\sqrt{13}\ t)$.
♢

Proposition 12.23: (Initial Velocity for Drumskin)

Let $\mathbb{X} = [0,\pi] \times [0,\pi]$, and let $f_1 : \mathbb{X} \longrightarrow \mathbb{R}$ be a function describing the initial velocity of the drumskin. Suppose f_1 has Fourier Sine Series $f_1(x,y) \underset{\text{unif}}{\equiv} \sum_{n,m=1}^{\infty} B_{n,m} \sin(nx)\sin(my)$, *such that*

$$\sum_{n,m=1}^{\infty} \sqrt{n^2+m^2} \cdot |B_{n,m}| < \infty. \tag{12.18}$$

Define:

$$v(x,y;t) \underset{\text{unif}}{\equiv} \sum_{n,m=1}^{\infty} \frac{B_{n,m}}{\sqrt{n^2+m^2}} \sin(nx)\cdot\sin(my)\cdot\sin\left(\sqrt{n^2+m^2}\ \cdot t\right), \quad \textit{for all } (x,y) \in \mathbb{X} \textit{ and } t \geq 0. \tag{12.19}$$

Then the series (12.19) *converges uniformly, and $v(x,y;t)$ is the unique solution to the Wave Equation* (12.14), *satisfying the Dirichlet boundary conditions* (14.30), *as well as*

$$\left.\begin{array}{rrcl} \textbf{Initial Position:} & v(x,y,0) & = & 0; \\ \textbf{Initial Velocity:} & \partial_t\, v(x,y,0) & = & f_1(x,y). \end{array}\right\} \quad \textit{for all } (x,y) \in \mathbb{X}.$$

Proof: **Exercise 12.16** (a) Use the hypothesis (12.18) and Proposition 1.7 on page 16 to conclude that

$$\partial_t^2\, w(x,y;t) \underset{\text{unif}}{\equiv} -\sum_{n,m=1}^{\infty} \sqrt{n^2+m^2}\cdot B_{n,m} \sin(nx)\cdot\sin(my)\cdot\cos\left(\sqrt{n^2+m^2}\ \cdot t\right) \underset{\text{unif}}{\equiv} \triangle w(x,y;t)$$

for all $(x,y) \in \mathbb{X}$ and $t > 0$.

(b) Check that the Fourier series (12.19) converges uniformly.

(c) Use Theorem 10.3(e) on page 183 to conclude that $u(x,y;t)$ satisfies Dirichlet boundary conditions.

(d) Set $t = 0$ to check the initial position.

(e) Set $t = 0$ and use Proposition 1.7 on page 16 to check initial velocity.

(f) Apply Theorem 6.18 on page 108 to show that this solution is unique □

Remark: Note that it is important in these theorems not only for the Fourier series (12.17) and (12.19) to converge uniformly, but also for their formal *second derivative* series to converge uniformly. This is not guaranteed. This is the reason for imposing the hypotheses (12.16) and (12.18).

Example 12.24: Suppose $f_1(x,y) = 1$. From Example 10.2 on page 180, we know that f_1 has two-dimensional Fourier sine series

$$1 \underset{\text{L2}}{\approx} \frac{16}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{1}{n \cdot m} \sin(nx)\sin(my)$$

Thus, the solution to the two-dimensional Wave equation, with homogeneous Dirichlet boundary conditions and initial velocity f_0, is given:

$$w(x, y;\ t) \quad \underset{L2}{\approx} \quad \frac{16}{\pi^2} \sum_{\substack{n,m=1 \\ \text{both odd}}}^{\infty} \frac{1}{n \cdot m \cdot \sqrt{n^2 + m^2}} \sin(nx) \sin(my) \sin\left(\sqrt{n^2 + m^2}\ \cdot t\right).$$

Remark: This example is somewhat bogus, because condition (12.18) is not satisfied, ♢

Question: For the solutions of the Heat Equation and Poisson equation, in Propositions 12.7, 12.9, and 12.14, we did not need to impose explicit hypotheses guaranteeing the uniform convergence of the given series (and its derivatives). But we *do* need explicit hypotheses to get convergence for the Wave Equation. Why is this?

12.5 Practice Problems

1. Let $f(y) \;=\; 4\sin(5y)$ for all $y \in [0, \pi]$.

   (a) Solve the **two-dimensional Laplace Equation** ($\triangle u \;=\; 0$) on the square domain $\mathbb{X} = [0, \pi] \times [0, \pi]$, with **nonhomogeneous Dirichlet boundary conditions:**

   $$\begin{array}{rclcrcll} u(x,0) &=& 0 & \text{and} & u(x,\pi) &=& 0, & \text{for all } x \in [0,\pi] \\ u(0,y) &=& 0 & \text{and} & u(\pi,y) &=& f(y), & \text{for all } y \in [0,\pi]. \end{array}$$

   (b) *Verify* your solution to part (a) (ie. check boundary conditions, Laplacian, etc.).

2. Let $f_1(x, y) \;=\; \sin(3x)\sin(4y)$.

   (a) Solve the **two-dimensional Wave Equation** ($\partial_t^2 u \;=\; \triangle u$) on the square domain $\mathbb{X} = [0, \pi] \times [0, \pi]$, with on the square domain $\mathbb{X} = [0, \pi] \times [0, \pi]$, with **homogeneous Dirichlet boundary conditions**, and initial conditions:

   $$\begin{array}{rrcll} \text{Initial position:} & u(x,y,0) &=& 0 & \text{for all } (x,y) \in \mathbb{X} \\ \text{Initial velocity:} & \partial_t\, u(x,y,0) &=& f_1(x,y) & \text{for all } (x,y) \in \mathbb{X} \end{array}$$

   (b) *Verify* your that solution in part (a) satisfies the required initial conditions (don't worry about boundary conditions or checking the Wave equation).

3. Solve the two-dimensional **Laplace Equation** $\triangle h \;=\; 0$ on the square domain $\mathbb{X} = [0, \pi]^2$, with **inhomogeneous Dirichlet** boundary conditions:

   (a) $h(\pi, y) \;=\; \sin(2y)$ and $h(0, y) \;=\; 0$, for all $y \in [0, \pi]$;
   $h(x, 0) \;=\; 0 \;=\; h(x, \pi)$ for all $x \in [0, \pi]$.

   (b) $h(\pi, y) \;=\; 0$ and $h(0, y) \;=\; \sin(4y)$, for all $y \in [0, \pi]$;
   $h(x, \pi) \;=\; \sin(3x)$; $h(x, 0) \;=\; 0$, for all $x \in [0, \pi]$.

4. Let $\mathbb{X} = [0, \pi]^2$ and let $q(x, y) = \sin(x) \cdot \sin(3y) + 7\sin(4x) \cdot \sin(2y)$. Solve the Poisson Equation $\triangle u(x, y) = q(x, y)$. with **homogeneous Dirichlet** boundary conditions.

5. Let $\mathbb{X} = [0, \pi]^2$. Solve the **Heat Equation** $\partial_t u(x, y; t) = \triangle u(x, y; t)$ on $\mathbb{X}$, with initial conditions $u(x, y; 0) = \cos(5x) \cdot \cos(y)$. and **homogeneous Neumann** boundary conditions.

6. Let $f(x, y) = \cos(2x)\cos(3y)$. Solve the following boundary value problems on the square domain $\mathbb{X} = [0, \pi]^2$ (**Hint:** see problem #3 of §10.3).

   (a) Solve the two-dimensional **Heat Equation** $\partial_t u = \triangle u$, with homogeneous **Neumann** boundary conditions, and initial conditions $u(x, y; 0) = f(x, y)$.

   (b) Solve the two-dimensional **Wave Equation** $\partial_t^2 u = \triangle u$, with homogeneous **Dirichlet** boundary conditions, initial **position** $w(x, y; 0) = f(x, y)$ and initial **velocity** $\partial_t w(x, y; 0) = 0$.

   (c) Solve the two-dimensional **Poisson Equation** $\triangle u = f$ with homogeneous **Neumann** boundary conditions.

   (d) Solve the two-dimensional **Poisson Equation** $\triangle u = f$ with homogeneous **Dirichlet** boundary conditions.

   (e) Solve the two-dimensional **Poisson Equation** $\triangle v = f$ with **inhomogeneous Dirichlet** boundary conditions:

$$v(\pi, y) = \sin(2y); \qquad v(0, y) = 0 \quad \text{for all } y \in [0, \pi].$$
$$v(x, 0) = 0 = v(x, \pi) \qquad \text{for all } x \in [0, \pi].$$

7. $\mathbb{X} = [0, \pi]^2$ be the **box** of sidelength π. Let $f(x, y) = \sin(3x) \cdot \sinh(3y)$. (**Hint:** see problem #4 of §10.3).

   (a) Solve the **Heat Equation** on $\mathbb{X}$, with **initial conditions** $u(x, y; 0) = f(x, y)$, and **homogeneous Dirichlet** boundary conditions.

   (b) Let $T(x) = \sin(3x)$. Solve the **Laplace Equation** $\triangle u(x, y) = 0$ on the box, with **inhomogeneous Dirichlet** boundary conditions: $u(x, \pi) = T(x)$ and $u(x, 0) = 0$ for $x \in [0, \pi]$; $u(0, y) = 0 = u(\pi, y)$, for $y \in [0, \pi]$.

   (c) Solve the **Heat Equation** on the box with initial conditions on the box $\mathbb{X}$, with **initial conditions** $u(x, y; 0) = 0$, and the same **inhomogeneous Dirichlet** boundary conditions as in part (b).

Notes: ....

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13 BVP's on a Cube

The Fourier series technique used to solve BVPs on a square box extends readily to 3-dimensional cubes, and indeed, to rectilinear domains in any number of dimensions. As in Chapter 12, we will confine our exposition to the cube $[0,\pi]^3$, and assume that the physical constants in the various equations are all set to one. Thus, the Heat Equation becomes "$\partial_t u = \triangle u$", the Wave Equation is "$\partial_t^2 u = \triangle u$", etc. This allows us to develop the solution methods with minimum technicalities. The extension of each solution method to equations with arbitrary physical constants on an arbitrary rectangular domain $[0,X]\times[0,Y]$ (for some $X,Y>0$) is left as a straightforward (but important!) exercise.

We will use the following notation:

- The cube of dimensions $\pi\times\pi\times\pi$ is denoted $\mathbb{X} \;=\; [0,\pi]\times[0,\pi]\times[0,\pi] \;=\; [0,\pi]^3$.
- A point in the cube will be indicated by a vector $\mathbf{x}=(x_1,x_2,x_3)$, where $0\le x_1,x_2,x_3\le \pi$.
- If $f:\mathbb{X}\longrightarrow\mathbb{R}$ is a function on the cube, then

$$\triangle f(\mathbf{x}) \;=\; \partial_1^2 f(\mathbf{x}) + \partial_2^2 f(\mathbf{x}) + \partial_3^2 f(\mathbf{x}).$$

- A triple of natural numbers will be denoted by $\mathbf{n}=(n_1,n_2,n_3)$, where $n_1,n_2,n_3\in\mathbb{N}=\{1,2,3,4,\ldots\}$. Let $\mathbb{N}^3$ be the set of all such triples. Thus, an expression of the form

$$\sum_{\mathbf{n}\in\mathbb{N}^3}(\text{something about } \mathbf{n})$$

should be read as: "$\displaystyle\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}(\text{something about } (n_1,n_2,n_3))$".

Let $\mathbb{N}_0=\{0,1,2,3,4,\ldots\}$, and let $\mathbb{N}_0^3$ be the set of all triples $\mathbf{n}=(n_1,n_2,n_3)$ in $\mathbb{N}_0$. Thus, an expression of the form

$$\sum_{\mathbf{n}\in\mathbb{N}_0^3}(\text{something about } \mathbf{n})$$

should be read as: "$\displaystyle\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}(\text{something about } (n_1,n_2,n_3))$".

- For any $\mathbf{n}\in\mathbb{N}^3$, $\mathbf{S_n}(\mathbf{x}) \;=\; \sin(n_1x_1)\cdot\sin(n_2x_2)\cdot\sin(n_3x_3)$. The Fourier *sine* series of a function $f(\mathbf{x})$ thus has the form: $f(\mathbf{x}) \;=\; \displaystyle\sum_{\mathbf{n}\in\mathbb{N}^3} B_{\mathbf{n}}\mathbf{S_n}(\mathbf{x})$
- For any $\mathbf{n}\in\mathbb{N}_0^3$, $\mathbf{C_n}(\mathbf{x}) \;=\; \cos(n_1x_1)\cdot\cos(n_2x_2)\cdot\cos(n_3x_3)$. The Fourier *cosine* series of a function $f(\mathbf{x})$ thus has the form: $f(\mathbf{x}) \;=\; \displaystyle\sum_{\mathbf{n}\in\mathbb{N}_0^3} A_{\mathbf{n}}\mathbf{C_n}(\mathbf{x})$

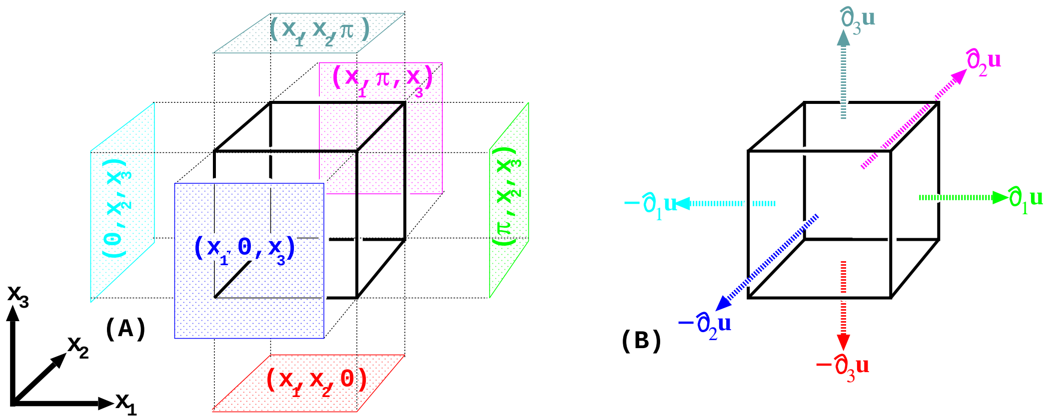

Figure 13.1: Boundary conditions on a cube: **(A)** Dirichlet. **(B)** Neumann.

- For any $\mathbf{n} \in \mathbb{N}_0^3$, let $\|\mathbf{n}\| = \sqrt{n_1^2 + n_2^2 + n_3^2}$. In particular, note that:

$$\triangle \mathbf{S_n} = -\|\mathbf{n}\|^2 \cdot \mathbf{S_n}, \qquad \text{and} \quad \triangle\, \mathbf{C_n} = -\|\mathbf{n}\|^2 \cdot \mathbf{C_n} \qquad (\textbf{Exercise 13.1})$$

13.1 The Heat Equation on a Cube

Prerequisites: §10.2, §6.4, §6.5, §2.2(b) **Recommended:** §11.1, §12.2(a), §8.3(e)

Proposition 13.1: (Heat Equation; homogeneous Dirichlet BC)

Consider the cube $\mathbb{X} = [0, \pi]^3$, and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be some function describing an initial heat distribution. Suppose f has Fourier sine series $f(\mathbf{x}) \underset{\text{L2}}{\approx} \sum_{\mathbf{n} \in \mathbb{N}^3} B_{\mathbf{n}} \mathbf{S_n}(\mathbf{x})$. Define:

$$u(\mathbf{x}; t) \underset{\text{L2}}{\approx} \sum_{\mathbf{n} \in \mathbb{N}^3} B_{\mathbf{n}} \mathbf{S_n}(\mathbf{x}) \cdot \exp\Big(-\|\mathbf{n}\|^2 \cdot t \Big).$$

Then $u(\mathbf{x}; t)$ is the unique solution to the Heat Equation "$\partial_t u = \triangle u$", with homogeneous **Dirichlet** *boundary conditions*

$$\begin{aligned} u(x_1, x_2, 0; t) &= u(x_1, x_2, \pi; t) = u(x_1, 0, x_3; t) \\ &= u(x_1, \pi, x_3; t) = u(0, x_2, x_3; t) = u(\pi, x_2, x_3, ; t) = 0, \qquad \textit{(see Figure 13.1A)} \end{aligned}$$

and initial conditions: $u(\mathbf{x}; 0) = f(\mathbf{x})$.

Furthermore, the series defining u converges semiuniformly on $\mathbb{X} \times (0, \infty)$.

Proof: **Exercise 13.2** □

Example: *An ice cube is removed from a freezer (ambient temperature* -10^o C*) and dropped into a pitcher of freshly brewed tea (initial temperature* $+90^o$ C*). We want to compute how long it takes the ice cube to melt.*

Answer: We will assume that the cube has an initially uniform temperature of -10^o C and is completely immersed in the tea[1]. We will also assume that the pitcher is large enough that its temperature doesn't change during the experiment.

We assume the outer surface of the cube takes the temperature of the surrounding tea. By subtracting 90 from the temperature of the cube and the water, we can set the water to have temperature 0 and the cube, -100. Hence, we assume homogeneous Dirichlet boundary conditions; the initial temperature distribution is a constant function: $f(\mathbf{x}) = -100$. The constant function -100 has Fourier sine series:

$$-100 \underset{L2}{\approx} \frac{-6400}{\pi^3} \sum_{\substack{\mathbf{n}\in\mathbb{N}^3 \\ n_1,n_2,n_3 \text{ all odd}}}^{\infty} \frac{1}{n_1 n_2 n_3} \mathbf{S_n}(\mathbf{x}) \qquad (\underline{\textbf{Exercise 13.3}})$$

Let κ be the thermal conductivity of the ice. Thus, the time-varying thermal profile of the cube is given[2]

$$\boxed{u(\mathbf{x};t) \underset{L2}{\approx} \frac{-6400}{\pi^3} \sum_{\substack{\mathbf{n}\in\mathbb{N}^3 \\ n_1,n_2,n_3 \text{ all odd}}}^{\infty} \frac{1}{n_1 n_2 n_3} \mathbf{S_n}(\mathbf{x}) \exp\Big(-\|\mathbf{n}\|^2 \cdot \kappa \cdot t\Big)}.$$

Thus, to determine how long it takes the cube to melt, we must solve for the minimum value of t such that $u(\mathbf{x}, t) > -90$ everywhere (recall than -90 corresponds to 0^o C.). The coldest point in the cube is always at its center (**Exercise 13.4**), which has coordinates $\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$, so we need to solve for t in the inequality $u\left(\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right); t\right) \geq -90$, which is equivalent to

$$\begin{aligned}
\frac{90 \cdot \pi^3}{6400} \quad &\geq \quad \sum_{\substack{\mathbf{n}\in\mathbb{N}^3 \\ n_1,n_2,n_3 \text{ all odd}}}^{\infty} \frac{1}{n_1 n_2 n_3} \mathbf{S_n}\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) \exp\Big(-\|\mathbf{n}\|^2 \cdot \kappa \cdot t\Big) \\
&= \quad \sum_{\substack{\mathbf{n}\in\mathbb{N}^3 \\ n_1,n_2,n_3 \text{ all odd}}}^{\infty} \frac{1}{n_1 n_2 n_3} \sin\left(\frac{n_1\pi}{2}\right) \sin\left(\frac{n_2\pi}{2}\right) \sin\left(\frac{n_3\pi}{2}\right) \exp\Big(-\|\mathbf{n}\|^2 \cdot \kappa \cdot t\Big) \\
&\overline{\overline{\scriptstyle (8.8)}} \quad \sum_{k_1,k_2,k_3\in\mathbb{N}} \frac{(-1)^{k_1+k_2+k_3} \exp\Big(-\kappa \cdot \big[(2k_1+1)^2 + (2k_2+1)^2 + (2k_3+1)^2\big] \cdot t\Big)}{(2k_1+1)\cdot(2k_2+1)\cdot(2k_3+1)}.
\end{aligned}$$

where **(8.8)** is by eqn. (8.8) on p. 156. The solution of this inequality is **Exercise 13.5** .

[1] Unrealistic, since actually the cube floats just at the surface.

[2] Actually, this is physically unrealistic for two reasons. First, as the ice melts, additional thermal energy is absorbed in the phase transition from solid to liquid. Second, once part of the ice cube has melted, its thermal properties change; liquid water has a different thermal conductivity, and in addition, transports heat through convection.

Exercise 13.6 Imitating Proposition 13.1, find a Fourier series solution to the initial value problem for the *free Schrödinger equation*

$$\mathbf{i}\partial_t \omega \quad = \quad \frac{-1}{2} \triangle \omega,$$

on the cube $\mathbb{X} = [0, \pi]^3$, with homogeneous Dirichlet boundary conditions. Prove that your solution converges, satisfies the desired initial conditions and boundary conditions, and satisfies the Schrödinger equation.

Proposition 13.2: (Heat Equation; homogeneous Neumann BC)

Consider the cube $\mathbb{X} = [0, \pi]^3$, and let $f : \mathbb{X} \longrightarrow \mathbb{R}$ be some function describing an initial heat distribution. Suppose f has Fourier Cosine Series $f(\mathbf{x}) \underset{L2}{\approx} \sum_{\mathbf{n} \in \mathbb{N}_0^3} A_{\mathbf{n}} \mathbf{C}_{\mathbf{n}}(\mathbf{x})$. Define:

$$u(\mathbf{x}; t) \underset{L2}{\approx} \sum_{\mathbf{n} \in \mathbb{N}_0^3} A_{\mathbf{n}} \mathbf{C}_{\mathbf{n}}(\mathbf{x}) \cdot \exp\Big(- \|\mathbf{n}\|^2 \cdot t \Big)$$

Then $u(\mathbf{x}; t)$ is the unique solution to the Heat Equation "$\partial_t u = \triangle u$", with homogeneous **Neumann** *boundary conditions*

$$\begin{aligned} \partial_3 u(x_1, x_2, 0; t) = \partial_3 u(x_1, x_2, \pi; t) = \partial_2 u(x_1, 0, x_3; t) &= \\ \partial_2 u(x_1, \pi, x_3; t) = \partial_1 u(0, x_2, x_3; t) = \partial_1 u(\pi, x_2, x_3, ; t) &= 0. \qquad \textit{(see Figure 13.1B)} \end{aligned}$$

and initial conditions: $u(\mathbf{x}; 0) = f(\mathbf{x})$.

Furthermore, the series defining u converges semiuniformly on $\mathbb{X} \times (0, \infty)$.

Proof: **Exercise 13.7** □

13.2 The (nonhomogeneous) Dirichlet problem on a Cube

Prerequisites: §10.2, §6.5(a), §2.3 **Recommended:** §8.3(e), §12.1

Proposition 13.3: (Laplace Equation; one constant nonhomogeneous Dirichlet BC)

Let $\mathbb{X} = [0, \pi]^3$, and consider the Laplace equation "$\triangle u = 0$", with nonhomogeneous **Dirichlet** *boundary conditions (see Figure 13.2A):*

$$u(x_1, x_2, 0) = u(x_1, 0, x_3) = u(x_1, \pi, x_3) = u(0, x_2, x_3) = u(\pi, x_2, x_3,) \quad = \quad 0; \qquad (13.1)$$

$$u(x_1, x_2, \pi) \quad = \quad 1. \qquad (13.2)$$

The unique solution to this problem is the function

$$u(x, y, z) \underset{L2}{\approx} \sum_{\substack{n,m=1 \\ n,m \text{ both odd}}}^{\infty} \frac{16}{nm\pi \sinh(\pi\sqrt{n^2 + m^2})} \sin(nx) \sin(my) \cdot \sinh(\sqrt{n^2 + m^2} \cdot z).$$

Furthermore, this series converges semiuniformly on $\mathrm{int}(\mathbb{X})$.

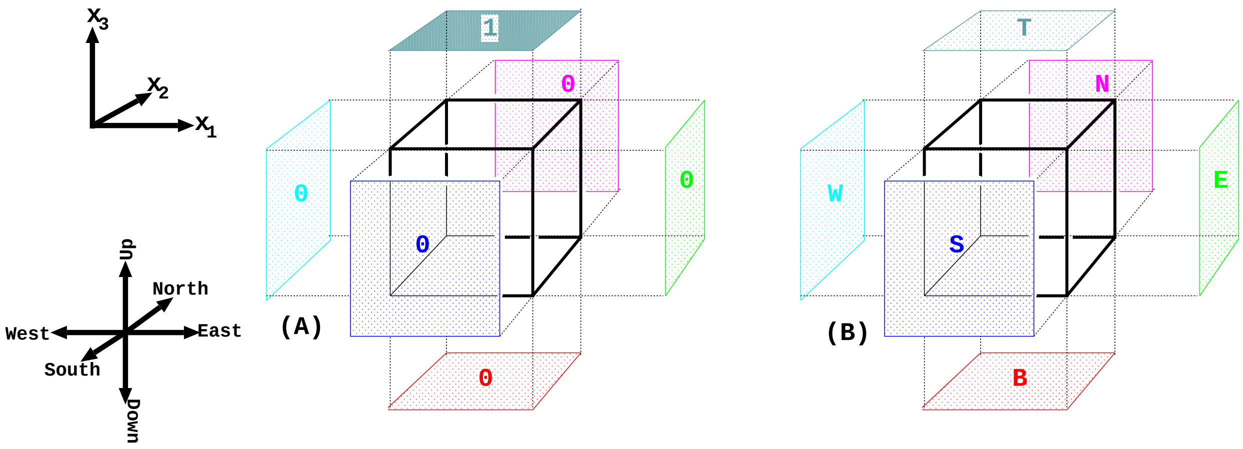

Figure 13.2: Dirichlet boundary conditions on a cube **(A)** Constant; Nonhomogeneous on one side only. **(B)** Arbitrary nonhomogeneous on all sides.

Proof: **Exercise 13.8** (a) Check that the series and its formal Laplacian both converge uniformly. (b) Check that each of the functions $u_{n,m}(\mathbf{x}) = \sin(nx)\sin(my)\cdot\sinh(\sqrt{n^2+m^2}z)$ satisfies the Laplace equation and the first boundary condition (13.1). (c) To check that the solution also satisfies the boundary condition (13.2), subsititute $y = \pi$ to get:

$$\begin{aligned} u(x,y,\pi) &= \sum_{\substack{n,m=1 \\ n,m \text{ both odd}}}^{\infty} \frac{16}{nm\pi\sinh(\pi\sqrt{n^2+m^2})}\sin(nx)\sin(my)\cdot\sinh(\sqrt{n^2+m^2}\pi) \\ &= \sum_{\substack{n,m=1 \\ n,m \text{ both odd}}}^{\infty} \frac{16}{nm\pi}\sin(nx)\sin(my) \underset{\text{L2}}{\approx} 1 \end{aligned}$$

because this is the Fourier sine series for the function $b(x,y) = 1$, by Example 10.2 on page 180.

(d) Apply Theorem 6.14(a) on page 106 to conclude that this solution is unique. □

Proposition 13.4: (Laplace Equation; arbitrary nonhomogeneous Dirichlet BC)

Let $\mathbb{X} = [0,\pi]^3$, and consider the Laplace equation "$\triangle h = 0$", with nonhomogeneous Dirichlet boundary conditions (see Figure 13.2B):

$$\begin{aligned} h(x_1,x_2,0) &= D(x_1,x_2) & h(x_1,x_2,\pi) &= U(x_1,x_2) \\ h(x_1,0,x_3) &= S(x_1,x_3) & h(x_1,\pi,x_3) &= N(x_1,x_3) \\ h(0,x_2,x_3) &= W(x_2,x_3) & h(\pi,x_2,x_3,) &= E(x_2,x_3) \end{aligned}$$

where $D(x_1,x_2)$, $U(x_1,x_2)$, $S(x_1,x_3)$, $N(x_1,x_3)$, $W(x_2,x_3)$, and $E(x_2,x_3)$ are six func-

tions. Suppose that these functions have two-dimensional Fourier sine series:

$$D(x_1,x_2) \underset{L2}{\approx} \sum_{n_1,n_2=1}^{\infty} D_{n_1,n_2} \sin(n_1x_1)\sin(n_2x_2); \qquad U(x_1,x_2) \underset{L2}{\approx} \sum_{n_1,n_2=1}^{\infty} U_{n_1,n_2} \sin(n_1x_1)\sin(n_2x_2);$$

$$S(x_1,x_3) \underset{L2}{\approx} \sum_{n_1,n_3=1}^{\infty} S_{n_1,n_3} \sin(n_1x_1)\sin(n_3x_3); \qquad N(x_1,x_3) \underset{L2}{\approx} \sum_{n_1,n_3=1}^{\infty} N_{n_1,n_3} \sin(n_1x_1)\sin(n_3x_3);$$

$$W(x_2,x_3) \underset{L2}{\approx} \sum_{n_2,n_3=1}^{\infty} W_{n_2,n_3} \sin(n_2x_2)\sin(n_3x_3); \qquad E(x_2,x_3) \underset{L2}{\approx} \sum_{n_2,n_3=1}^{\infty} E_{n_2,n_3} \sin(n_2x_2)\sin(n_3x_3).$$

Then the unique solution to this problem is the function:

$$h(\mathbf{x}) = d(\mathbf{x}) + u(\mathbf{x}) + s(\mathbf{x}) + n(\mathbf{x}) + w(\mathbf{x}) + e(\mathbf{x})$$

$$d(x_1,x_2,x_3) \underset{L2}{\approx} \sum_{n_1,n_2=1}^{\infty} \frac{D_{n_1,n_2}}{\sinh\left(\pi\sqrt{n_1^2+n_2^2}\right)} \sin(n_1x_1)\sin(n_2x_2)\sinh\left(\sqrt{n_1^2+n_2^2}\cdot x_3\right);$$

$$u(x_1,x_2,x_3) \underset{L2}{\approx} \sum_{n_1,n_2=1}^{\infty} \frac{U_{n_1,n_2}}{\sinh\left(\pi\sqrt{n_1^2+n_2^2}\right)} \sin(n_1x_1)\sin(n_2x_2)\sinh\left(\sqrt{n_1^2+n_2^2}\cdot(\pi-x_3)\right);$$

$$s(x_1,x_2,x_3) \underset{L2}{\approx} \sum_{n_1,n_3=1}^{\infty} \frac{S_{n_1,n_3}}{\sinh\left(\pi\sqrt{n_1^2+n_3^2}\right)} \sin(n_1x_1)\sin(n_3x_3)\sinh\left(\sqrt{n_1^2+n_3^2}\cdot x_2\right);$$

$$n(x_1,x_2,x_3) \underset{L2}{\approx} \sum_{n_1,n_3=1}^{\infty} \frac{N_{n_1,n_3}}{\sinh\left(\pi\sqrt{n_1^2+n_3^2}\right)} \sin(n_1x_1)\sin(n_3x_3)\sinh\left(\sqrt{n_1^2+n_3^2}\cdot(\pi-x_2)\right);$$

$$w(x_1,x_2,x_3) \underset{L2}{\approx} \sum_{n_2,n_3=1}^{\infty} \frac{W_{n_2,n_3}}{\sinh\left(\pi\sqrt{n_2^2+n_3^2}\right)} \sin(n_2x_2)\sin(n_3x_3)\sinh\left(\sqrt{n_2^2+n_3^2}\cdot x_1\right);$$

$$e(x_1,x_2,x_3) \underset{L2}{\approx} \sum_{n_2,n_3=1}^{\infty} \frac{E_{n_2,n_3}}{\sinh\left(\pi\sqrt{n_2^2+n_3^2}\right)} \sin(n_2x_2)\sin(n_3x_3)\sinh\left(\sqrt{n_2^2+n_3^2}\cdot(\pi-x_1)\right).$$

Furthermore, these six series converge semiuniformly on $\mathsf{int}(\mathbb{X})$.

Proof: **Exercise 13.9** □

13.3 The Poisson Problem on a Cube

Prerequisites: §10.2, §6.5, §2.4 **Recommended:** §11.3, §12.3, §8.3(e)

Proposition 13.5: Poisson Problem on Cube; Homogeneous Dirichlet BC

Let $\mathbb{X} = [0,\pi]^3$, *and let* $q : \mathbb{X} \longrightarrow \mathbb{R}$ *be some function. Suppose* q *has Fourier sine series:* $q(\mathbf{x}) \underset{L2}{\approx} \sum_{\mathbf{n}\in\mathbb{N}^3} Q_{\mathbf{n}} \mathbf{S}_{\mathbf{n}}(\mathbf{x})$, *and define the function* $u(\mathbf{x})$ *by*

$$u(\mathbf{x}) \underset{\text{unif}}{\equiv} \sum_{\mathbf{n}\in\mathbb{N}^3} \frac{-Q_{\mathbf{n}}}{\|\mathbf{n}\|^2} \cdot \mathbf{S}_{\mathbf{n}}(\mathbf{x}).$$

Then $u(\mathbf{x})$ *is the unique solution to the* **Poisson equation** *"*$\triangle u(\mathbf{x}) = q(\mathbf{x})$*", satisfying homogeneous Dirichlet boundary conditions* $u(x_1,x_2,0) = u(x_1,x_2,\pi) = u(x_1,0,x_3) = u(x_1,\pi,x_3) = u(0,x_2,x_3) = u(\pi,x_2,x_3,) = 0$.

Proof: **Exercise 13.10** □

Proposition 13.6: Poisson Problem on Cube; Homogeneous Neumann BC

Let $\mathbb{X} = [0,\pi]^3$, *and let* $q : \mathbb{X} \longrightarrow \mathbb{R}$ *be some function. Suppose* q *has Fourier cosine series:* $q(\mathbf{x}) \underset{L2}{\approx} \sum_{\mathbf{n}\in\mathbb{N}^3} Q_{\mathbf{n}} \mathbf{C}_{\mathbf{n}}(\mathbf{x})$, *and suppose that* $Q_{0,0,0} = 0$.

Fix some constant $K \in \mathbb{R}$, *and define the function* $u(\mathbf{x})$ *by*

$$u(\mathbf{x}) \underset{\text{unif}}{\equiv} \sum_{\substack{\mathbf{n}\in\mathbb{N}_0^3 \\ n_1,n_2,n_3 \text{ not all zero}}} \frac{-Q_{\mathbf{n}}}{\|\mathbf{n}\|^2} \cdot \mathbf{C}_{\mathbf{n}}(\mathbf{x}) \quad + \quad K. \tag{13.3}$$

Then $u(\mathbf{x})$ *is a solution to the* **Poisson equation** *"*$\triangle u(\mathbf{x}) = q(\mathbf{x})$*", satisfying homogeneous Neumann boundary conditions* $\partial_3 u(x_1,x_2,0) = \partial_3 u(x_1,x_2,\pi) = \partial_2 u(x_1,0,x_3) = \partial_2 u(x_1,\pi,x_3) = \partial_1 u(0,x_2,x_3) = \partial_1 u(\pi,x_2,x_3,) = 0$.

Furthermore, all *solutions to this Poisson equation with these boundary conditions have the form* (13.3).

If $Q_{0,0,0} \neq 0$, *however, the problem has no solution.*

Proof: **Exercise 13.11** □

Notes: ....
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V BVPs in other Coordinate Systems

In Chapters 11 to 13, we used Fourier series to solve partial differential equations. This worked because the orthogonal trigonometric functions $\mathbf{C}_n$ and $\mathbf{S}_n$ were eigenfunctions of the Laplacian. Furthermore, these functions were "well-adapted" to domains like the interval $[0, \pi]$ or the square $[0, \pi]^2$, for two reasons:

- The trigonometric functions and the domains are both easily expressed in a Cartesian coordinate system.
- The trigonometric functions satisfied desirable boundary conditions (e.g. homogeneous Dirichlet/Neumann) on the boundaries of these domains.

When we consider other domains (e.g. disks, annuli, balls, etc.), the trigonometric functions are no longer so "well-adapted". Thus, instead of using trigonometric functions, we must find some other *orthogonal system of eigenfunctions* with which to construct something analogous to a Fourier series. This system of eigenfunctions should be constructed so as to be "well-adapted" to the domain in question, in the above sense, so that we can mimic the solution methods of Chapters 11 to 13. This is the strategy which we will explore in Chapter 14.

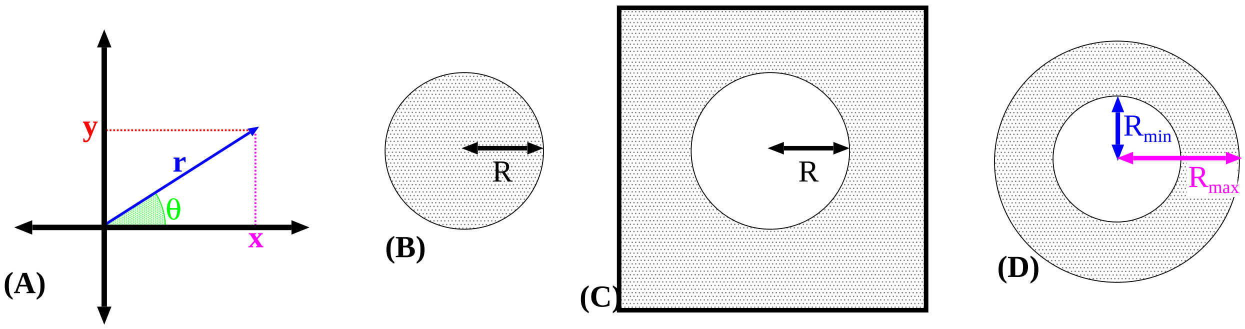


Figure 14.1: **(A)** Polar coordinates; **(B)** The disk $\mathbb{D}$; **(C)** The codisk $\mathbb{D}^{\complement}$; **(D)** The annulus $\mathbb{A}$.

14 BVPs in Polar Coordinates

14.1 Introduction

Prerequisites: §1.6(b)

When solving a boundary value problem, the shape of the domain dictates the choice of coordinate system. Seek the coordinate system yielding the simplest description of the boundary. For rectangular domains, Cartesian coordinates are the most convenient. For disks and annuli in the plane, *polar* coordinates are a better choice. Recall that polar coordinates (r, θ) on $\mathbb{R}^2$ are defined by the transformation:

$$x = r \cdot \cos(\theta) \quad \text{and} \quad y = r \cdot \sin(\theta). \qquad \text{(Figure 14.1A)}$$

with reverse transformation:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan\left(\frac{y}{x}\right).$$

Here, the coordinate r ranges over $[0, \infty)$, while the variable θ ranges over $[-\pi, \pi)$. (Clearly, we could let θ range over *any* interval of length 2π; we just find $[-\pi, \pi)$ the most convenient).

The three domains we will examine are:

- $\mathbb{D} = \{(r, \theta)\ ;\ r \leq R\}$, the **disk** of radius R; see Figure 14.1B. For simplicity we will usually assume $R = 1$.
- $\mathbb{D}^{\complement} = \{(r, \theta)\ ;\ R \leq r\}$, the **codisk** or **punctured plane** of radius R; see Figure 14.1C. For simplicity we will usually assume $R = 1$.
- $\mathbb{A} = \{(r, \theta)\ ;\ R_{\min} \leq r \leq R_{\max}\}$, the **annulus**, of inner radius ρ and outer radius R; see Figure 14.1D.

The boundaries of these domains are circles. For example, the boundary of the disk $\mathbb{D}$ of radius R is the **circle**:

$$\partial\mathbb{D} \;=\; \mathbb{S} \;=\; \{(r,\theta)\;;\; r = R\}$$

The circle can be parameterized by a single angular coordinate $\theta \in [-\pi, \pi)$. Thus, the boundary conditions will be specified by a function $b : [-\pi, \pi) \longrightarrow \mathbb{R}$. Note that, if $b(\theta)$ is to be *continuous* as a function on the circle, then it must be 2π-*periodic* as a function on $[-\pi, \pi)$.

In polar coordinates, the Laplacian is written:

$$\triangle u \quad = \quad \partial_r^2 u \;+\; \frac{1}{r}\,\partial_r u \;+\; \frac{1}{r^2}\,\partial_\theta^2 u \qquad (\underline{\textbf{Exercise 14.1}}) \tag{14.1}$$

14.2 The Laplace Equation in Polar Coordinates

14.2(a) Polar Harmonic Functions

Prerequisites: §1.6(b), §2.3

The following important harmonic functions *separate* in polar coordinates:

$$\Phi_n(r,\theta) \;=\; \cos(n\theta)\cdot r^n; \qquad \Psi_n(r,\theta) \;=\; \sin(n\theta)\cdot r^n; \quad \text{for } n = 1,2,3,\ldots \qquad \text{(Figure 14.2)}$$

$$\phi_n(r,\theta) \;=\; \frac{\cos(n\theta)}{r^n}; \qquad \psi_n(r,\theta) \;=\; \frac{\sin(n\theta)}{r^n}; \quad \text{for } n = 1,2,3,\ldots \qquad \text{(Figure 14.3)}$$

$$\Phi_0(r,\theta) \;=\; 1. \qquad \text{and} \qquad \phi_0(r,\theta) \;=\; \log(r) \qquad \text{(Figure 14.4)}$$

Proposition 14.1: *The functions Φ_n, Ψ_n, ϕ_n, and ψ_n are harmonic, for all $n \in \mathbb{N}$.*

Proof: See practice problems #1 to #5 in §14.9. □

Exercise 14.2 (a) Show that $\Phi_1(r,\theta) = x$ and $\Psi_1(r,\theta) = y$ in Cartesian coordinates.
(b) Show that $\Phi_2(r,\theta) = x^2 - y^2$ and $\Psi_2(r,\theta) = 2xy$ in Cartesian coordinates.
(c) Define $F_n : \mathbb{C} \longrightarrow \mathbb{C}$ by $F_n(z) := z^n$. Show that $\Phi_n(x,y) \;=\; \mathbf{re}\,[F_n(x+y\mathbf{i})]$ and $\Psi_n(x,y) \;=\; \mathbf{im}\,[F_n(x+y\mathbf{i})]$.
(d) (Hard) Show that Φ_n can be written as a homogeneous polynomial of degree n in x and y.
(e) Show that, if $(x,y) \in \partial\mathbb{D}$ (i.e. if $x^2+y^2 = 1$), then $\Phi_N(x,y) \;=\; \zeta_N(x)$, where

$$\zeta_N(x) \quad := \quad 2^{(N-1)}x^N \;+\; \sum_{n=1}^{\lfloor \frac{N}{2} \rfloor} (-1)^n 2^{(N-1-2n)}\,\frac{N}{n}\binom{N-n-1}{n-1}\,x^{(N-2n)}.$$

is the Nth **Chebyshev polynomial**. (To learn more about Chebyshev polynomials, see [Bro89, §3.4].

We will solve the Laplace equation in polar coordinates by representing solutions as sums of these simple functions. Note that Φ_n and Ψ_n are *bounded* at zero, but *unbounded* at infinity (Figure 14.5(A) shows the radial growth of Φ_n and Ψ_n). Conversely, ϕ_n and ψ_n are *unbounded*

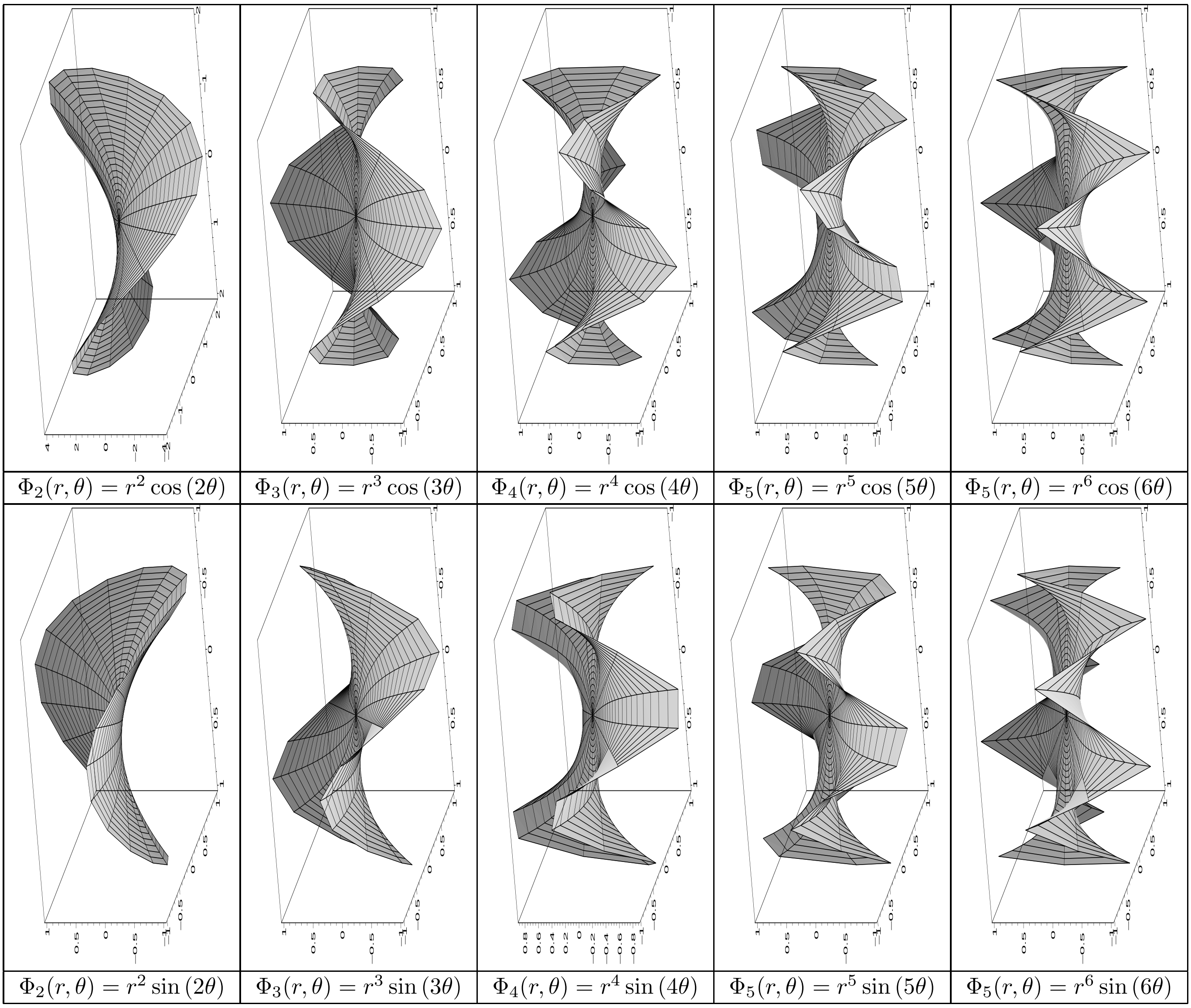


Figure 14.2: Φ_n and Ψ_n for $n = 2..6$ (rotate page).

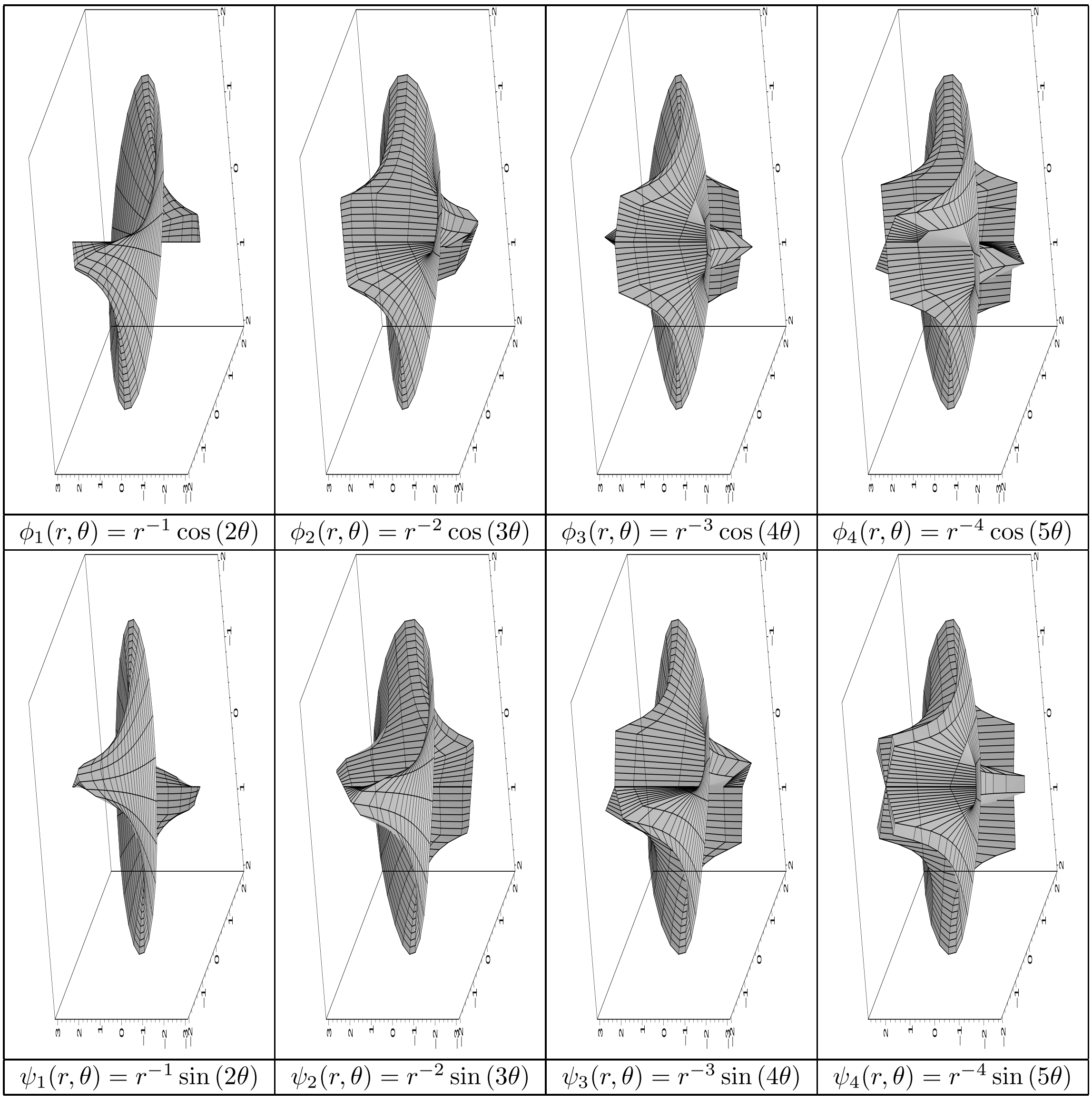


Figure 14.3: ϕ_n and ψ_n for $n = 1..4$ (rotate page). Note that these plots have been 'truncated' to have vertical bounds ± 3, because these functions explode to $\pm\infty$ at zero.

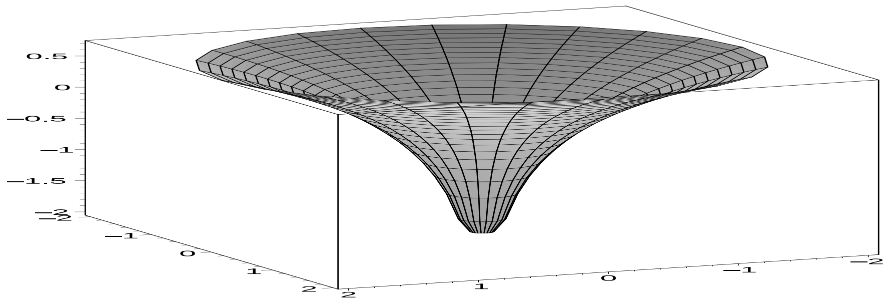
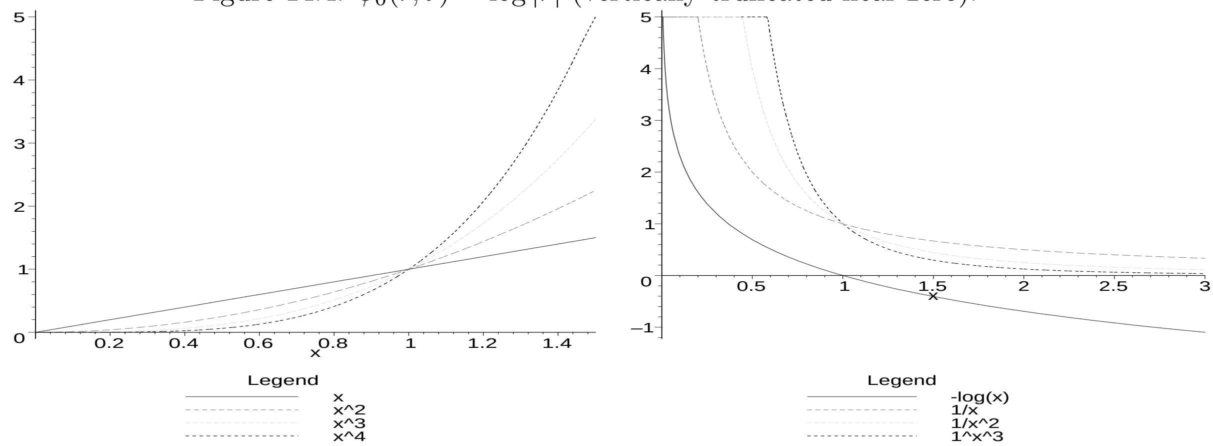


Figure 14.4: $\phi_0(r, \theta) = \log|r|$ (vertically truncated near zero).



(A): x^n, for $n = 1, 2, 3, 4$;

(B): $-\log(x)$ and $\frac{1}{x^n}$, for $n = 1, 2, 3$ (these four plots are vertically truncated).

Figure 14.5: Radial growth/decay of polar-separated harmonic functions.

at zero, but *bounded* at infinity) (Figure 14.5(B) shows the radial decay of ϕ_n and ψ_n). Finally, Φ_0 being constant, is bounded everywhere, while ϕ_0 is unbounded at both 0 and ∞ (see Figure 14.5B). Hence, when solving BVPs in a neighbourhood around zero (eg. the disk), it is preferable to use Φ_0, Φ_n and Ψ_n. When solving BVPs on an unbounded domain (ie. one "containing infinity") it is preferable to use Φ_0, ϕ_n and ψ_n. When solving BVP's on a domain containing neither zero nor infinity (eg. the annulus), we use all of Φ_n, Ψ_n, ϕ_n, ψ_n, Φ_0, and ϕ_0

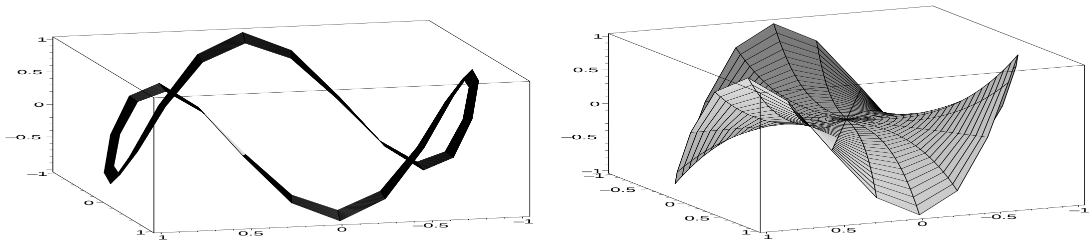
14.2(b) Boundary Value Problems on a Disk

Prerequisites: §6.5, §14.1, §14.2(a), §9.1, §1.7

Proposition 14.2: (Laplace Equation on Unit Disk; nonhomogeneous Dirichlet BC)

Let $\mathbb{D} = \{(r, \theta)\ ;\ r \leq 1\}$ be the unit disk, and let $b \in \mathbf{L}^2[-\pi, \pi)$ be some function. Consider the Laplace equation "$\triangle u = 0$", with nonhomogeneous <u>*Dirichlet*</u> *boundary conditions:*

$$u(1, \theta) \;=\; b(\theta), \qquad \textit{for all } \theta \in [-\pi, \pi). \tag{14.2}$$



(A): A bent circular wire frame: $b(\theta) = \sin(3\theta)$.

(B): A bubble in the frame: $u(r, \theta) \;=\; r^3 \sin(3\theta)$.

Figure 14.6: A soap bubble in a bent wire frame.

Suppose b has real Fourier series: $b(\theta) \underset{\text{L2}}{\approx} A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) \;+\; \sum_{n=1}^{\infty} B_n \sin(n\theta)$.

Then the unique bounded solution to this problem is the function

$$\begin{aligned} u(r,\theta) \quad &\underset{\text{L2}}{\approx} \quad A_0 + \sum_{n=1}^{\infty} A_n \Phi_n(r,\theta) \;+\; \sum_{n=1}^{\infty} B_n \Psi_n(r,\theta) \\ &= \quad A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta)\cdot r^n \;+\; \sum_{n=1}^{\infty} B_n \sin(n\theta)\cdot r^n \end{aligned} \tag{14.3}$$

Furthermore, the series (14.3) *converges semiuniformly to u on* $\mathsf{int}(\mathbb{D})$.

Proof: **Exercise 14.3** (a) To show that u is harmonic, apply eqn.(14.1) on page 236 to get

$$\begin{aligned} \triangle\, u(r,\theta) \quad = \quad & \partial_r^2 \left(\sum_{n=1}^{\infty} A_n \cos(n\theta)\cdot r^n \;+\; \sum_{n=1}^{\infty} B_n \sin(n\theta)\cdot r^n \right) \;+\; \frac{1}{r}\partial_r \left(\sum_{n=1}^{\infty} A_n \cos(n\theta)\cdot r^n \;+\; \sum_{n=1}^{\infty} B_n \sin(n\theta)\cdot r^n \right) \\ & + \frac{1}{r^2}\partial_\theta^2 \left(\sum_{n=1}^{\infty} A_n \cos(n\theta)\cdot r^n \;+\; \sum_{n=1}^{\infty} B_n \sin(n\theta)\cdot r^n \right). \end{aligned} \tag{14.4}$$

Now let $R < 1$. Check that, on the domain $\mathbb{D}(R) = \{(r,\theta)\;;\; r < R\}$, the conditions of Proposition 1.7 on page 16 are satisfied; use this to simplify the expression (14.4). Finally, apply Proposition 14.1 on page 236 to deduce that $\triangle u(r,\theta) \;=\; 0$ for all $r \leq R$. Since this works for any $R < 1$, conclude that $\triangle u \equiv 0$ on $\mathbb{D}$.

(b) To check that u also satisfies the boundary condition (14.2), substitute $r = 1$ into (14.3) to get: $u(1,\theta) \underset{\text{L2}}{\approx} A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) \;+\; \sum_{n=1}^{\infty} B_n \sin(n\theta) \;=\; b(\theta)$.

(c) Use Proposition 6.14(a) on page 106 to conclude that this solution is unique. □

Example 14.3: *Take a circular wire frame of radius* 1*, and warp it so that its vertical distortion is described by the function $b(\theta) = \sin(3\theta)$, shown in Figure 14.6(A). Dip the frame into a soap solution to obtain a bubble with the bent wire as its boundary. What is the shape of the bubble?*

Solution: A soap bubble suspended from the wire is a *minimal surface*[1]. Minimal surfaces of low curvature are well-approximated by harmonic functions, so it is a good approximation to model the bubble by a function with zero Laplacian.

Let $u(r,\theta)$ be a function describing the bubble surface. As long as the distortion $b(\theta)$ is relatively small, $u(r,\theta)$ will be a solution to Laplace's equation, with boundary conditions $u(1,\theta) = b(\theta)$. Thus, as shown in Figure 14.6(B), $u(r,\theta) = r^3 \sin(3\theta)$. ♢

Remark: There is another "formal solution" to the Dirichlet problem on the disk, given by:

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n \frac{\cos(n\theta)}{r^n} + \sum_{n=1}^{\infty} B_n \frac{\sin(n\theta)}{r^n}$$

The problem with this solution is that it is *unbounded* at zero, which is unphysical, and also means we can't use Proposition 1.7 differentiate the series and verify that it satisfies the Laplace equation. This is why Proposition 14.2 specifically remarks that (14.3) is a *bounded* solution.

<u>Exercise 14.4</u> Let $u(x,\theta)$ be a solution to the Dirichlet problem with boundary conditions $u(1,\theta) = b(\theta)$. Use Proposition 14.2 to prove that $u(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} b(\theta)\, d\theta$.

Proposition 14.4: (Laplace Equation on Unit Disk; nonhomogeneous Neumann BC)

Let $\mathbb{D} = \{(r,\theta) \,;\, r \leq 1\}$ be the unit disk, and let $b \in \mathbf{L}^2[-\pi,\pi)$. Consider the Laplace equation "$\triangle u = 0$", with nonhomogeneous <u>Neumann</u> boundary conditions:

$$\partial_r\, u(1,\theta) = b(\theta) \tag{14.5}$$

Suppose b has real Fourier series: $b(\theta) \underset{L2}{\approx} A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) + \sum_{n=1}^{\infty} B_n \sin(n\theta)$.

If $A_0 = 0$, then the <u>bounded</u> solutions to this problem are all functions of the form

$$u(r,\theta) \underset{L2}{\approx} C + \sum_{n=1}^{\infty} \frac{A_n}{n}\Phi_n(r,\theta) + \sum_{n=1}^{\infty} \frac{B_n}{n}\Psi_n(r,\theta)$$

$$= C + \sum_{n=1}^{\infty} \frac{A_n}{n}\cos(n\theta)\cdot r^n + \sum_{n=1}^{\infty} \frac{B_n}{n}\sin(n\theta)\cdot r^n \tag{14.6}$$

where C is any constant. Furthermore, the series (14.6) converges semiuniformly to u on int$(\mathbb{D})$.

However, if $A_0 \neq 0$, then there is no bounded solution.

Proof:

Claim 1: *For any $r < 1$,* $\sum_{n=1}^{\infty} n^2 \frac{|A_n|}{n}\cdot r^n + \sum_{n=1}^{\infty} n^2 \frac{|B_n|}{n}\cdot r^n < \infty$.

[1] This means that it is the surface with the *minimum possible area*, given that it must span the wire frame.

Proof: Let $M = \max\left\{ \max\{|A_n|\}_{n=1}^\infty,\ \max\{|B_n|\}_{n=1}^\infty \right\}$. Then

$$\sum_{n=1}^\infty n^2 \frac{|A_n|}{n} \cdot r^n + \sum_{n=1}^\infty n^2 \frac{|B_n|}{n} \cdot r^n \quad \leq \quad \sum_{n=1}^\infty n^2 \frac{M}{n} \cdot r^n + \sum_{n=1}^\infty n^2 \frac{M}{n} \cdot r^n \quad = \quad 2M \sum_{n=1}^\infty n r^n. \tag{14.7}$$

Let $f(r) = \dfrac{1}{1-r}$. Then $f'(r) = \dfrac{1}{(1-r)^2}$. Recall that, for $|r| < 1$, $f(r) = \sum_{n=0}^\infty r^n$. Thus, $f'(r) = \sum_{n=1}^\infty n r^{n-1} = \frac{1}{r}\sum_{n=1}^\infty n r^n$. Hence, the right hand side of eqn.(14.7) is equal to

$$2M \sum_{n=1}^\infty n r^n \quad = \quad 2Mr \cdot f'(r) \quad = \quad 2Mr \cdot \frac{1}{(1-r)^2} \quad < \quad \infty,$$

for any $r < 1$. $\diamond_{\text{Claim 1}}$

Let $R < 1$ and let $\mathbb{D}(R) = \{(r,\theta)\ ;\ r \leq R\}$ be the disk of radius R. If $u(r,\theta) = C + \sum_{n=1}^\infty \frac{A_n}{n}\Phi_n(r,\theta) + \sum_{n=1}^\infty \frac{B_n}{n}\Psi_n(r,\theta)$, then

$$\triangle u(r,\theta) \underset{\text{unif}}{\equiv} \sum_{n=1}^\infty \frac{A_n}{n}\triangle\Phi_n(r,\theta) + \sum_{n=1}^\infty \frac{B_n}{n}\triangle\Psi_n(r,\theta) \underset{(*)}{=} \sum_{n=1}^\infty \frac{A_n}{n}(0) + \sum_{n=1}^\infty \frac{B_n}{n}(0) = 0,$$

on $\mathbb{D}(R)$. Here, "$\underset{\text{unif}}{\equiv}$" is by Proposition 1.7 on page 16 and Claim 1, while $(*)$ is by Proposition 14.1 on page 236.

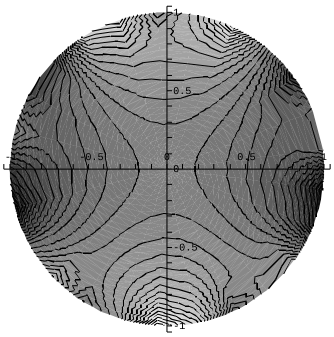
To check boundary conditions, observe that, for all $(r,\theta) \in \mathbb{D}(R)$,

$$\begin{aligned}
\partial_r u(r,\theta) &\underset{\text{unif}}{\equiv} \sum_{n=1}^\infty \frac{A_n}{n}\partial_r \Phi_n(r,\theta) + \sum_{n=1}^\infty \frac{B_n}{n}\partial_r \Psi_n(r,\theta) \\
&= \sum_{n=1}^\infty \frac{A_n}{n} n r^{n-1}\cos(n\theta) + \sum_{n=1}^\infty \frac{B_n}{n} n r^{n-1}\sin(n\theta) \\
&= \sum_{n=1}^\infty A_n r^{n-1}\cos(n\theta) + \sum_{n=1}^\infty B_n r^{n-1}\sin(n\theta).
\end{aligned}$$

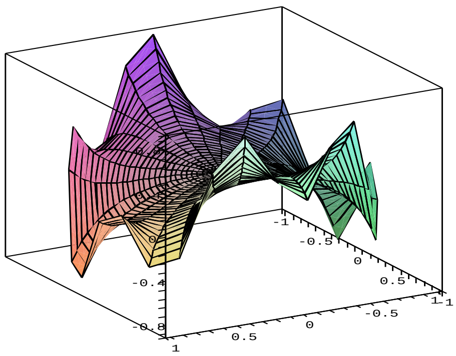
Here "$\underset{\text{unif}}{\equiv}$" is by Proposition 1.7 on page 16. Hence, letting $R \to 1$, we get

$$\begin{aligned}
\partial_\perp u(1,\theta) = \partial_r u(1,\theta) &= \sum_{n=1}^\infty A_n \cdot (1)^{n-1}\cos(n\theta) + \sum_{n=1}^\infty B_n \cdot (1)^{n-1}\sin(n\theta) \\
&= \sum_{n=1}^\infty A_n \cos(n\theta) + \sum_{n=1}^\infty B_n \sin(n\theta) \underset{\text{L2}}{\approx} b(\theta),
\end{aligned}$$

as desired. Here, "$\underset{\text{L2}}{\approx}$" is because this is the Fourier Series for $b(\theta)$, assuming $A_0 = 0$. (If $A_0 \neq 0$, then this solution doesn't work.)



(A) Contour plot



(B) Surface plot

Figure 14.7: The electric potential deduced from Scully's voltage measurements in Example 14.5.

Finally, Proposition 6.14(c) on page 106 implies that this solution is unique up to addition of a constant. □

Remark: *Physically speaking, why must* $A_0 = 0$*?*

If $u(r, \theta)$ is an electric potential, then $\partial_r u$ is the *radial component* of the *electric field*. The requirement that $A_0 = 0$ is equivalent to requiring that the *net electric flux* entering the disk is zero, which is equivalent (via Gauss's law) to the assertion that the *net electric charge* contained in the disk is zero. If $A_0 \neq 0$, then the net electric charge within the disk must be nonzero. Thus, if $q : \mathbb{D} \longrightarrow \mathbb{R}$ is the charge density field, then we must have $q \not\equiv 0$. However, $q = \triangle u$ (see Example 2.8 on page 29), so this means $\triangle u \neq 0$, which means u is not harmonic.

Example 14.5: *While covertly investigating mysterious electrical phenomena on a top-secret military installation in the Nevada desert, Mulder and Scully are trapped in a cylindrical concrete silo by the Cancer Stick Man. Scully happens to have a voltimeter, and she notices an electric field in the silo. Walking around the (circular) perimeter of the silo, Scully estimates the radial component of the electric field to be the function* $b(\theta) = 3\sin(7\theta) - \cos(2\theta)$*. Estimate the electric potential field inside the silo.*

Solution: The electric potential will be a solution to Laplace's equation, with boundary conditions $\partial_r u(1, \theta) = 3\sin(7\theta) - \cos(2\theta)$. Thus,

$$\boxed{u(r,\theta) \;=\; C \;+\; \frac{3}{7}\sin(7\theta)\cdot r^7 \;-\; \frac{1}{2}\cos(2\theta)\cdot r^2} \qquad \text{(see Figure 14.7)}$$

Question: *Moments later, Mulder repeats Scully's experiment, and finds that the perimeter field has changed to* $b(\theta) = 3\sin(7\theta) - \cos(2\theta) + 6$*. He immediately suspects that an Alien Presence has entered the silo. Why?* ♢

14.2(c) Boundary Value Problems on a Codisk

Prerequisites: §6.5, §14.1, §14.2(a), §9.1, §1.7 **Recommended:** §14.2(b)

We will now solve the Dirichlet problem on an *unbounded* domain: the **codisk**

$$\mathbb{D}^{\complement} \quad := \quad \{(r,\theta)\ ;\ 1 \leq r\}.$$

Physical Interpretations:

Chemical Concentration: Suppose there is an unknown source of some chemical hidden inside the disk, and that this chemical diffuses into the surrounding medium. Then the solution function $u(r,\theta)$ represents the *equilibrium concentration* of the chemical. In this case, it is reasonable to expect $u(r,\theta)$ to be *bounded at infinity*, by which we mean:

$$\lim_{r\to\infty} |u(r,\theta)| \quad \neq \quad \infty. \tag{14.8}$$

Otherwise the chemical concentration would become very large far away from the center, which is not realistic.

Electric Potential: Suppose there is an unknown charge distribution inside the disk. Then the solution function $u(r,\theta)$ represents the *electric potential field* generated by this charge. Even though we don't know the exact charge distribution, we can use the boundary conditions to extrapolate the shape of the potential field outside the disk.

If the net charge within the disk is zero, then the electric potental far away from the disk should be bounded (because from far away, the charge distribution inside the disk 'looks' neutral); hence, the solution $u(r,\theta)$ will again satisfy the Boundedness Condition (14.8).

However, if there is a *nonzero* net charge within the the disk, then the electric potential will *not* be bounded (because, even from far away, the disk still 'looks' charged). Nevertheless, the electric *field* generated by this potential should still be decay to zero (because the influence of the charge should be weak at large distances). This means that, while the potential is unbounded, the *gradient* of the potential must decay to zero near infinity. In other words, we must impose the *decaying gradient condition*:

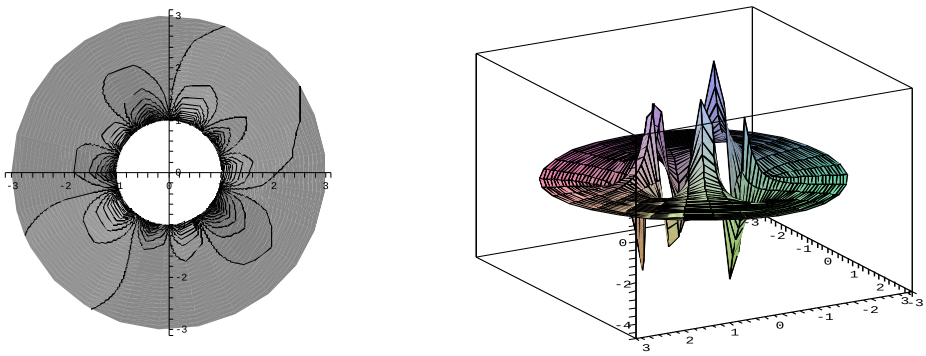
$$\lim_{r\to\infty} \nabla u(r,\theta) \quad = \quad 0. \tag{14.9}$$

Proposition 14.6: (Laplace equation on codisk; nonhomogeneous Dirichlet BC)

Let $\mathbb{D}^{\complement} = \{(r,\theta)\ ;\ 1 \leq r\}$ *be the codisk, and let* $b \in \mathbf{L}^2[-\pi,\pi)$. *Consider the Laplace equation "*$\triangle u = 0$*", with nonhomogeneous* <u>*Dirichlet*</u> *boundary conditions:*

$$u(1,\theta) \;=\; b(\theta) \tag{14.10}$$

Suppose b has real Fourier series: $b(\theta) \underset{\mathrm{L2}}{\approx} A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) \;+\; \sum_{n=1}^{\infty} B_n \sin(n\theta).$



(A) Contour plot (unit disk occulted) **(B)** Surface plot (unit disk deleted)

Figure 14.8: The electric potential deduced from voltage measurements in Example 14.7.

Then the unique solution to this problem which is <u>*bounded at infinity*</u> *as in* (14.8) *is the function*

$$u(r,\theta) \underset{L2}{\approx} A_0 + \sum_{n=1}^{\infty} A_n \frac{\cos(n\theta)}{r^n} + \sum_{n=1}^{\infty} B_n \frac{\sin(n\theta)}{r^n} \qquad (14.11)$$

Furthermore, the series (14.11) *converges semiuniformly to* u *on* $\mathsf{int}\left(\mathbb{D}^{\complement}\right)$.

Proof: **<u>Exercise 14.5</u>** (a) To show that u is harmonic, apply eqn.(14.1) on page 236 to get

$$\triangle u(r,\theta) = \partial_r^2 \left(\sum_{n=1}^{\infty} A_n \frac{\cos(n\theta)}{r^n} + \sum_{n=1}^{\infty} B_n \frac{\sin(n\theta)}{r^n} \right) + \frac{1}{r}\partial_r \left(\sum_{n=1}^{\infty} A_n \frac{\cos(n\theta)}{r^n} + \sum_{n=1}^{\infty} B_n \frac{\sin(n\theta)}{r^n} \right)$$
$$+ \frac{1}{r^2}\partial_\theta^2 \left(\sum_{n=1}^{\infty} A_n \frac{\cos(n\theta)}{r^n} + \sum_{n=1}^{\infty} B_n \frac{\sin(n\theta)}{r^n} \right). \qquad (14.12)$$

Now let $R > 1$. Check that, on the domain $\mathbb{D}^{\complement}(R) = \{(r,\theta)\ ;\ r > R\}$, the conditions of Proposition 1.7 on page 16 are satisfied; use this to simplify the expression (14.12). Finally, apply Proposition 14.1 on page 236 to deduce that $\triangle u(r,\theta) = 0$ for all $r \geq R$. Since this works for any $R > 1$, conclude that $\triangle u \equiv 0$ on $\mathbb{D}^{\complement}$.

(b) To check that the solution also satisfies the boundary condition (14.10), subsititute $r = 1$ into (14.11) to get: $u(1,\theta) \underset{L2}{\approx} A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) + \sum_{n=1}^{\infty} B_n \sin(n\theta) = b(\theta)$.

(c) Use Proposition 6.14(a) on page 106 to conclude that this solution is unique. □

Example 14.7: *An unknown distribution of electric charges lies inside the unit disk in the plane. Using a voltimeter, the electric potential is measured along the perimeter of the circle, and is approximated by the function* $b(\theta) = \sin(2\theta) + 4\cos(5\theta)$. *Far away from the origin, the potential is found to be close to zero. Estimate the electric potential field.*

Solution: The electric potential will be a solution to Laplace's equation, with boundary conditions $u(1,\theta) = \sin(2\theta)+4\cos(5\theta)$. Far away, the potential apparently remains bounded. Thus, as shown in Figure 14.8,

$$\boxed{u(r,\theta) \;=\; \frac{\sin(2\theta)}{r^2} + \frac{4\cos(5\theta)}{r^5}} \qquad \diamondsuit$$

Remark: Note that, for any constant $C \in \mathbb{R}$, another solution to the Dirichlet problem with boundary conditions (14.10) is given by the function

$$u(r,\theta) \;=\; A_0 \;+\; C\log(r) \;+\; \sum_{n=1}^{\infty} A_n \frac{\cos(n\theta)}{r^n} \;+\; \sum_{n=1}^{\infty} B_n \frac{\sin(n\theta)}{r^n}. \qquad (\textbf{Exercise 14.6})$$

However, unless $C = 0$, this will *not* be bounded at infinity.

Proposition 14.8: (Laplace equation on codisk; nonhomogeneous Neumann BC)

Let $\mathbb{D}^{\complement} = \{(r,\theta)\,;\, 1 \le r\}$ *be the codisk, and let* $b \in \mathbf{L}^2[-\pi,\pi)$. *Consider the Laplace equation "*$\triangle u = 0$*", with nonhomogeneous* *Neumann* *boundary conditions:*

$$-\,\partial_\perp\, u(1,\theta) \;=\; \partial_r\, u(1,\theta) \;=\; b(\theta) \qquad (14.13)$$

Suppose b *has real Fourier series:* $b(\theta) \underset{L2}{\approx} A_0 + \sum_{n=1}^{\infty} A_n\cos(n\theta) \;+\; \sum_{n=1}^{\infty} B_n\sin(n\theta)$.

Fix a constant $C \in \mathbb{R}$, *and define* $u(r,\theta)$ *by:*

$$u(r,\theta) \underset{L2}{\approx} C \;+\; A_0\log(r) \;+\; \sum_{n=1}^{\infty} \frac{-A_n}{n}\frac{\cos(n\theta)}{r^n} \;+\; \sum_{n=1}^{\infty} \frac{-B_n}{n}\frac{\sin(n\theta)}{r^n} \qquad (14.14)$$

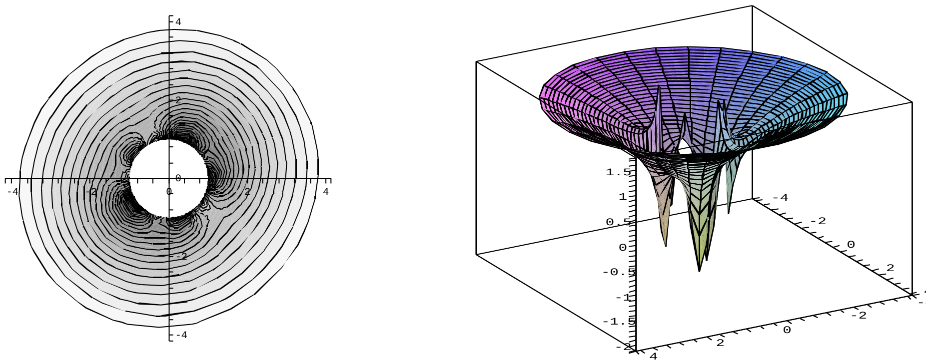
Then u *is a solution to the Laplace equation, with nonhomogeneous Neumann boundary conditions* (14.13)*, and furthermore, obeys the* *Decaying Gradient Condition* (14.9) *on p.244. Furthermore,* *all* *harmonic functions satisfying equations* (14.13) *and* (14.9) *must be of the form* (14.14)*. However, the solution* (14.14) *is* *bounded at infinity* *as in* (14.8) *if and only if* $A_0 = 0$.

Finally, the series (14.14) *converges semiuniformly to* u *on* $\mathsf{int}\left(\mathbb{D}^{\complement}\right)$.

Proof: **Exercise 14.7** (a) To show that u is harmonic, apply eqn.(14.1) on page 236 to get

$$\triangle u(r,\theta) \;=\; \partial_r^2\left(A_0\log(r) \;-\; \sum_{n=1}^{\infty}\frac{A_n}{n}\frac{\cos(n\theta)}{r^n} \;-\; \sum_{n=1}^{\infty}\frac{B_n}{n}\frac{\sin(n\theta)}{r^n}\right)$$
$$+\;\frac{1}{r}\partial_r\left(A_0\log(r) \;-\; \sum_{n=1}^{\infty}\frac{A_n}{n}\frac{\cos(n\theta)}{r^n} \;-\; \sum_{n=1}^{\infty}\frac{B_n}{n}\frac{\sin(n\theta)}{r^n}\right) \;+\; \frac{1}{r^2}\partial_\theta^2\left(A_0\log(r) \;-\; \sum_{n=1}^{\infty}\frac{A_n}{n}\frac{\cos(n\theta)}{r^n} \;-\; \sum_{n=1}^{\infty}\frac{B_n}{n}\frac{\sin(n\theta)}{r^n}\right). \quad (14.15)$$

Now let $R > 1$. Check that, on the domain $\mathbb{D}^{\complement}(R) = \{(r,\theta)\,;\, r > R\}$, the conditions of Proposition 1.7 on page 16 are satisfied; use this to simplify the expression (14.15). Finally, apply Proposition 14.1 on page 236 to deduce that $\triangle u(r,\theta) \;=\; 0$ for all $r \ge R$. Since this works for any $R > 1$, conclude that $\triangle u \equiv 0$ on $\mathbb{D}^{\complement}$.



(A) Contour plot (unit disk occulted) **(B)** Surface plot (unit disk deleted)

Figure 14.9: The electric potential deduced from field measurements in Example 14.9.

(b) To check that the solution also satisfies the boundary condition (14.13), subsititute $r = 1$ into (14.14) and compute the radial derivative (using Proposition 1.7 on page 16) to get: $\partial_r u(1,\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) + \sum_{n=1}^{\infty} B_n \sin(n\theta) \underset{L2}{\approx} b(\theta)$.

(c) Use Proposition 6.14(c) on page 106 to show that this solution is unique up to addition of a constant.

(d) What is the physical interpretation of $A_0 = 0$? □

Example 14.9: *An unknown distribution of electric charges lies inside the unit disk in the plane. The* radial *component of the electric field is measured along the perimeter of the circle, and is approximated by the function* $b(\theta) = 0.9 + \sin(2\theta) + 4\cos(5\theta)$. *Estimate the electric potential potential (up to a constant).*

Solution: The electric potential will be a solution to Laplace's equation, with boundary conditions $\partial_r\ u(1,\theta) = 0.9 + \sin(2\theta) + 4\cos(5\theta)$. Thus, as shown in Figure 14.9,

$$\boxed{u(r,\theta) \;=\; C \;+\; 0.9\log(r) + \frac{-\sin(2\theta)}{2\cdot r^2} + \frac{-4\cos(5\theta)}{5\cdot r^5}} \qquad \diamondsuit$$

14.2(d) Boundary Value Problems on an Annulus

Prerequisites: §6.5, §14.1, §14.2(a), §9.1, §1.7 **Recommended:** §14.2(b), §14.2(c)

Proposition 14.10: (Laplace Equation on Annulus; nonhomogeneous Dirichlet BC)

Let $\mathbb{A} = \{(r,\theta) \; ; \; R_{\min} \leq r \leq R_{\max}\}$ be an annulus, and let $b, B : [-\pi, \pi) \longrightarrow \mathbb{R}$ be two functions. Consider the Laplace equation "$\triangle u = 0$", with nonhomogeneous Dirichlet boundary conditions:

$$u(R_{\min}, \theta) \;=\; b(\theta); \qquad \text{and } u(R_{\max}, \theta) \;=\; B(\theta); \tag{14.16}$$

Suppose b and B have real Fourier series:

$$b(\theta) \;\underset{L2}{\approx}\; a_0 + \sum_{n=1}^{\infty} a_n \cos(n\theta) \;+\; \sum_{n=1}^{\infty} b_n \sin(n\theta);$$

$$B(\theta) \;\underset{L2}{\approx}\; A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) \;+\; \sum_{n=1}^{\infty} B_n \sin(n\theta);$$

Then the unique solution to this problem is the function

$$u(r,\theta) \;=\underset{L2}{\approx}\; U_0 \;+\; u_0 \log(r) \;+\; \sum_{n=1}^{\infty} \left(U_n r^n + \frac{u_n}{r^n} \right) \cos(n\theta)$$
$$+ \; \sum_{n=1}^{\infty} \left(V_n r^n + \frac{v_n}{r^n} \right) \sin(n\theta) \tag{14.17}$$

where the coefficients $\{u_n, U_n, v_n, V_N\}_{n=1}^{\infty}$ are the unique solutions to the equations:

$$U_0 + u_0 \log(R_{\min}) \;=\; a_0; \qquad U_0 + u_0 \log(R_{\max}) \;=\; A_0;$$

$$U_n R_{\min}^n + \frac{u_n}{R_{\min}^n} \;=\; a_n; \qquad U_n R_{\max}^n + \frac{u_n}{R_{\max}^n} \;=\; A_n;$$

$$V_n R_{\min}^n + \frac{v_n}{R_{\min}^n} \;=\; b_n; \qquad V_n R_{\max}^n + \frac{v_n}{R_{\max}^n} \;=\; B_n.$$

Furthermore, the series (14.17) *converges semiuniformly to u on* $\mathsf{int}(\mathbb{A})$.

Proof: **Exercise 14.8** (a) To check that u is harmonic, generalize the strategies used to prove Proposition 14.2 on page 239 and Proposition 14.6 on page 244.

(b) To check that the solution also satisfies the boundary condition (14.16), subsititute $r = 1$ into (14.17) to get the Fourier series for b and B.

(c) Use Proposition 6.14(a) on page 106 to show that this solution is unique. □

Example: *Consider an annular bubble spanning two concentric circular wire frames. The inner wire has radius $R_{\min} = 1$, and is unwarped, but is elevated to a height of* 4cm, *while the outer wire has radius $R_{\max} = 2$, and is twisted to have shape $B(\theta) = \cos(3\theta) - 2\sin(\theta)$. Estimate the shape of the bubble between the two wires.*

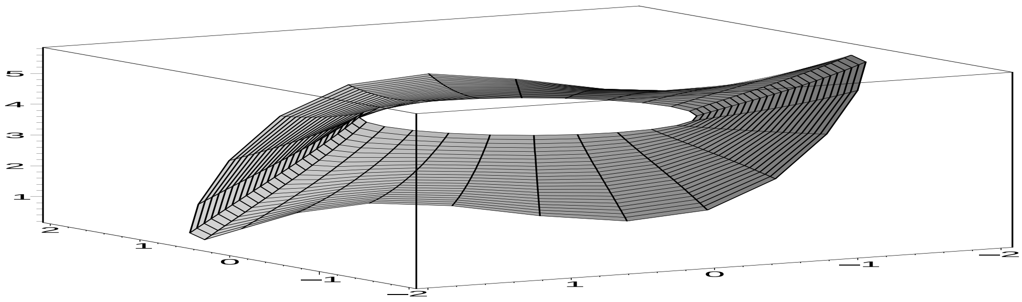


Figure 14.10: A bubble between two concentric circular wires

Solution: We have $b(\theta) = 4$, and $B(\theta) = \cos(3\theta) - 2\sin(\theta)$. Thus:

$$a_0 \quad = \quad 4; \qquad A_3 \quad = \quad 1; \qquad \text{and } B_1 \quad = \quad -2$$

and all other coefficients of the boundary conditions are zero. Thus, our solution will have the form:

$$u(r,\theta) \quad = \quad U_0 \; + \; u_0 \log(r) \quad + \quad \left(U_3 r^3 + \frac{u_3}{r^3}\right) \cdot \cos(3\theta) \quad + \quad \left(V_1 r + \frac{v_1}{r}\right) \cdot \sin(\theta),$$

where U_0, u_0, U_3, u_3, V_1, and v_1 are chosen to solve the equations:

$$U_0 + u_0 \log(1) \quad = \quad 4; \qquad U_0 + u_0 \log(2) \quad = \quad 0;$$

$$U_3 + u_3 \quad = \quad 0; \qquad U_3 + \frac{u_3}{8} \quad = \quad 1;$$

$$V_1 + v_1 \quad = \quad 0; \qquad 2V_1 + \frac{v_1}{2} \quad = \quad -2.$$

which is equivalent to:

$$U_0 \quad = \quad 4; \qquad u_0 \quad = \quad \frac{-U_0}{\log(2)} \quad = \frac{-4}{\log(2)};$$

$$u_3 \quad = \quad -U_3; \qquad \left(1 - \frac{1}{8}\right) U_3 \quad = \quad 1, \qquad \text{and thus} \quad U_3 \quad = \quad \frac{8}{63};$$

$$v_1 \quad = \quad -V_1; \qquad \left(2 - \frac{1}{2}\right) V_1 \quad = \quad -2, \qquad \text{and thus} \quad V_1 \quad = \quad \frac{-4}{3}.$$

so that $\boxed{u(r,\theta) \quad = \quad 4 \quad - \quad \frac{4\log(r)}{\log(2)} \quad + \quad \frac{8}{63}\left(r^3 - \frac{1}{r^3}\right) \cdot \cos(3\theta) \quad - \quad \frac{4}{3}\left(r - \frac{1}{r}\right) \cdot \sin(\theta)}$.

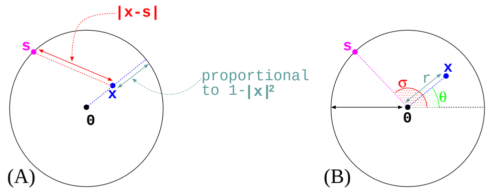


Figure 14.11: The Poisson kernel (see also Figure 16.25 on page 331)

14.2(e) Poisson's Solution to the Dirichlet Problem on the Disk

Prerequisites: §14.2(b) **Recommended:** §16.7

Let $\mathbb{D} = \{(r,\theta)\ ;\ r \leq R\}$ be the disk of radius R, and let $\partial\mathbb{D} = \mathbb{S} = \{(r,\theta)\ ;\ r = R\}$ be its boundary, the circle of radius R. Recall the *Dirichlet problem on the disk* from § 16.7 on page 330 In §16.7, we solved this problem using the **Poisson kernel**, $\mathcal{P} : \mathbb{D}\times\mathbb{S} \longrightarrow \mathbb{R}$, defined:

$$\mathcal{P}(\mathbf{x},\mathbf{s}) \quad = \quad \frac{R^2 - \|\mathbf{x}\|^2}{\|\mathbf{x}-\mathbf{s}\|^2} \quad \text{for any } \mathbf{x}\in\mathbb{D} \text{ and } \mathbf{s}\in\mathbb{S}$$

In polar coordinates (Figure 14.11B), we can parameterize $\mathbf{s} \in \mathbb{S}$ with a single angular coordinate $\sigma \in [-\pi,\pi)$, and assign $\mathbf{x}$ the coordinates (r,θ). Poisson's kernel then takes the form:

$$\mathcal{P}(\mathbf{x},\mathbf{s}) \;=\; \mathcal{P}(r,\theta;\ \sigma) \;=\; \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta-\sigma) + r^2} \qquad (\textbf{Exercise 14.9})$$

In § 16.7 on page 330, we stated the following theorem, and sketched a proof using 'impulse-response' methods. Now we are able to offer a rigorous proof using the methods of §14.2(b).

Proposition 14.11: Poisson's Integral Formula

Let $\mathbb{D} = \{(r,\theta)\ ;\ r \leq R\}$ be the disk of radius R, and let $B \in \mathbf{L}^2[-\pi,\pi)$. Consider the Laplace equation "$\triangle u = 0$", with nonhomogeneous Dirichlet boundary conditions $u(R,\theta) = B(\theta)$. The unique bounded solution to this problem satisfies:

$$\text{For any } r\in[0,R) \text{ and } \theta\in[-\pi,\pi), \qquad u(r,\theta) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{P}(r,\theta;\sigma)\cdot B(\sigma)\ d\sigma. \qquad (14.18)$$

$$\text{or, more abstractly,} \quad u(\mathbf{x}) \;=\; \frac{1}{2\pi}\int_{\mathbb{S}} \mathcal{P}(\mathbf{x},\mathbf{s})\cdot B(\mathbf{s})\ d\mathbf{s}, \quad \text{for any } \mathbf{x}\in \mathsf{int}(\mathbb{D}).$$

Proof: For simplicity, assume $R = 1$ (the general case can be obtained by rescaling). From Proposition 14.2 on page 239, we know that

$$u(r,\theta) \;\underset{\mathrm{L2}}{\approx}\; A_0 + \sum_{n=1}^{\infty} A_n\cos(n\theta)\cdot r^n \;+\; \sum_{n=1}^{\infty} B_n\sin(n\theta)\cdot r^n,$$

where A_n and B_n are the (real) Fourier coefficients for $B(\theta)$. Substituting in the definition of these coefficients (see § 9.1 on page 169), we get:

$$\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} B(\sigma)\, d\sigma \;+\; \sum_{n=1}^{\infty}\cos(n\theta)\cdot r^n\cdot\left(\frac{1}{\pi}\int_{-\pi}^{\pi} B(\sigma)\cos(n\sigma)\, d\sigma\right)\\
&\qquad + \sum_{n=1}^{\infty}\sin(n\theta)\cdot r^n\cdot\left(\frac{1}{\pi}\int_{-\pi}^{\pi} B(\sigma)\sin(n\sigma)\, d\sigma\right)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} B(\sigma)\left(1 \;+\; 2\sum_{n=1}^{\infty} r^n\cdot\cos(n\theta)\cos(n\sigma) + 2\sum_{n=1}^{\infty} r^n\cdot\sin(n\theta)\sin(n\sigma)\right)\, d\sigma\\
&\underset{(*)}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} B(\sigma)\left(1 \;+\; 2\sum_{n=1}^{\infty} r^n\cdot\cos\Big(n(\theta-\sigma)\Big)\right) \qquad (14.19)
\end{aligned}$$

where $(*)$ is because $\cos(n\theta)\cos(n\sigma)+\sin(n\theta)\sin(n\sigma) \;=\; \cos\Big(n(\theta-\sigma)\Big)$.

It now suffices to prove:

Claim 1: $1 \;+\; 2\sum_{n=1}^{\infty} r^n\cdot\cos\Big(n(\theta-\sigma)\Big) \;=\; \mathcal{P}(r,\theta;\sigma)$.

Proof: By de Moivre's formula[2], $2\cos\Big(n(\theta-\sigma)\Big) \;=\; e^{\mathbf{i}n(\theta-\sigma)} + e^{-\mathbf{i}n(\theta-\sigma)}$. Hence,

$$1 \;+\; 2\sum_{n=1}^{\infty} r^n\cdot\cos\Big(n(\theta-\sigma)\Big) \;=\; 1 \;+\; \sum_{n=1}^{\infty} r^n\cdot\left(e^{\mathbf{i}n(\theta-\sigma)} + e^{-\mathbf{i}n(\theta-\sigma)}\right). \qquad (14.20)$$

Now define complex number $z = r\cdot e^{\mathbf{i}(\theta-\sigma)}$; then observe that $r^n\cdot e^{\mathbf{i}n(\theta-\sigma)} \;=\; z^n$ and $r^n\cdot e^{-\mathbf{i}n(\theta-\sigma)} \;=\; \overline{z}^n$. Thus, we can rewrite the right hand side of (14.20) as:

$$\begin{aligned}
&1 \;+\; \sum_{n=1}^{\infty} r^n\cdot e^{\mathbf{i}n(\theta-\sigma)} \;+\; \sum_{n=1}^{\infty} r^n\cdot e^{-\mathbf{i}n(\theta-\sigma)}\\
&= 1 + \sum_{n=1}^{\infty} z^n \;+\; \sum_{n=1}^{\infty}\overline{z}^n \;\underset{(a)}{=}\; 1 + \frac{z}{1-z} + \frac{\overline{z}}{1-\overline{z}}\\
&= 1 + \frac{z - z\overline{z} + \overline{z} - z\overline{z}}{1 - z - \overline{z} + z\overline{z}} \;\underset{(b)}{=}\; 1 + \frac{2\mathbf{re}[z] - 2|z|^2}{1 - 2\mathbf{re}[z] + |z|^2}\\
&= \frac{1 - 2\mathbf{re}[z] + |z|^2}{1 - 2\mathbf{re}[z] + |z|^2} + \frac{2\mathbf{re}[z] - 2|z|^2}{1 - 2\mathbf{re}[z] + |z|^2}\\
&= \frac{1 - |z|^2}{1 - 2\mathbf{re}[z] + |z|^2} \;\underset{(c)}{=}\; \frac{1 - r^2}{1 \;-\; 2r\cos(\theta-\sigma) \;+\; r^2} \;=\; \mathcal{P}(r,\theta;\sigma).
\end{aligned}$$

(a) is because $\sum_{n=1}^{\infty} x^n \;=\; \frac{x}{1-x}$ for any $x\in\mathbb{C}$ with $|x|<1$. (b) is because $z+\overline{z} = 2\mathbf{re}[z]$ and $z\overline{z} = |z|^2$ for any $z\in\mathbb{C}$. (c) is because $|z| = r$ and $\mathbf{re}[z] = \cos(\theta-\sigma)$ by definition of z. $\diamond_{\text{Claim 1}}$

[2]See the formula sheet.

Friedrich Wilhelm Bessel
Born: July 22, 1784 in Minden, Westphalia
Died: March 17,1846 in Königsberg, Prussia

Now, use Claim 1 to substitute $\mathcal{P}(r,\theta;\sigma)$ into (14.19); this yields the Poisson integral formula (16.20). □

14.3 Bessel Functions

14.3(a) Bessel's Equation; Eigenfunctions of $\triangle$ in Polar Coordinates

Prerequisites: §5.2, §14.1 **Recommended:** §15.3

Fix $n \in \mathbb{N}$. The (2-dimensional) **Bessel's Equation** (of **order** n) is the ordinary differential equation

$$x^2\mathcal{R}''(x) \;+\; x\mathcal{R}'(x) \;+\; (x^2-n^2)\cdot\mathcal{R}(x) \;=\; 0 \qquad (14.21)$$

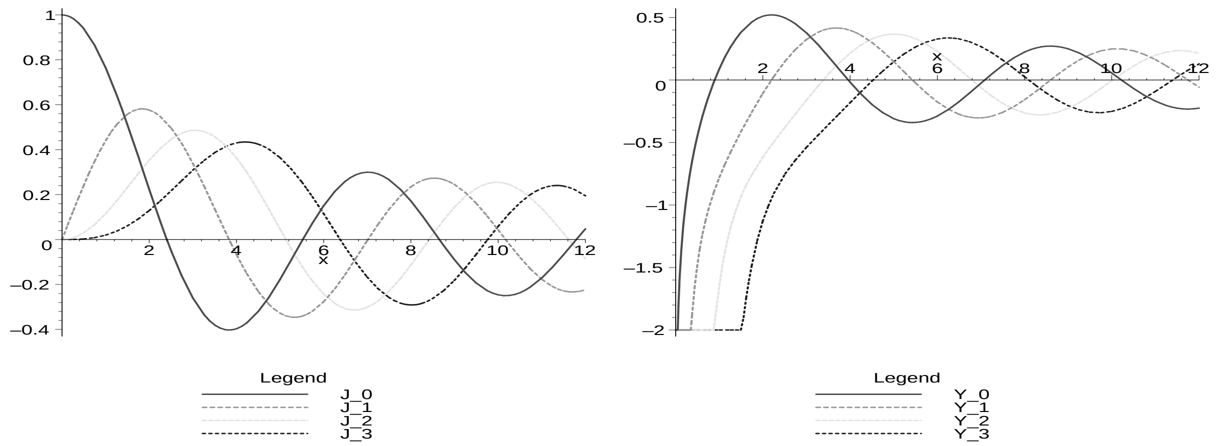
where $\mathcal{R} : [0,\infty] \longrightarrow \mathbb{R}$ is an unknown function. In §15.3, we will explain how this equation was first derived. In the present section, we will investigate its mathematical consequences.

The Bessel equation has two solutions:

$\mathcal{R}(x) = \mathcal{J}_n(x)$ the nth order **Bessel function** of the **first kind**. [See Figures 14.12(A) and 14.13(A)]

$\mathcal{R}(x) = \mathcal{Y}_n(x)$ the nth order **Bessel function** of the **second kind**, or **Neumann function**. [See Figures 14.12(B) and 14.13(B)]

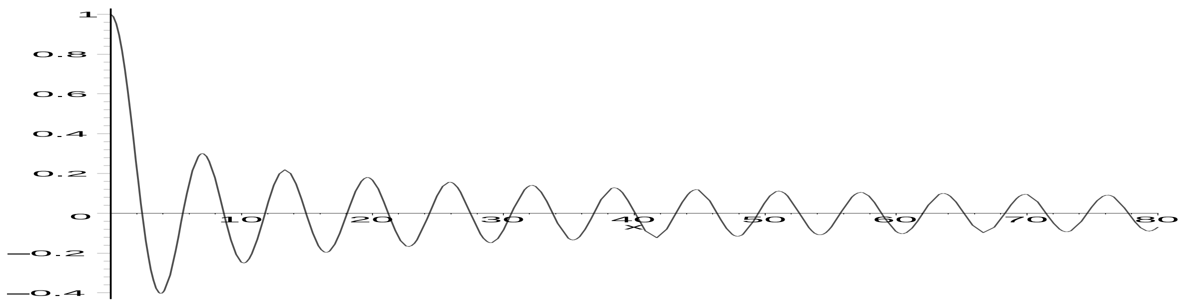
Bessel functions are like trigonometric or logarithmic functions; the 'simplest' expression for them is in terms of a power series. Hence, you should treat the functions "$\mathcal{J}_n$" and "$\mathcal{Y}_n$" the same way you treat elementary functions like "sin", "tan" or "log". In §14.7 we will derive an explicit power-series for Bessel's functions, and in §14.8, we will derive some of their important properties. However, for now, we will simply take for granted that some solution functions



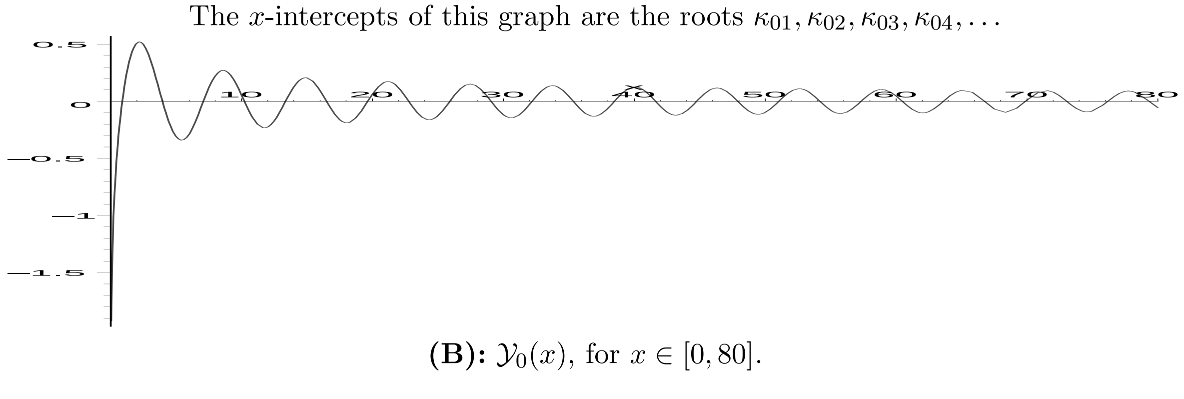
(A)
$\mathcal{J}_0(x)$, $\mathcal{J}_1(x)$, $\mathcal{J}_2(x)$, and $\mathcal{J}_3(x)$, for $x \in [0, 12]$;

(B)
$\mathcal{Y}_0(x)$, $\mathcal{Y}_1(x)$, $\mathcal{Y}_2(x)$, and $\mathcal{Y}_3(x)$ for $x \in [0, 12]$ (these four plots are vertically truncated).

Figure 14.12: Bessel functions near zero.



(A): $\mathcal{J}_0(x)$, for $x \in [0, 100]$.
The x-intercepts of this graph are the roots $\kappa_{01}, \kappa_{02}, \kappa_{03}, \kappa_{04}, \ldots$



(B): $\mathcal{Y}_0(x)$, for $x \in [0, 80]$.

Figure 14.13: Bessel functions are asymptotically periodic.

$\mathcal{J}_n$ exists, and discuss how we can us these functions to build eigenfunctions for the Laplacian which *separate* in polar coordinates....

Proposition 14.12: *Fix $\lambda > 0$. For any $n \in \mathbb{N}$, define*

$$\begin{array}{rclcrcl} \Phi_{n,\lambda}(r,\theta) & = & \mathcal{J}_n(\lambda \cdot r) \cdot \cos(n\theta); & & \Psi_{n,\lambda}(r,\theta) & = & \mathcal{J}_n(\lambda \cdot r) \cdot \sin(n\theta); \\ \phi_{n,\lambda}(r,\theta) & = & \mathcal{Y}_n(\lambda \cdot r) \cdot \cos(n\theta); & \textit{and} & \psi_{n,\lambda}(r,\theta) & = & \mathcal{Y}_n(\lambda \cdot r) \cdot \sin(n\theta). \end{array}$$

(see Figures 14.14 and 14.15). *Then $\Phi_{n,\lambda}$, $\Psi_{n,\lambda}$, $\phi_{n,\lambda}$, and $\psi_{n,\lambda}$ are all eigenfunctions of the Laplacian with eigenvalue $-\lambda^2$:*

$$\triangle\Phi_{n,\lambda} = -\lambda^2\Phi_{n,\lambda}; \quad \triangle\Psi_{n,\lambda} = -\lambda^2\Psi_{n,\lambda}; \quad \triangle\phi_{n,\lambda} = -\lambda^2\phi_{n,\lambda}; \quad \textit{and} \quad \triangle\,\psi_{n,\lambda} = -\lambda^2\psi_{n,\lambda}.$$

Proof: See practice problems #12 to #15 of §14.9. □

We can now use these eigenfunctions to solve PDEs in polar coordinates. Notice that $\mathcal{J}_n$ —and thus, eigenfunctions $\Phi_{n,\lambda}$ and $\Psi_{n,\lambda}$ —are *bounded* around zero (see Figure 14.12A). On the other hand, $\mathcal{Y}_n$ —and thus, eigenfunctions $\phi_{n,\lambda}$ and $\psi_{n,\lambda}$ —are *unbounded* at zero (see Figure 14.12B). Hence, when solving BVPs in a neighbourhood around zero (eg. the disk), it is preferable to use $\mathcal{J}_n$, $\Phi_{n,\lambda}$ and $\Psi_{n,\lambda}$. When solving BVPs on a domain *away* from zero (eg. the annulus), we can also use $\mathcal{Y}_n$, $\phi_{n,\lambda}$ and $\psi_{n,\lambda}$.

14.3(b) Boundary conditions; the roots of the Bessel function

Prerequisites: §6.5, §14.3(a)

To obtain *homogeneous Dirichlet boundary conditions* on a disk of radius R, we need an eigenfunction of the form $\Phi_{n,\lambda}$ (or $\Psi_{n,\lambda}$) such that $\Phi_{n,\lambda}(R,\theta) = 0$ for all $\theta \in [-\pi, \pi)$. Hence, we need:

$$\mathcal{J}_n(\lambda \cdot R) = 0 \tag{14.22}$$

The **roots** of the Bessel function are the values $\kappa \in [0, \infty)$ such that $\mathcal{J}_n(\kappa) = 0$. These roots form an increasing sequence

$$0 \leq \kappa_{n1} < \kappa_{n2} < \kappa_{n3} < \kappa_{n4} < \ldots$$

of irrational values[3]. Thus, to solve (14.22), we must set $\lambda = \kappa_{nm}/R$ for some $m \in \mathbb{N}$. This yields an increasing sequence of eigenvalues:

$$\lambda_{n1}^2 = \left(\frac{\kappa_{n1}}{R}\right)^2 < \lambda_{n2}^2 = \left(\frac{\kappa_{n2}}{R}\right)^2 < \lambda_{n3}^2 = \left(\frac{\kappa_{n3}}{R}\right)^2 < \lambda_{n4}^2 = \left(\frac{\kappa_{n4}}{R}\right)^2 < \ldots$$

which are the eigenvalues which we can expect to see in this problem. The corresponding eigenfunctions will then have the form:

$$\Phi_{n,m}(r,\theta) = \mathcal{J}_n(\lambda_{n,m} \cdot r) \cdot \cos(n\theta) \qquad \Psi_{n,m}(r,\theta) = \mathcal{J}_n(\lambda_{n,m} \cdot r) \cdot \sin(n\theta) \tag{14.23}$$

(see Figures 14.14 and 14.15).

[3]Computing these roots is difficult; tables of κ_{nm} can be found in most standard references on PDEs.

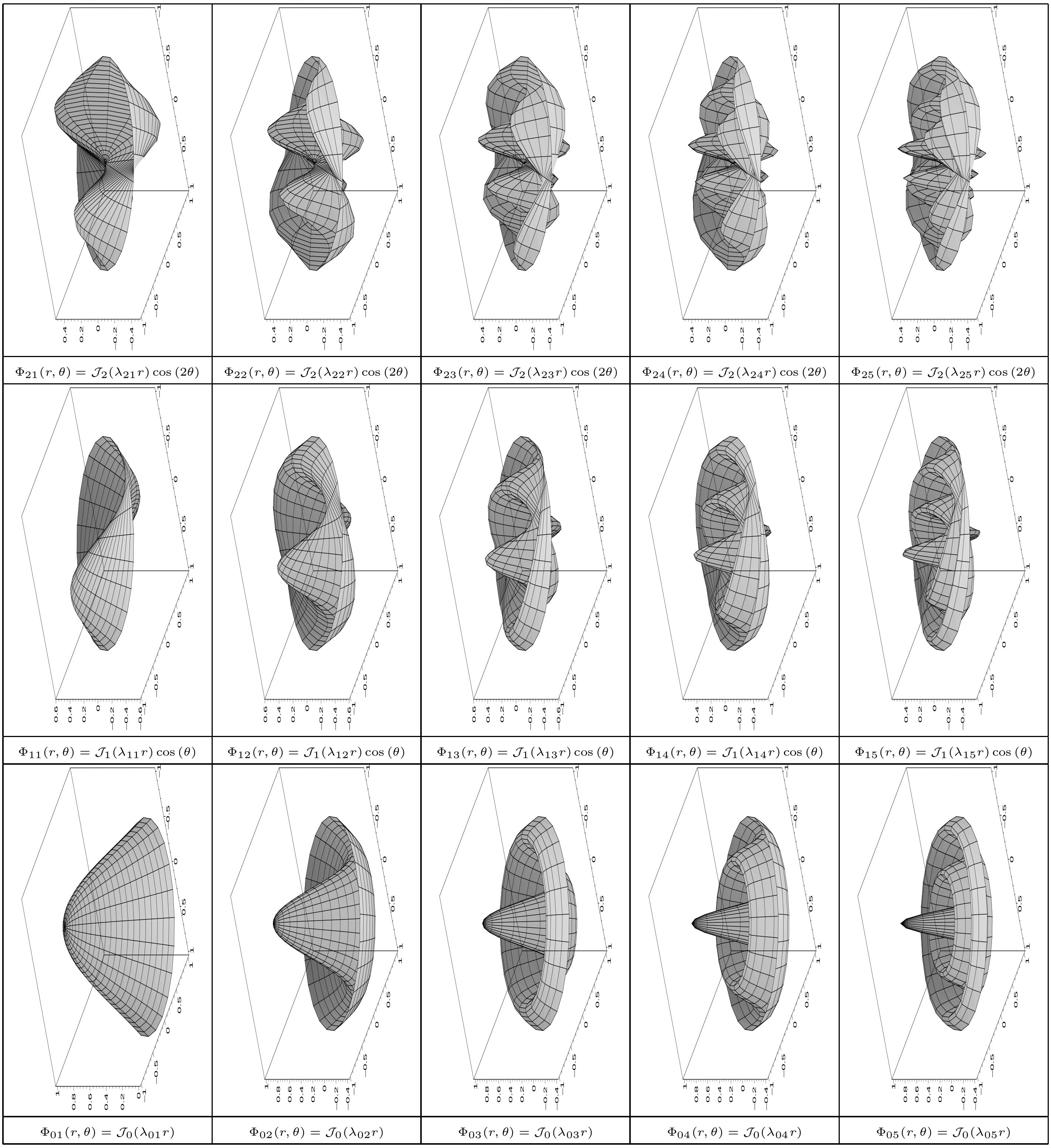


Figure 14.14: $\Phi_{n,m}$ for $n = 0, 1, 2$ and for $m = 1, 2, 3, 4, 5$ (rotate page).

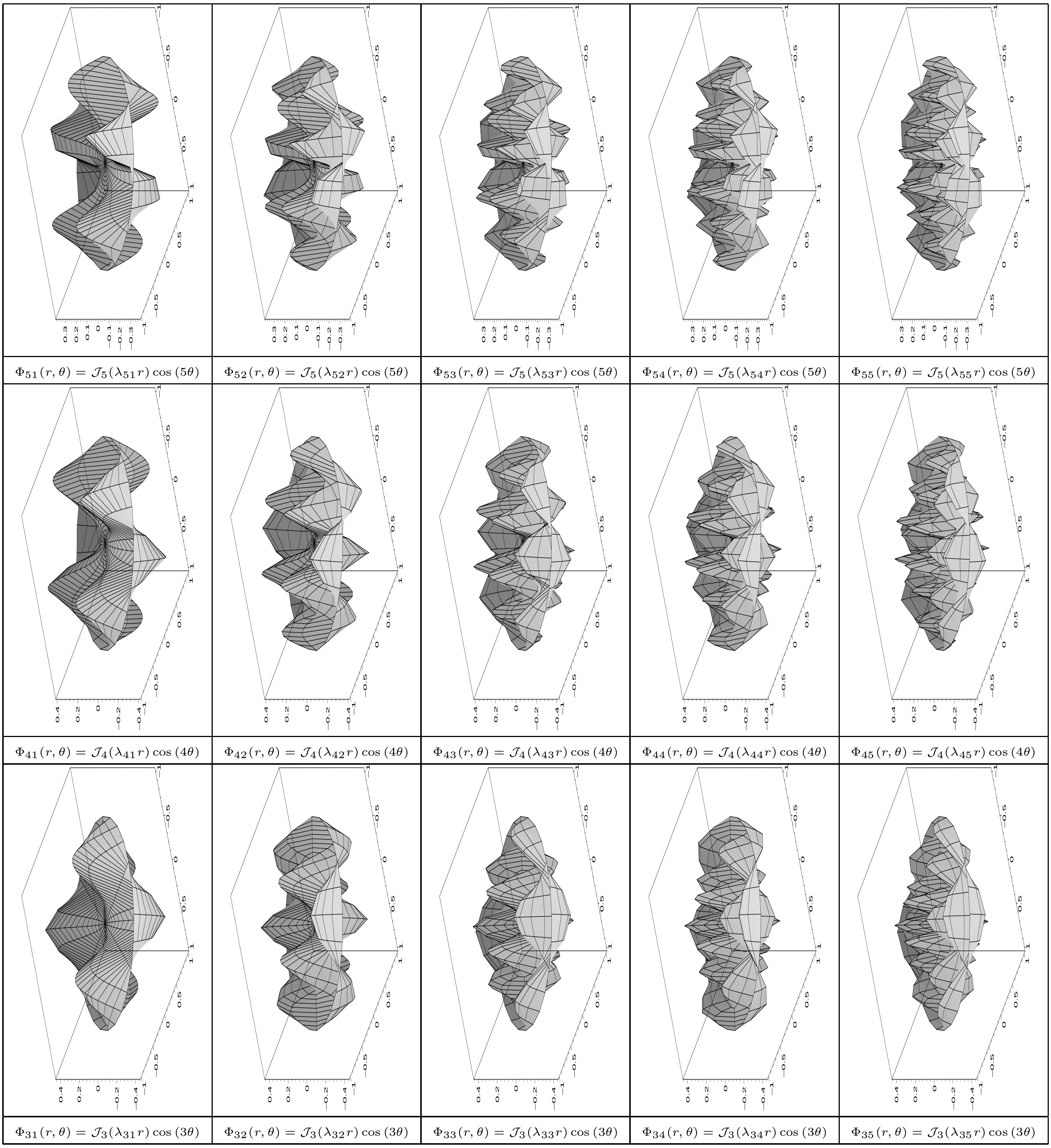


Figure 14.15: $\Phi_{n,m}$ for $n = 3, 4, 5$ and for $m = 1, 2, 3, 4, 5$ (rotate page).

14.3(c) Initial conditions; Fourier-Bessel Expansions

Prerequisites: §6.4, §7.5, §14.3(b)

To solve an initial value problem, while satisfying the desired boundary conditions, we express our initial conditions as a sum of the eigenfunctions from expression (14.23). This is called a **Fourier-Bessel Expansion:**

$$f(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot\Phi_{nm}(r,\theta) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot\Psi_{nm}(r,\theta) + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_{nm}\cdot\phi_{nm}(r,\theta) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\cdot\psi_{nm}(r,\theta), \qquad (14.24)$$

where A_{nm}, B_{nm}, a_{nm}, and b_{nm} are all real-valued coefficients. Suppose we are considering boundary value problems on the unit disk $\mathbb{D}$. Then we want this expansion to be bounded at 0, so we don't want the second two types of eigenfunctions. Thus, expression (14.24) simplifies to:

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot\Phi_{nm}(r,\theta) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot\Psi_{nm}(r,\theta). \qquad (14.25)$$

If we substitute the explicit expressions from (14.23) for $\Phi_{nm}(r,\theta)$ and $\Psi_{nm}(r,\theta)$ into expression (14.25), we get:

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta). \qquad (14.26)$$

Now, if $f : \mathbb{D} \longrightarrow \mathbb{R}$ is some function describing initial conditons, is it always possile to express f using an expansion like (14.26)? If so, how do we compute the coefficients A_{nm} and B_{nm} in expression (14.26)? The answer to these questions lies in the following result:

Theorem 14.13: *The collection* $\{\Phi_{n,m},\ \Psi_{\ell,m}\ ;\ n = 0...\infty,\ \ell \in \mathbb{N}, m \in \mathbb{N}\}$ *is an* <u>*orthogonal basis*</u> *for* $\mathbf{L}^2(\mathbb{D})$*. Thus, suppose* $f \in \mathbf{L}^2(\mathbb{D})$*, and for all* $n, m \in \mathbb{N}$*, we define*

$$A_{nm} := \frac{\langle f,\ \Phi_{nm}\rangle}{\|\Phi_n m\|_2^2} = \frac{2}{\pi R^2\cdot\mathcal{J}_{n+1}^2(\kappa_{nm})}\cdot\int_{-\pi}^{\pi}\int_0^R f(r,\theta)\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta)\cdot r\ dr\ d\theta.$$

$$B_{nm} := \frac{\langle f,\ \Psi_{nm}\rangle}{\|\Psi_n m\|_2^2} = \frac{2}{\pi R^2\cdot\mathcal{J}_{n+1}^2(\kappa_{nm})}\cdot\int_{-\pi}^{\pi}\int_0^R f(r,\theta)\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta)\cdot r\ dr\ d\theta.$$

Then the Fourier-Bessel series (14.26) *converges to* f *in* $\mathbf{L}^2$*-norm.*

Proof: (sketch) The fact that the collection $\{\Phi_{n,m},\ \Psi_{\ell,m}\ ;\ n = 0...\infty,\ \ell \in \mathbb{N}, m \in \mathbb{N}\}$ is an orthogonal set will be verified in Proposition 14.29 on page 271 of §14.8. The fact that this orthogonal set is actually a *basis* of $\mathbf{L}^2(\mathbb{D})$ is too complicated for us to prove here. Given that this is true, if we define $A_{nm} := \langle f,\ \Phi_{nm}\rangle/\|\Phi_n m\|_2^2$ and $B_{nm} := \langle f,\ \Psi_{nm}\rangle/\|\Psi_n m\|_2^2$, then the Fourier-Bessel series (14.26) converges to f in $\mathbf{L}^2$-norm, by definition of "orthogonal basis" (see § 7.5 on page 133).

It remains to verify the integral expressions given for the two inner products. To do this, recall that

$$\begin{aligned}
\langle f,\ \Phi_{nm}\rangle &= \frac{1}{\mathbf{Area}[\mathbb{D}]}\int_{\mathbb{D}} f(\mathbf{x})\cdot\Phi_{nm}(\mathbf{x})\ d\mathbf{x} \\
&= \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R f(r,\theta)\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta)\cdot r\ dr\ d\theta
\end{aligned}$$

and

$$\begin{aligned}
\|\Phi_{nm}\|_2^2 &= \langle\Phi_{nm},\ \Phi_{nm}\rangle = \frac{1}{\pi R^2}\int_{-\pi}^{\pi}\int_0^R \mathcal{J}_n^2\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos^2(n\theta)\cdot r\ dr\ d\theta \\
&= \left(\frac{1}{R^2}\int_0^R \mathcal{J}_n^2\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot r\ dr\right)\cdot\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(n\theta)\ d\theta\right) \\
&= \frac{1}{R^2}\int_0^R \mathcal{J}_n^2\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot r\ dr. \qquad \text{(By Proposition 7.6 on page 116)} \\
&= \int_0^1 \mathcal{J}_n^2(\kappa_{nm}\cdot s)\cdot s\ ds. \qquad \left(s = \tfrac{r}{R};\quad dr = R\,ds\right) \\
&\underset{(*)}{=} \frac{1}{2}\mathcal{J}_{n+1}^2(\kappa_{nm})
\end{aligned}$$

here, $(*)$ is by Lemma 14.28(b) on page 269 of §14.8. □

To compute the integrals in Theorem 14.13, one generally uses 'integration by parts' techniques similar to those used to compute trigonometic Fourier coefficients. However, instead of the convenient trigonometric facts that $\sin' = \cos$ and $\cos' = -\sin$, one must make use of slightly more complicated recurrence relations of Proposition 14.26 on page 268 of §14.8. See Remark 14.27 on page 269.

We will do not have time to properly develop integration techniques for computing Fourier-Bessel coefficients in this book. Instead, in the remaining discussion, we will simply assume that f is given to us in the form (14.26).

14.4 The Poisson Equation in Polar Coordinates

Prerequisites: §2.4, §14.3(b), §1.7 **Recommended:** §11.3, §12.3, §13.3, §14.2

Proposition 14.14: (Poisson Equation on Disk; homogeneous Dirichlet BC)

Let $\mathbb{D} = \{(r,\theta)\ ;\ r \leq R\}$ be a disk, and let $q \in \mathbf{L}^2(\mathbb{D})$ be some function. Consider the Posson equation "$\triangle u(r,\theta) = q(r,\theta)$", with homogeneous Dirichlet boundary conditions.

Suppose q has Fourier-Bessel series:

$$q(r,\theta) \underset{\text{L2}}{\approx} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta)$$

Then the unique solution to this problem is the function

$$u(r,\theta) \underset{\text{unif}}{\equiv} -\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{R^2 \cdot A_{nm}}{\kappa_{nm}^2} \cdot \mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right) \cdot \cos(n\theta) \;-\; \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{R^2 \cdot B_{nm}}{\kappa_{nm}^2} \cdot \mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right) \cdot \sin(n\theta)$$

Proof: **Exercise 14.10** □

Remark: If $R = 1$, then the initial conditions simplify to:

$$q(r,\theta) \underset{\text{L2}}{\approx} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm} \cdot \mathcal{J}_n(\kappa_{nm}\cdot r) \cdot \cos(n\theta) \;+\; \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm} \cdot \mathcal{J}_n(\kappa_{nm}\cdot r) \cdot \sin(n\theta)$$

and the solution simplifies to

$$u(r,\theta) \underset{\text{unif}}{\equiv} -\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{A_{nm}}{\kappa_{nm}^2} \cdot \mathcal{J}_n(\kappa_{nm}\cdot r) \cdot \cos(n\theta) \;-\; \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{B_{nm}}{\kappa_{nm}^2} \cdot \mathcal{J}_n(\kappa_{nm}\cdot r) \cdot \sin(n\theta)$$

Example 14.15: Suppose $R = 1$, and $q(r,\theta) = \mathcal{J}_0(\kappa_{0,3}\cdot r) + \mathcal{J}_5(\kappa_{2,5}\cdot r)\cdot\sin(2\theta)$. Then

$$u(r,\theta) = \frac{-\mathcal{J}_0(\kappa_{0,3}\cdot r)}{\kappa_{0,3}^2} - \frac{\mathcal{J}_5(\kappa_{2,5}\cdot r)\cdot\sin(2\theta)}{\kappa_{2,5}^2}.$$ ◇

Proposition 14.16: (Poisson Equation on Disk; nonhomogeneous Dirichlet BC)

Let $\mathbb{D} = \{(r,\theta)\ ;\ r \le R\}$ *be a disk. Let* $b \in \mathbf{L}^2[-\pi,\pi)$ *and* $q \in \mathbf{L}^2(\mathbb{D})$. *Consider the Poisson equation "*$\triangle u(r,\theta) = q(r,\theta)$*", with nonhomogeneous Dirichlet boundary conditions:*

$$u(R,\theta) = b(\theta) \qquad \text{for all } \theta \in [-\pi,\pi) \tag{14.27}$$

1. *Let* $w(r,\theta)$ *be the solution*[4] *to the Laplace Equation; "*$\triangle w(r,\theta) = 0$*", with the nonhomogeneous Dirichlet BC* (14.27).
2. *Let* $v(r,\theta)$ *be the solution*[5] *to the Poisson Equation; "*$\triangle v(r,\theta) = q(r,\theta)$*", with the homogeneous Dirichlet BC.*
3. *Define* $u(r,\theta) = v(r,\theta;t) + w(r,\theta)$. *Then* $u(r,\theta)$ *is a solution to the Poisson Equation with inhomogeneous Dirichlet BC* (14.27).

Proof: **Exercise 14.11** □

[4]Obtained from Proposition 14.2 on page 239, for example.

[5]Obtained from Proposition 14.14, for example.

Example 14.17: Suppose $R = 1$, and $q(r,\theta) = \mathcal{J}_0(\kappa_{0,3} \cdot r) + \mathcal{J}_2(\kappa_{2,5} \cdot r) \cdot \sin(2\theta)$. Let $b(\theta) = \sin(3\theta)$.

From Example 14.3 on page 240, we know that the (bounded) solution to the Laplace equation with Dirichlet boundary $w(1,\theta) = b(\theta)$ is:

$$w(r,\theta) = r^3 \sin(3\theta).$$

From Example 14.15, we know that the solution to the Poisson equation "$\triangle v = q$", with homogeneous Dirichlet boundary is:

$$v(r,\theta) = \frac{\mathcal{J}_0(\kappa_{0,3} \cdot r)}{\kappa_{0,3}^2} + \frac{\mathcal{J}_2(\kappa_{2,5} \cdot r) \cdot \sin(2\theta)}{\kappa_{2,5}^2}.$$

Thus, by Proposition 14.16, the the solution to the Poisson equation "$\triangle u = q$", with Dirichlet boundary $w(1,\theta) = b(\theta)$, is given:

$$u(r,\theta) = v(r,\theta) + w(r,\theta) = \frac{\mathcal{J}_0(\kappa_{0,3} \cdot r)}{\kappa_{0,3}^2} + \frac{\mathcal{J}_2(\kappa_{2,5} \cdot r) \cdot \sin(2\theta)}{\kappa_{2,5}^2} + r^3 \sin(3\theta). \quad \diamondsuit$$

14.5 The Heat Equation in Polar Coordinates

Prerequisites: §2.2, §14.3(c), §1.7 **Recommended:** §11.1, §12.2, §13.1, §14.2

Proposition 14.18: (Heat Equation on Disk; homogeneous Dirichlet BC)

Let $\mathbb{D} = \{(r,\theta)\ ;\ r \leq R\}$ be a disk, and consider the Heat equation "$\partial_t u = \triangle u$", with homogeneous Dirichlet boundary conditions, and initial conditions $u(r,\theta;0) = f(r,\theta)$.

Suppose f has Fourier-Bessel series:

$$f(r,\theta) \underset{L2}{\approx} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} A_{nm} \cdot \mathcal{J}_n\left(\frac{\kappa_{nm} \cdot r}{R}\right) \cdot \cos(n\theta) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{nm} \cdot \mathcal{J}_n\left(\frac{\kappa_{nm} \cdot r}{R}\right) \cdot \sin(n\theta)$$

Then the unique solution to this problem is the function

$$u(r,\theta;\ t) \underset{L2}{\approx} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} A_{nm} \cdot \mathcal{J}_n\left(\frac{\kappa_{nm} \cdot r}{R}\right) \cdot \cos(n\theta) \exp\left(\frac{-\kappa_{nm}^2}{R^2} t\right) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{nm} \cdot \mathcal{J}_n\left(\frac{\kappa_{nm} \cdot r}{R}\right) \cdot \sin(n\theta) \exp\left(\frac{-\kappa_{nm}^2}{R^2} t\right)$$

Furthermore, the series defining u converges semiuniformly on $\mathbb{D} \times (0,\infty)$.

Proof: **Exercise 14.12** □

Remark: If $R = 1$, then the initial conditions simplify to:

$$f(r,\theta) \underset{L2}{\approx} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot \mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\cos(n\theta) \quad + \quad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot \mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\sin(n\theta)$$

and the solution simplifies to:

$$u(r,\theta;\ t) \underset{L2}{\approx} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot \mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\cos(n\theta)\cdot e^{-\kappa_{nm}^2 t} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot \mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\sin(n\theta)\cdot e^{-\kappa_{nm}^2 t}$$

Example 14.19: Suppose $R = 1$, and $f(r,\theta) = \mathcal{J}_0(\kappa_{0,7}\cdot r) - 4\mathcal{J}_3(\kappa_{3,2}\cdot r)\cdot\cos(3\theta)$. Then

$$u(r,\theta;\ t) \quad = \quad \mathcal{J}_0(\kappa_{0,7}\cdot r)\cdot e^{-\kappa_{0,7}^2 t} \ - \ 4\mathcal{J}_3(\kappa_{3,2}\cdot r)\cdot\cos(3\theta)\cdot e^{-\kappa_{32}^2 t}.$$ ♢

Proposition 14.20: (Heat Equation on Disk; nonhomogeneous Dirichlet BC)

Let $\mathbb{D} = \{(r,\theta)\ ;\ r \le R\}$ be a disk, and let $f : \mathbb{D} \longrightarrow \mathbb{R}$ and $b : [-\pi,\pi) \longrightarrow \mathbb{R}$ be given functions. Consider the heat equation

$$\partial_t u(r,\theta;t) \quad = \quad \triangle u(r,\theta;t)$$

with initial conditions $u(r,\theta) = f(r,\theta)$, and nonhomogeneous Dirichlet boundary conditions:

$$u(R,\theta) \ = \ b(\theta) \qquad \text{for all } \theta \in [-\pi,\pi) \tag{14.28}$$

1. *Let $w(r,\theta)$ be the solution*[6] *to the Laplace Equation; "$\triangle v(r,\theta) = 0$", with the nonhomogeneous Dirichlet BC* (14.28).

2. *Define $g(r,\theta) = f(r,\theta) - w(r,\theta)$. Let $v(r,\theta;t)$ be the solution*[7] *to the heat equation "$\partial_t v(r,\theta;t) = \triangle v(r,\theta;t)$" with initial conditions $v(r,\theta) = g(r,\theta)$, and homogeneous Dirichlet BC.*

3. *Define $u(r,\theta;t) = v(r,\theta;t) + w(r,\theta)$. Then $u(r,\theta;t)$ is a solution to the Heat Equation with initial conditions $u(r,\theta) = f(r,\theta)$, and inhomogeneous Dirichlet BC* (14.28).

Proof: **Exercise 14.13** □

[6]Obtained from Proposition 14.2 on page 239, for example.
[7]Obtained from Proposition 14.18, for example.

14.6 The Wave Equation in Polar Coordinates

Prerequisites: §3.2, §14.3(b), §14.3(c), §1.7 **Recommended:** §11.2, §12.4, §14.5

Imagine a drumskin stretched tightly over a circular frame. At equilibrium, the drumskin is perfectly flat, but if we strike the skin, it will vibrate, meaning that the membrane will experience vertical displacements from equilibrium. Let $\mathbb{D} = \{(r,\theta) \; ; \; r \leq R\}$ represent the round skin, and for any point $(r,\theta) \in \mathbb{D}$ on the drumskin and time $t > 0$, let $u(r,\theta;t)$ be the vertical displacement of the drum. Then u will obey the two-dimensional Wave Equation:

$$\partial_t^2 u(r,\theta;t) \;=\; \triangle u(r,\theta;t). \tag{14.29}$$

However, since the skin is held down along the edges of the circle, the function u will also exhibit homogeneous Dirichlet boundary conditions:

$$u(R,\theta;t) \;=\; 0, \qquad \text{for all } \theta \in [-\pi,\pi) \text{ and } t \geq 0. \tag{14.30}$$

Proposition 14.21: (Wave Equation on Disk; homogeneous Dirichlet BC)

Let $\mathbb{D} = \{(r,\theta) \; ; \; r \leq R\}$ be a disk, and consider the wave equation "$\partial_t^2 u = \triangle u$", with homogeneous **Dirichlet** *boundary conditions, and*

$$\begin{array}{rrcl} \textbf{Initial position:} & u(r,\theta;0) & = & f_0(r,\theta); \\ \textbf{Initial velocity:} & \partial_t u(r,\theta;0) & = & f_1(r,\theta) \end{array}$$

Suppose f_0 and f_1 have Fourier-Bessel series:

$$f_0(r,\theta) \;\underset{L2}{\approx}\; \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta) \;+\; \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta);$$

$$\text{and } f_1(r,\theta) \;\underset{L2}{\approx}\; \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A'_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta) \;+\; \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B'_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta).$$

Assume that

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\kappa_{nm}^2|A_{nm}| \;+\; \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\kappa_{nm}^2|B_{nm}| \;<\; \infty,$$

$$\text{and } \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\kappa_{nm}|A'_{nm}| \;+\; \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\kappa_{nm}|B'_{nm}| \;<\; \infty.$$

Then the unique solution to this problem is the function

$$u(r,\theta;\; t) \;\underset{L2}{\approx}\; \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta)\cdot\cos\left(\frac{\kappa_{nm}}{R}t\right)$$

$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta)\cdot\cos\left(\frac{\kappa_{nm}}{R}t\right)$$
$$+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{R\cdot A'_{nm}}{\kappa_{nm}}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta)\cdot\sin\left(\frac{\kappa_{nm}}{R}t\right)$$
$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{R\cdot B'_{nm}}{\kappa_{nm}}\cdot\mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta)\cdot\sin\left(\frac{\kappa_{nm}}{R}t\right).$$

Proof: <u>**Exercise 14.14**</u> □

Remark: If $R = 1$, then the initial conditions would be:

$$f_0(r,\theta) \underset{L2}{\approx} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot\mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\cos(n\theta) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot\mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\sin(n\theta),$$

and $$f_1(r,\theta) \underset{L2}{\approx} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A'_{nm}\cdot\mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\cos(n\theta) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B'_{nm}\cdot\mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\sin(n\theta).$$

and the solution simplifies to:

$$u(r,\theta;\ t) \underset{L2}{\approx} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cdot\mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\cos(n\theta)\cdot\cos(\kappa_{nm}t)$$
$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\cdot\mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\sin(n\theta)\cdot\cos(\kappa_{nm}t)$$
$$+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{A'_{nm}}{\kappa_{nm}}\cdot\mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\cos(n\theta)\cdot\sin(\kappa_{nm}t)$$
$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{B'_{nm}}{\kappa_{nm}}\cdot\mathcal{J}_n(\kappa_{nm}\cdot r)\cdot\sin(n\theta)\cdot\sin(\kappa_{nm}t).$$

Acoustic Interpretation: The vibration of the drumskin is a superposition of distinct **modes** of the form

$$\Phi_{nm}(r,\theta) = \mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\cos(n\theta) \quad \text{and} \quad \Psi_{nm}(r,\theta) = \mathcal{J}_n\left(\frac{\kappa_{nm}\cdot r}{R}\right)\cdot\sin(n\theta),$$

for all $m, n \in \mathbb{N}$. For fixed m and n, the modes Φ_{nm} and and Ψ_{nm} vibrate at (temporal) frequency $\lambda_{nm} = \frac{\kappa_{nm}}{R}$. In the case of the *vibrating string*, all the different modes vibrated at frequences that were *integer multiples* of the fundamental frequency; musically speaking, this means that they 'harmonized'. In the case of a *drum*, however, the frequencies are all *irrational* multiples (because the roots κ_{nm} are all irrationally related). Acoustically speaking, this means we expect a drum to sound somewhat more 'discordant' than a string.

Notice also that, as the radius R gets *larger*, the frequency $\lambda_{nm} = \dfrac{\kappa_{nm}}{R}$ gets *smaller*. This means that larger drums vibrate at *lower* frequencies, which matches our experience.

Example 14.22: *A circular membrane of radius $R = 1$ is emitting a pure pitch at frequency κ_{35}. Roughly describe the space-time profile of the solution (as a pattern of distortions of the membrane).*

Answer: The spatial distortion of the membrane must be a combination of modes vibrating at this frequency. Thus, we expect it to be a function of the form:

$$u(r,\theta;\ t) \quad = \quad \mathcal{J}_3\left(\kappa_{35}\cdot r\right)\left[\Big(A\cdot\cos(3\theta) + \ B\cdot\sin(3\theta)\Big)\cdot\cos\left(\kappa_{35}t\right) \ + \left(\frac{A'}{\kappa_{35}}\cdot\cos(3\theta) + \ \frac{B'}{\kappa_{35}}\cdot\sin(3\theta)\right)\cdot\sin\left(\kappa_{35}t\right)\right]$$

By introducing some constant angular phase-shifts ϕ and ϕ', as well as new constants C and C', we can rewrite this (**Exercise 14.15**) as:

$$u(r,\theta;\ t) \quad = \quad \mathcal{J}_3\left(\kappa_{35}\cdot r\right)\left(C\cdot\cos(3(\theta+\phi))\cdot\cos\left(\kappa_{35}t\right) \quad + \quad \frac{C'}{\kappa_{35}}\cdot\cos(3(\theta+\phi'))\cdot\sin\left(\kappa_{35}t\right)\right).$$

♢

Example 14.23: *An initially silent circular drum of radius $R = 1$ is struck in its exact center with a drumstick having a spherical head. Describe the resulting pattern of vibrations.*

Solution: This is a problem with *nonzero* initial *velocity* and *zero* initial *position*. Since the initial velocity (the impact of the drumstick) is rotationally symmetric (dead centre, spherical head), we can write it as a Fourer-Bessel expansion with no angular dependence:

$$f_1(r,\theta) \quad = \quad f(r) \quad \underset{L2}{\approx} \quad \sum_{m=1}^{\infty} A'_m\cdot\mathcal{J}_0\left(\kappa_{0m}\cdot r\right) \qquad (A'_1, A'_2, A'_3, \ldots \text{ some constants})$$

(all the higher-order Bessel functions disappear, since $\mathcal{J}_n$ is always associated with terms of the form $\sin(n\theta)$ and $\cos(n\theta)$, which depend on θ.) Thus, the solution must have the form:

$$u(r,\theta;\ t) \quad = \quad u(r,t) \quad \underset{L2}{\approx} \quad \sum_{m=1}^{\infty}\frac{A'_m}{\kappa_{0m}}\cdot\mathcal{J}_0\left(\kappa_{0m}\cdot r\right)\cdot\sin\left(\kappa_{0m}t\right). \qquad \diamondsuit$$

14.7 The power series for a Bessel Function

Recommended: §14.3(a)

In §14.3-§14.6, we claimed that Bessel's equation had certain solutions called *Bessel functions*, and showed how to use these Bessel functions to solve differential equations in polar coordinates. Now we will derive an an explicit formula for these Bessel functions in terms of their power series.

Proposition 14.24: *Set $\lambda := 1$. For any fixed $m \in \mathbb{N}$ there is a solution $\mathcal{J}_m : [0,\infty) \longrightarrow \mathbb{R}$ to the Bessel Equation*

$$x^2 \mathcal{J}''(x) + x \cdot \mathcal{J}'(x) + (x^2 - m^2) \cdot \mathcal{J}(x) \quad = \quad 0, \quad \textit{for all } x > 0. \tag{14.31}$$

with a power series expansion:

$$\mathcal{J}_m(x) \quad = \quad \left(\frac{x}{2}\right)^m \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}\, k!\, (m+k)!} x^{2k} \tag{14.32}$$

($\mathcal{J}_m$ is called the mth order **Bessel function** of the **first kind.**)

Proof: We will apply the *Method of Frobenius* to solve (14.31). Suppose that $\mathcal{J}$ is is a solution, with an (unknown) power series $\mathcal{J}(x) = x^M \sum_{k=0}^{\infty} a_k x^k$, where $a_0, a_1, \ldots$ are unknown coefficients, and $M \geq 0$. We assume that $a_0 \neq 0$. We substitute this power series into eqn.(14.31) to get equations relating the coefficients. If

$$\begin{array}{rclclclcl}
\mathcal{J}(x) & = & a_0 x^M & + & a_1 x^{M+1} & + & a_2 x^{M+2} & + \cdots + & a_k x^{M+k} & + \cdots \\
\text{Then} & & & & & & & & & \\
-m^2 \mathcal{J}(x) & = & -m^2 a_0 x^M & - & m^2 a_1 x^{M+1} & - & m^2 a_2 x^{M+2} & - \cdots - & m^2 a_k x^{M+k} & - \cdots \\
x^2 \mathcal{J}(x) & = & & & & & a_0 x^{M+2} & + \cdots + & a_{k-2} x^{M+k} & + \cdots \\
x \mathcal{J}'(x) & = & M a_0 x^M & + & (M+1) a_1 x^{M+1} & + & (M+2) a_2 x^{M+2} & + \cdots + & (M+k) a_k x^{M+k} & + \cdots \\
x^2 \mathcal{J}''(x) & = & M(M-1) a_0 x^M & + & (M+1) M a_1 x^{M+1} & + & (M+2)(M+1) a_2 x^{M+2} & + \cdots + & (M+k)(M+k-1) a_k x^{M+k} & + \cdots \\
\text{Thus } 0 & = & \multicolumn{8}{l}{x^2 \mathcal{J}''(x) + x \cdot \mathcal{J}'(x) + (x^2 - m^2) \cdot \mathcal{J}(x)} \\
 & = & \underbrace{(M^2 - m^2)}_{(a)} a_0 x^M & + & \underbrace{((M+1)^2 - m^2)}_{(b)} a_1 x^{M+1} & + & \left({a_0 + \atop ((M+2)^2 - m^2) a_2} \right) x^{M+2} & + \cdots + & b_k x^{M+k} & + \cdots
\end{array}$$

where

$$\begin{aligned}
b_k \quad &:= \quad a_{k-2} + (M+k) a_k + (M+k)(M+k-1) a_k - m^2 a_k \\
&= \quad a_{k-2} + (M+k)(1+M+k-1) a_k - m^2 a_k \quad = \quad a_{k-2} + \left((M+k)^2 - m^2\right) a_k.
\end{aligned}$$

Claim 1: $M = m$.

Proof: If the Bessel equation is to be satisfied, the power series in the bottom row of the tableau must be identically zero. In particular, this means that the coefficient labeled '(a)' must be zero; in other words $a_0(M^2 - m^2) = 0$.

Since we know that $a_0 \neq 0$, this means $(M^2 - m^2) = 0$ —ie. $M^2 = m^2$. But $M \geq 0$, so this means $M = m$. $\diamond_{\text{Claim 1}}$

Claim 2: $a_1 = 0$.

Proof: If the Bessel equation is to be satisfied, the power series in the bottom row of the tableau must be identically zero. In particular, this means that the coefficient labeled '(b)' must be zero; in other words, $a_1 \left[(M+1)^2 - m^2\right] = 0$.

Claim 1 says that $M = m$; hence this is equivalent to $a_1 \left[(m+1)^2 - m^2\right] = 0$. Clearly, $\left[(m+1)^2 - m^2\right] \neq 0$; hence we conclude that $a_1 = 0$. $\diamond_{\text{Claim 2}}$

Claim 3: *For all $k \geq 2$, the coefficients $\{a_2, a_3, a_4, \ldots\}$ must satisfy the following recurrence relation:*

$$a_k \quad = \quad \frac{-1}{(m+k)^2 - m^2} a_{k-2}, \qquad \text{for all even } k \in \mathbb{N} \text{ with } k \geq 2. \tag{14.33}$$

In particular, $a_k = 0$ for all odd $k \in \mathbb{N}$.

Proof: If the Bessel equation is to be satisfied, the power series in the bottom row of the tableau must be identically zero. In particular, this means that all the coefficients b_k must be zero. In other words, $a_{k-2} + \left((M+k)^2 - m^2\right) a_k \;=\; 0$.

From Claim 1, we know that $M = m$; hence this is equivalent to $a_{k-2} + \left((m+k)^2 - m^2\right) a_k \;=\; 0$. In other words, $a_k = -a_{k-2} / \left((m+k)^2 - m^2\right) a_k$.

From Claim 2, we know that $a_1 = 0$. It follows from this equation that $a_3 = 0$; hence $a_5 = 0$, etc. Inductively, $a_n = 0$ for all odd n. $\diamond_{\text{Claim 3}}$

Claim 4: *Assume we have fixed a value for a_0. Define*

$$a_{2j} \quad := \quad \frac{(-1)^j \cdot a_0}{2^{2j} j! (m+1)(m+2)\cdots(m+j)}, \qquad \text{for all } j \in \mathbb{N}.$$

Then the sequence $\{a_0, a_2, a_4, \ldots\}$ satisfies the recurrence relation (14.33).

Proof: Set $k = 2j$ in eqn.(14.33). For any $j \geq 2$, we must show that $a_{2j} = \dfrac{-a_{2j-2}}{(m+2j)^2 - m^2}$. Now, by definition,

$$a_{2j-2} \quad = \quad a_{2(j-1)} \quad := \quad \frac{(-1)^{j-1} \cdot a_0}{2^{2j-2} (j-1)! (m+1)(m+2)\cdots(m+j-1)},$$

Also,

$$(m+2j)^2 - m^2 \quad = \quad m^2 + 4jm + 4j^2 - m^2 \quad = \quad 4jm + 4j^2 \quad = \quad 2^2 j(m+j).$$

Hence

$$\begin{aligned}
\frac{-a_{2j-2}}{(m+2j)^2 - m^2} \quad &= \quad \frac{-a_{2j-2}}{2^2 j(m+j)} \quad = \quad \frac{(-1)(-1)^{j-1} \cdot a_0}{2^2 j(m+j) \;\cdot\; 2^{2j-2} (j-1)! (m+1)(m+2)\cdots(m+j-1)} \\
&= \quad \frac{(-1)^j \cdot a_0}{2^{2j-2+2} \cdot j(j-1)! \cdot (m+1)(m+2)\cdots(m+j-1)(m+j)} \\
&= \quad \frac{(-1)^j \cdot a_0}{2^{2j} j! (m+1)(m+2)\cdots(m+j-1)(m+j)} \quad = \quad a_{2j},
\end{aligned}$$

as desired. $\diamond_{\text{Claim 4}}$

By convention we define $a_0 := \dfrac{1}{2^m} \dfrac{1}{m!}$. We claim that that the resulting coefficients yield the Bessel function $\mathcal{J}_m(x)$ defined by (14.32) To see this, let b_{2k} be the $2k$th coefficient of

Ferdinand Georg Frobenius
Born: October 26, 1849 in Berlin-Charlottenburg, Prussia
Died: August 3, 1917 in Berlin

the Bessel series. By definition,

$$\begin{aligned} b_{2k} &:= \frac{1}{2^m} \cdot \frac{(-1)^k}{2^{2k}\, k!\,(m+k)!} \quad = \quad \frac{1}{2^m} \cdot \frac{(-1)^k}{2^{2k}\, k!\, m!(m+1)(m+2)\cdots(m+k-1)(m+k)} \\ &= \frac{1}{2^m m!} \cdot \frac{(-1)^k}{2^{2k}\, k!\,(m+1)(m+2)\cdots(m+k-1)(m+k)} \\ &= a_0 \cdot \left(\frac{(-1)^{k+1}}{2^{2k} k!(m+1)(m+2)\cdots(m+k-1)(m+k)} \right) \quad = \quad a_{2k}, \end{aligned}$$

as desired. □

Corollary 14.25: *Fix $m \in \mathbb{N}$. For any $\lambda > 0$, the Bessel Equation* (15.12) *has solution $\mathcal{R}(r) := \mathcal{J}_m(\lambda r)$.*

Proof: Exercise 14.16 . □

Remarks: (a) We can generalize the Bessel Equation be replacing m with an arbitrary real number $\mu \in \mathbb{R}$ with $\mu \geq 0$. The solution to this equation is the Bessel function

$$\mathcal{J}_\mu(x) \quad = \quad \left(\frac{x}{2}\right)^\mu \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}\, k!\, \Gamma(\mu+k+1)} x^{2k}$$

Here, Γ is the *Gamma function*; if $\mu = m \in \mathbb{N}$, then $\Gamma(m+k+1) = (m+k)!$, so this expression agrees with (14.32).

(b) There is a second solution to (14.31); a function $\mathcal{Y}_m(x)$ which is *unbounded* at zero. This is called a **Neumann function** (or a *Bessel function of the second kind* or a *Weber-Bessel function*). It's derivation is too complicated to discuss here. See [Bro89, §6.8, p.115] or [CB87, §68, p.233].

14.8 Properties of Bessel Functions

Prerequisites: §14.7 **Recommended:** §14.3(a)

Let $\mathcal{J}_n(x)$ be the Bessel function defined by eqn.(14.32) on page 265 of §14.7. In this section, we will develop some computational tools to work with these functions. First, we will define Bessel functions with *negative* order as follows: for any $n \in \mathbb{N}$, we define

$$\mathcal{J}_{-n}(x) \quad := \quad (-1)^n \mathcal{J}_n(x). \tag{14.34}$$

We can now state the following useful recurrence relations

Proposition 14.26: *For any $m \in \mathbb{Z}$,*

(a) $\dfrac{2m}{x}\mathcal{J}_m(x) \;=\; \mathcal{J}_{m-1}(x) + \mathcal{J}_{m+1}(x)$.

(b) $2\mathcal{J}'_m(x) \;=\; \mathcal{J}_{m-1}(x) - \mathcal{J}_{m+1}(x)$.

(c) $\mathcal{J}'_0(x) \;=\; -\mathcal{J}_1(x)$.

(d) $\partial_x \left(x^m \cdot \mathcal{J}_m(x)\right) \;=\; x^m \cdot \mathcal{J}_{m-1}(x)$.

(e) $\partial_x \left(\dfrac{1}{x^m}\mathcal{J}_m(x)\right) \;=\; \dfrac{-1}{x^m} \cdot \mathcal{J}_{m+1}(x)$.

(f) $\mathcal{J}'_m(x) \;=\; \mathcal{J}_{m-1}(x) - \dfrac{m}{x}\mathcal{J}_m(x)$.

(g) $\mathcal{J}'_m(x) \;=\; -\mathcal{J}_{m+1}(x) + \dfrac{m}{x}\mathcal{J}_m(x)$.

Proof: **Exercise 14.17** (i) Prove (d) for $m \geq 1$ by substituting in the power series (14.32) and differentiating.

(ii) Prove (e) for $m \geq 0$ by substituting in the power series (14.32) and differentiating.

(iii) Use the definition (14.34) and (i) and (ii) to prove (d) for $m \leq 0$ and (e) for $m \leq -1$.

(iv) Set $m = 0$ in (e) to obtain (c).

(v) Deduce (f) and (g) from (d) and (e).

(vi) Compute the sum and difference of (f) and (g) to get (a) and (b). □

Remark 14.27: (Integration with Bessel functions)
The recurrence relations of Proposition 14.26 can be used to simplify integrals involving Bessel functions. For example, parts (d) and (e) immediately imply that

$$\int x^m \cdot \mathcal{J}_{m-1}(x)\, dx \;=\; x^m \cdot \mathcal{J}_m(x) + C \qquad \text{and} \qquad \int \frac{1}{x^m} \cdot \mathcal{J}_{m+1}(x)\, dx \;=\; \frac{-1}{x^m}\mathcal{J}_m(x) \; + C.$$

The other relations are sometimes useful in an 'integration by parts' strategy. ♢

For any $n \in \mathbb{N}$, let $0 \leq \lambda_{n,1} < \lambda_{n,2} < \lambda_{n,3} < \cdots$ be the zeros of the nth Bessel function $\mathcal{J}_n$ (ie. $\mathcal{J}_n(\lambda_{n,m}) = 0$ for all $m \in \mathbb{N}$). Proposition 14.12 on page 254 of §14.3(a) says we can use Bessel functions to define a sequence of polar-separated eigenfunctions of the Laplacian:

$$\Phi_{n,m}(r,\theta) \;:=\; \mathcal{J}_n(\lambda_{n,m} \cdot r) \cdot \cos(n\theta); \qquad \Psi_{n,m}(r,\theta) \;:=\; \mathcal{J}_n(\lambda_{n,m} \cdot r) \cdot \sin(n\theta).$$

In the proof of Theorem 14.13 on page 257 of §14.3(c), we claimed that these eigenfunctions were *orthogonal* as elements of $\mathbf{L}^2(\mathbb{D})$. We will now verify this claim. First we must prove a technical lemma.

Lemma 14.28: *Fix $n \in \mathbb{N}$.*

(a) *If $m \neq M$, then* $\displaystyle\int_0^1 \mathcal{J}_n(\lambda_{n,m} \cdot r) \cdot \mathcal{J}_n(\lambda_{n,M} \cdot r)\, r\, dr \;=\; 0.$

(b) $\displaystyle\int_0^1 \mathcal{J}_n(\lambda_{n,m} \cdot r)^2 \cdot\; r\, dr \;=\; \frac{1}{2}\mathcal{J}_{n+1}(\lambda_{n,m})^2.$

Proof: **(a)** Let $\alpha = \lambda_{n,m}$ and $\beta = \lambda_{n,M}$. Define $f(x) := \mathcal{J}_m(\alpha x)$ and $g(x) := \mathcal{J}_m(\beta x)$. Hence we want to show that

$$\int_0^1 f(x) g(x) x\, dx \;=\; 0.$$

Define $h(x) = x \cdot \Big(f(x)g'(x) - g(x)f'(x)\Big)$.

Claim 1: $h'(x) \;=\; (\alpha^2 - \beta^2) f(x) g(x) x.$

Proof: First observe that

$$\begin{aligned} h'(x) &= x \cdot \partial_x \Big(f(x)g'(x) - g(x)f'(x)\Big) \;+\; \Big(f(x)g'(x) - g(x)f'(x)\Big) \\ &= x \cdot \Big(f(x)g''(x) + f'(x)g'(x) - g'(x)f'(x) - g(x)f''(x)\Big) \;+\; \Big(f(x)g'(x) - g(x)f'(x)\Big) \\ &= x \cdot \Big(f(x)g''(x) - g(x)f''(x)\Big) \;+\; \Big(f(x)g'(x) - g(x)f'(x)\Big). \end{aligned}$$

By setting $\mathcal{R} = f$ or $\mathcal{R} = g$ in Corollary 14.25, we obtain:

$$\begin{aligned} x^2 f''(x) + x f'(x) + (\alpha^2 x^2 - n^2) f(x) &= 0, \\ \text{and} \quad x^2 g''(x) + x g'(x) + (\beta^2 x^2 - n^2) g(x) &= 0. \end{aligned}$$

We multiply the first equation by $g(x)$ and the second by $f(x)$ to get

$$\begin{aligned} x^2f''(x)g(x) + xf'(x)g(x) + \alpha^2x^2f(x)g(x) - n^2f(x)g(x) &= 0, \\ \text{and}\quad x^2g''(x)f(x) + xg'(x)f(x) + \beta^2x^2g(x)f(x) - n^2g(x)f(x) &= 0. \end{aligned}$$

We then subtract these two equations to get

$$x^2\Big(f''(x)g(x) - g''(x)f(x)\Big) + x\Big(f'(x)g(x) - g'(x)f(x)\Big) + \Big(\alpha^2 - \beta^2\Big)f(x)g(x)x^2 = 0.$$

Divide by x to get

$$x\Big(f''(x)g(x) - g''(x)f(x)\Big) + \Big(f'(x)g(x) - g'(x)f(x)\Big) + \Big(\alpha^2 - \beta^2\Big)f(x)g(x)x = 0.$$

Hence we conclude

$$\Big(\alpha^2 - \beta^2\Big)f(x)g(x)x = x\Big(g''(x)f(x) - f''(x)g(x)\Big) + \Big(g'(x)f(x) - f'(x)g(x)\Big) = h'(x),$$

as desired $\diamond_{\text{Claim 1}}$

It follows from Claim 1 that

$$(\alpha^2 - \beta^2)\cdot\int_0^1 f(x)g(x)x\,dx = \int_0^1 h'(x)\,dx = h(1) - h(0) \underset{(*)}{=} 0 - 0 = 0.$$

To see $(*)$, observe that $h(0) = 0\cdot\Big(f(0)g'(0) - g(0)f'(0)\Big) = 0$, Also,

$$h(1) = (1)\cdot\Big(f(1)g'(1) - g(1)f'(1)\Big)$$

But $f(1) = \mathcal{J}_n(\lambda_{n,m}) = 0$ and $g(1) = \mathcal{J}_n(\lambda_{n,N}) = 0$.

(b) Let $\alpha = \lambda_{n,m}$ and $f(x) := \mathcal{J}_m(\alpha x)$. Hence we want to evaluate

$$\int_0^1 f(x)^2 x\,dx.$$

Define $h(x) := x^2(f'(x))^2 + (\alpha^2x^2 - n^2)f^2(x)$

Claim 2: $h'(x) = 2\alpha^2 f(x)^2 x$.

Proof: By setting $\mathcal{R} = f$ in Corollary 14.25, we obtain:

$$0 = x^2f''(x) + xf'(x) + (\alpha^2x^2 - n^2)f(x)$$

We multiply by $f'(x)$ to get

$$\begin{aligned} 0 &= x^2f'(x)f''(x) + x(f'(x))^2 + (\alpha^2x^2 - n^2)f(x)f'(x) \\ &= x^2f'(x)f''(x) + x(f'(x))^2 + (\alpha^2x^2 - n^2)f(x)f'(x) + \alpha^2xf^2(x) - \alpha^2xf^2(x) \\ &= \tfrac{1}{2}\partial_x\Big[x^2(f'(x))^2 + (\alpha^2x^2 - n^2)f^2(x)\Big] - \alpha^2xf^2(x) = \tfrac{1}{2}h'(x) - \alpha^2xf^2(x). \end{aligned}$$

$\diamond_{\text{Claim 2}}$

It follows from Claim 2 that

$$\begin{aligned}
2\alpha^2 \int_0^1 f(x)^2 x\, dx &= \int_0^1 h'(x)\, dx = h(1) - h(0) \\
&= 1^2 (f'(1))^2 + (\alpha^2 1^2 - n^2) \underbrace{\cdot f^2(1)}_{\substack{\mathcal{J}_n^2(\lambda_{n,m}) \\ =0}} - \underbrace{0^2 (f'(0))^2}_{0} + \underbrace{(\alpha^2 0^2 - n^2)}_{[0 \text{ if } n=0]} \underbrace{f^2(0)}_{[0 \text{ if } n \neq 0]} \\
&= f'(1)^2 = \Big(\alpha \mathcal{J}_n'(\alpha)\Big)^2 = \alpha^2 \mathcal{J}_n'(\alpha)^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^1 f(x)^2 x\, dx &= \frac{1}{2}\mathcal{J}_n'(\alpha)^2 \underset{(*)}{=\!=} \frac{1}{2}\Big(\frac{n}{\alpha}\mathcal{J}_n(\alpha) - \mathcal{J}_{n+1}(\alpha)\Big)^2 \\
&\underset{(\dagger)}{=\!=} \frac{1}{2}\left(\frac{n}{\lambda_{n,m}} \underbrace{\mathcal{J}_n(\lambda_{n,m})}_{=0} - \mathcal{J}_{n+1}(\lambda_{n,m})\right)^2 = \frac{1}{2}\mathcal{J}_{n+1}(\lambda_{n,m})^2
\end{aligned}$$

where $(*)$ is by Proposition 14.26(g) and $(\dagger)$ is because $\alpha := \lambda_{n,m}$. □

Proposition 14.29: *Let $\mathbb{D} = \{(r,\theta)\ ;\ r \leq 1\}$ be the unit disk. Then the collection*

$$\{\Phi_{n,m},\ \Psi_{\ell,m}\ ;\ n = 0...\infty,\ \ell \in \mathbb{N}, m \in \mathbb{N}\}$$

is an **orthogonal set** *for $\mathbf{L}^2(\mathbb{D})$. In other words, for any $n, m, N, M \in \mathbb{N}$,*

$$\textbf{(a)} \qquad \langle \Phi_{n,m}, \Psi_{N,M}\rangle = \frac{1}{\pi}\int_0^1 \int_{-\pi}^{\pi} \Phi_{n,m}(r,\theta)\cdot\Psi_{N,M}(r,\theta)\, d\theta\ r\ dr = 0.$$

Furthermore, if $(n,m) \neq (N,M)$, then

$$\textbf{(b)} \qquad \langle \Phi_{n,m}, \Phi_{N,M}\rangle = \frac{1}{\pi}\int_0^1 \int_{-\pi}^{\pi} \Phi_{n,m}(r,\theta)\cdot\Phi_{N,M}(r,\theta)\, d\theta\ r\ dr = 0.$$

$$\textbf{(c)} \qquad \langle \Psi_{n,m}, \Psi_{N,M}\rangle = \frac{1}{\pi}\int_0^1 \int_{-\pi}^{\pi} \Psi_{n,m}(r,\theta)\cdot\Psi_{N,M}(r,\theta)\, d\theta\ r\ dr = 0.$$

Finally, for any (n,m),

$$\textbf{(d)} \qquad \|\Phi_{n,m}\|_2 = \frac{1}{\pi}\int_0^1 \int_{-\pi}^{\pi} \Phi_{n,m}(r,\theta)^2\, d\theta\ r\ dr = \frac{1}{2}\mathcal{J}_{n+1}(\lambda_{n,m})^2.$$

$$\textbf{(e)} \qquad \|\Psi_{n,m}\|_2 = \frac{1}{\pi}\int_0^1 \int_{-\pi}^{\pi} \Psi_{n,m}(r,\theta)^2\, d\theta\ r\ dr = \frac{1}{2}\mathcal{J}_{n+1}(\lambda_{n,m})^2.$$

Proof: **(a)** $\Phi_{n,m}$ and $\Psi_{N,M}$ separate in the coordinates (r, θ), so the integral splits in two:

$$\int_0^1 \int_{-\pi}^{\pi} \Phi_{n,m}(r,\theta) \cdot \Psi_{N,M}(r,\theta) \; d\theta \; r \; dr$$
$$= \int_0^1 \int_{-\pi}^{\pi} \mathcal{J}_n(\lambda_{n,m} \cdot r) \cdot \cos(n\theta) \cdot \mathcal{J}_N(\lambda_{N,M} \cdot r) \cdot \sin(N\theta) \; d\theta \; r \; dr$$
$$= \int_0^1 \mathcal{J}_n(\lambda_{n,m} \cdot r) \cdot \mathcal{J}_N(\lambda_{N,M} \cdot r) \; r \; dr \cdot \underbrace{\int_{-\pi}^{\pi} \cos(n\theta) \cdot \sin(N\theta) \; d\theta}_{= 0 \text{ by Prop. 9.3(c), p.147}} \quad = \quad 0.$$

(b) or **(c)** (Case $n \neq N$). Likewise, if $n \neq N$, then

$$\int_0^1 \int_{-\pi}^{\pi} \Phi_{n,m}(r,\theta) \cdot \Phi_{N,M}(r,\theta) \; d\theta \; r \; dr$$
$$= \int_0^1 \int_{-\pi}^{\pi} \mathcal{J}_n(\lambda_{n,m} \cdot r) \cdot \cos(n\theta) \cdot \mathcal{J}_N(\lambda_{N,M} \cdot r) \cdot \cos(N\theta) \; d\theta \; r \; dr$$
$$= \int_0^1 \mathcal{J}_n(\lambda_{n,m} \cdot r) \cdot \mathcal{J}_N(\lambda_{N,M} \cdot r) \; r \; dr \cdot \underbrace{\int_{-\pi}^{\pi} \cos(n\theta) \cdot \cos(N\theta) \; d\theta}_{= 0 \text{ by Prop. 9.3(a), p.147}} \quad = \quad 0.$$

the case **(c)** is proved similarly.

(b) or **(c)** (Case $n = N$ but $m \neq M$). If $n = N$, then

$$\int_0^1 \int_{-\pi}^{\pi} \Phi_{n,m}(r,\theta) \cdot \Phi_{n,M}(r,\theta) \; d\theta \; r \; dr$$
$$= \int_0^1 \int_{-\pi}^{\pi} \mathcal{J}_n(\lambda_{n,m} \cdot r) \cdot \cos(n\theta) \cdot \mathcal{J}_n(\lambda_{n,M} \cdot r) \cdot \cos(n\theta) \; d\theta \; r \; dr$$
$$= \underbrace{\int_0^1 \mathcal{J}_n(\lambda_{n,m} \cdot r) \cdot \mathcal{J}_n(\lambda_{n,M} \cdot r) \; r \; dr}_{= 0 \text{ by Lemma 14.28(a)}} \cdot \underbrace{\int_{-\pi}^{\pi} \cos(n\theta)^2 \; d\theta}_{\substack{= \pi \text{ by Prop 9.3(d)} \\ \text{on p. 147.}}} \quad = \quad 0 \cdot \pi \quad = \quad 0.$$

(d) and **(e)**: If $n = N$ and $m = M$ then

$$\int_0^1 \int_{-\pi}^{\pi} \Phi_{n,m}(r,\theta)^2 \; d\theta \; r \; dr \quad = \quad \int_0^1 \int_{-\pi}^{\pi} \mathcal{J}_n(\lambda_{n,m} \cdot r)^2 \cdot \cos(n\theta)^2 \; d\theta \; r \; dr$$
$$= \underbrace{\int_0^1 \mathcal{J}_n(\lambda_{n,m} \cdot r)^2 \; r \; dr}_{\substack{= \frac{1}{2}\mathcal{J}_{n+1}(\lambda_{n,m})^2 \\ \text{by Lemma 14.28(b)}}} \cdot \underbrace{\int_{-\pi}^{\pi} \cos(n\theta)^2 \; d\theta}_{\substack{= \pi \text{ by Prop 7.6(d)} \\ \text{on p. 116 of §7.3.}}} \quad = \quad \frac{\pi}{2} \mathcal{J}_{n+1}(\lambda_{n,m})^2.$$

□

Exercise 14.18 (a) Use a 'separation of variables' argument (similar to Proposition 15.5) to prove:

Proposition: *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a harmonic function —in other words suppose $\triangle f = 0$. Suppose f* **separates** *in polar coordinates, meaning that there is a function $\Theta : [-\pi, \pi] \longrightarrow \mathbb{R}$ (satisfying periodic boundary conditions) and a function $\mathcal{R} : [0, \infty) \longrightarrow \mathbb{R}$ such that*

$$f(r, \theta) = \mathcal{R}(r) \cdot \Theta(\theta), \quad \text{for all } r \geq 0 \text{ and } \theta \in [-\pi, \pi].$$

Then there is some $m \in \mathbb{N}$ so that

$$\Theta(\theta) = A \cos(m\theta) + B \sin(m\theta), \qquad \text{(for constants } A, B \in \mathbb{R}.)$$

and $\mathcal{R}$ is a solution to the **Cauchy-Euler Equation:**

$$r^2 \mathcal{R}''(r) + r \cdot \mathcal{R}'(r) - m^2 \cdot \mathcal{R}(r) = 0, \quad \text{for all } r > 0. \tag{14.35}$$

(b) Let $\mathcal{R}(r) = r^\alpha$ where $\alpha = \pm m$. Show that $\mathcal{R}(r)$ is a solution to the Cauchy-Euler equation (14.35).

(c) Deduce that $\Psi_m(r, \theta) = r^m \cdot \sin(m\theta)$; $\Phi_m(r, \theta) = r^m \cdot \cos(m\theta)$; $\psi_m(r, \theta) = r^{-m} \cdot \sin(m\theta)$; and $\phi_m(r, \theta) = r^{-m} \cdot \cos(m\theta)$ are harmonic functions in $\mathbb{R}^2$.

14.9 Practice Problems

1. For all (r, θ), let $\Phi_n(r, \theta) = r^n \cos(n\theta)$. Show that Φ_n is harmonic.
2. For all (r, θ), let $\Psi_n(r, \theta) = r^n \sin(n\theta)$. Show that Ψ_n is harmonic.
3. For all (r, θ) with $r > 0$, let $\phi_n(r, \theta) = r^{-n} \cos(n\theta)$. Show that ϕ_n is harmonic.
4. For all (r, θ) with $r > 0$, let $\psi_n(r, \theta) = r^{-n} \sin(n\theta)$. Show that ψ_n is harmonic.
5. For all (r, θ) with $r > 0$, let $\phi_0(r, \theta) = \log |r|$. Show that ϕ_0 is harmonic.
6. Let $b(\theta) = \cos(3\theta) + 2\sin(5\theta)$ for $\theta \in [-\pi, \pi)$.
   (a) Find the bounded solution(s) to the **Laplace equation** on $\mathbb{D}$, with nonhomogeneous **Dirichlet** boundary conditions $u(1, \theta) = b(\theta)$. Is the solution unique?
   (b) Find the bounded solution(s) to the **Laplace equation** on $\mathbb{D}^\complement$, with nonhomogeneous **Dirichlet** boundary conditions $u(1, \theta) = b(\theta)$. Is the solution unique?
   (c) Find the 'decaying gradient' solution(s) to the **Laplace equation** on $\mathbb{D}^\complement$, with nonhomogeneous **Neumann** boundary conditions $\partial_r u(1, \theta) = b(\theta)$. Is the solution unique?
7. Let $b(\theta) = 2\cos(\theta) - 6\sin(2\theta)$, for $\theta \in [-\pi, \pi)$.
   (a) Find the bounded solution(s) to the **Laplace equation** on $\mathbb{D}$, with nonhomogeneous **Dirichlet** boundary conditions: $u(1, \theta) = b(\theta)$ for all $\theta \in [-\pi, \pi)$. Is the solution unique?
   (b) Find the bounded solution(s) to the **Laplace equation** on $\mathbb{D}$, with nonhomogeneous **Neumann** boundary conditions: $\partial_r u(1, \theta) = b(\theta)$ for all $\theta \in [-\pi, \pi)$. Is the solution unique?

8. Let $b(\theta) = 4\cos(5\theta)$ for $\theta \in [-\pi, \pi)$.

   (a) Find the bounded solution(s) to the **Laplace equation** on the disk $\mathbb{D} = \{(r,\theta)\ ;\ r \le 1\}$, with nonhomogeneous **Dirichlet** boundary conditions $u(1,\theta) = b(\theta)$. Is the solution unique?

   (b) *Verify* your answer in part (a) (ie. check that the solution is harmonic and satisfies the prescribed boundary conditions.)
   (**Hint:** *Recall that* $\triangle \ =\ \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$.)

9. Let $b(\theta) \ = \ 5 + 4\sin(3\theta)$ for $\theta \in [-\pi, \pi)$.

   (a) Find the 'decaying gradient' solution(s) to the **Laplace equation** on the codisk $\mathbb{D}^{\complement} = \{(r,\theta)\ ;\ r \ge 1\}$, with nonhomogeneous **Neumann** boundary conditions $\partial_r\, u(1,\theta) = b(\theta)$. Is the solution unique?

   (b) *Verify* that your answer in part (a) satisfies the prescribed boundary conditions. (Forget about the Laplacian).

10. Let $b(\theta) \ = \ 2\cos(5\theta) + \sin(3\theta)$, for $\theta \in [-\pi, \pi)$.

    (a) Find the solution(s) (if any) to the **Laplace equation** on the disk $\mathbb{D} \ = \ \{(r,\theta)\ ;\ r \le 1\}$, with nonhomogeneous **Neumann** boundary conditions: $\partial_\perp\, u(1,\theta) \ = \ b(\theta)$, for all $\theta \in [-\pi, \pi)$.
    Is the solution unique? Why or why not?

    (b) Find the *bounded* solution(s) (if any) to the **Laplace equation** on the codisk $\mathbb{D}^{\complement} = \{(r,\theta)\ ;\ r \ge 1\}$, with nonhomogeneous **Dirichlet** boundary conditions: $u(1,\theta) = b(\theta)$, for all $\theta \in [-\pi, \pi)$.
    Is the solution unique? Why or why not?

11. Let $\mathbb{D}$ be the unit disk. Let $b : \partial\mathbb{D} \longrightarrow \mathbb{R}$ be some function, and let $u : \mathbb{D} \longrightarrow \mathbb{R}$ be the solution to the corresponding Dirichlet problem with boundary conditions $b(\sigma)$. Prove that
$$u(0,0) \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} b(\sigma)\ d\sigma.$$

    Remark: This is a special case of the Mean Value Theorem for Harmonic Functions (Theorem 2.13 on page 33), but do *not* simply 'quote' Theorem 2.13 to solve this problem. Instead, apply Proposition 14.11 on page 250.

12. Let $\Phi_{n,\lambda}(r,\theta) \ := \ \mathcal{J}_n(\lambda\cdot r)\cdot\cos(n\theta)$. Show that $\triangle\Phi_{n,\lambda} \ = \ -\lambda^2\Phi_{n,\lambda}$.

13. Let $\Psi_{n,\lambda}(r,\theta) \ := \ \mathcal{J}_n(\lambda\cdot r)\cdot\sin(n\theta)$. Show that $\triangle\Psi_{n,\lambda} \ = \ -\lambda^2\Psi_{n,\lambda}$.

14. Let $\phi_{n,\lambda}(r,\theta) \ := \ \mathcal{Y}_n(\lambda\cdot r)\cdot\cos(n\theta)$. Show that $\triangle\phi_{n,\lambda} \ = \ -\lambda^2\phi_{n,\lambda}$.

15. $\psi_{n,\lambda}(r,\theta) \ := \ \mathcal{Y}_n(\lambda\cdot r)\cdot\sin(n\theta)$. Show that $\triangle\psi_{n,\lambda} \ = \ -\lambda^2\psi_{n,\lambda}$.

Notes: ....

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VI Miscellaneous Solution Methods

In Chapters 11 to 14, we saw how initial/boundary value problems for linear partial differential equations could be solved by first identifying an orthogonal basis of eigenfunctions for the relevant differential operator (usually the Laplacian), and then representing the desired initial conditions or boundary conditions as an infinite summation of these eigenfunctions. For each bounded domain, each boundary condition, and each coordinate system we considered, we found a system of eigenfunctions that was 'adapted' to that domain, boundary conditions, and coordinate system.

This method is extremely powerful, but it raises several questions:

1. What if you are confronted with a new domain or coordinate system, where none of the known eigenfunction bases is applicable? Theorem 7.31 on page 139 of §7.6 says that a suitable eigenfunction basis for this domain always exists, *in principle.* But how do you go about discovering such a basis *in practice*? For that matter, how were eigenfunctions bases like the Fourier-Bessel functions discovered in the first place? Where did Bessel's equation come from?
2. What if you are dealing with an *unbounded* domain, such as diffusion in $\mathbb{R}^3$? In this case, Theorem 7.31 is not applicable, and in general, it may not be possible (or at least, not feasible) to represent initial/boundary conditions in terms of eigenfunctions. What alternative methods are available?
3. The eigenfunction method is difficult to connect to our physical intuitions. For example, intuitively, heat 'seaps' slowly through space, and temperature distributions gradually and irreversibly decay towards uniformity. It is thus impossible to send a long-distance 'signal' using heat. On the other hand, waves maintain their shape and propagate across great distances with a constant velocity; hence they can be used to send signals through space. These familiar intiutions are not explained or justified by the eigenfunction method. Is there an alternative solution method where these intuitions have a clear mathematical expression?

Part VI provides answers to these questions. In Chapter 15, we introduce a powerful and versatile technique called *separation of variables*, to construct eigenfunctions adapted to any coordinate system. In Chapter 16, we develop the entirely different solution technology of *impulse-response functions*, which allows you to solve differential equations on unbounded domains, and which has an an appealing intuitive interpretation.

15 Separation of Variables

15.1 Separation of variables in Cartesian coordinates on $\mathbb{R}^2$

Prerequisites: §2.2, §2.3

A function $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is said the **separate** if we can write $u(x, y) = X(x) \cdot Y(y)$ for some functions $X, Y : \mathbb{R} \longrightarrow \mathbb{R}$. If u is a solution to some partial differential equation, we call say u is a **separated solution**.

Example 15.1: *The Heat Equation on* $\mathbb{R}$

We wish to find $u(x, y)$ so that $\partial_t u = \partial_x^2 u$. Suppose $u(x; t) = X(x) \cdot T(t)$, where

$$X(x) = \exp(\mathbf{i}\mu x) \quad \text{and} \quad T(t) = \exp(-\mu^2 t).$$

Then $u(x; t) = \exp(\mu\mathbf{i}x - \mu^2 t)$, so that $\partial_x^2 u = -\mu^2 \cdot u = \partial_t u$. Thus, u is a separated solution to the Heat equation. ♢

Separation of variables is a strategy for for solving partial differential equations by specifically looking for separated solutions. At first, it seem like we are making our lives harder by insisting on a solution in separated form. However, often, we can use the hypothesis of separation to actually *simplify* the problem.

Suppose we are given some PDE for a function $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ of two variables. *Separation of variables* is the following strategy:

1. Hypothesize that u can be written as a product of two functions, $X(x)$ and $Y(y)$, each depending on only one coordinate; in other words, assume that

$$u(x, y) = X(x) \cdot Y(y) \tag{15.1}$$

2. When we evaluate the PDE on a function of type (15.1), we may find that the PDE decomposes into two separate, *ordinary* differential equations for each of the two functions X and Y. Thus, we can solve these ODEs independently, and combine the resulting solutions to get a solution for u.

Example 15.2: *Laplace's Equation in* $\mathbb{R}^2$

Suppose we want to find a function $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $\triangle u \equiv 0$. If $u(x, y) = X(x) \cdot Y(y)$, then

$$\triangle u = \partial_x^2 (X \cdot Y) + \partial_y^2 (X \cdot Y) = \left(\partial_x^2 X\right) \cdot Y + X \cdot \left(\partial_y^2 Y\right) = X'' \cdot Y + X \cdot Y'',$$

where we denote $X'' = \partial_x^2 X$ and $Y'' = \partial_y^2 Y$. Thus,

$$\triangle u(x, y) = X''(x) \cdot Y(y) + X(x) \cdot Y''(y) = \Big(X''(x) \cdot Y(y) + X(x) \cdot Y''(y)\Big) \frac{X(x)Y(y)}{X(x)Y(y)}$$
$$= \left(\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}\right) \cdot u(x, y).$$

Thus, dividing by $u(x,y)$, Laplace's equation is equivalent to:

$$0 \quad = \quad \frac{\triangle u(x,y)}{u(x,y)} \quad = \quad \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}.$$

This is a sum of two functions which depend on *different* variables. The only way the sum can be identically zero is if each of the component functions is constant:

$$\frac{X''}{X} \equiv \lambda, \qquad \frac{Y''}{Y} \equiv -\lambda$$

So, pick some **separation constant** λ, and then solve the two ordinary differential equations:

$$X''(x) = \lambda \cdot X(x) \qquad \text{and} \qquad Y''(y) = -\lambda \cdot Y(y) \tag{15.2}$$

The (real-valued) solutions to (15.2) depends on the sign of λ. Let $\mu = \sqrt{|\lambda|}$. Then the solutions of (15.2) have the form:

$$X(x) \quad = \quad \begin{cases} A\sinh(\mu x) + B\cosh(\mu x) & \text{if } \lambda > 0 \\ Ax + B & \text{if } \lambda = 0 \\ A\sin(\mu x) + B\cos(\mu x) & \text{if } \lambda < 0 \end{cases}$$

where A and B are arbitrary constants. Assuming $\lambda < 0$, and $\mu = \sqrt{|\lambda|}$, we get:

$$X(x) \;=\; A\sin(\mu x) + B\cos(\mu x) \qquad \text{and} \qquad Y(y) \;=\; C\sinh(\mu x) + D\cosh(\mu x).$$

This yields the following separated solution to Laplace's equation:

$$u(x,y) \;=\; X(x)\cdot Y(y) \;=\; \Big(A\sin(\mu x) + B\cos(\mu x)\Big)\cdot\Big(C\sinh(\mu x) + D\cosh(\mu x)\Big) \tag{15.3}$$

Alternately, we could consider the general *complex* solution to (15.2), given by:

$$X(x) \;=\; \exp\left(\sqrt{\lambda}\cdot x\right),$$

where $\sqrt{\lambda} \in \mathbb{C}$ is some complex number. For example, if $\lambda < 0$ and $\mu = \sqrt{|\lambda|}$, then $\sqrt{\lambda} = \pm\mu\mathbf{i}$ are imaginary, and

$$X_1(x) \;=\; \exp(\mathbf{i}\mu x) \;=\; \cos(\mu x) + \mathbf{i}\sin(\mu x) \qquad \text{and} \qquad X_2(x) \;=\; \exp(-\mathbf{i}\mu x) \;=\; \cos(\mu x) - \mathbf{i}\sin(\mu x)$$

are two solutions to (15.2). The general solution is then given by:

$$X(x) \;=\; a\cdot X_1(x) + b\cdot X_2(x) \;=\; (a+b)\cdot\cos(\mu x) \;+\; \mathbf{i}\cdot(a-b)\cdot\sin(\mu x).$$

Meanwhile, the general form for $Y(y)$ is

$$Y(y) \;=\; c\cdot\exp(\mu y) + d\cdot\exp(-\mu y) \;=\; (c+d)\cosh(\mu y) + (c-d)\sinh(\mu y)$$

The corresponding separated solution to Laplace's equation is:

$$u(x,y) \;=\; X(x)\cdot Y(y) \;=\; \Big(A\sin(\mu x) + B\mathbf{i}\cos(\mu x)\Big)\cdot\Big(C\sinh(\mu x) + D\cosh(\mu x)\Big) \tag{15.4}$$

where $A = (a+b)$, $B = (a-b)$, $C = (c+d)$, and $D = (c-d)$. In this case, we just recover solution (15.3). However, we could also construct separated solutions where $\lambda \in \mathbb{C}$ is an arbitrary complex number, and $\sqrt{\lambda}$ is one of its square roots. ♢

15.2 Separation of variables in Cartesian coordinates on $\mathbb{R}^D$

Recommended: §15.1

Given some PDE for a function $u : \mathbb{R}^D \longrightarrow \mathbb{R}$, we apply the strategy of *separation of variables* as follows:

1. Hypothesize that u can be written as a product of D functions, each depending on only one coordinate; in other words, assume that

$$u(x_1, \ldots, x_D) \;=\; u_1(x_1) \cdot u_2(x_2) \ldots u_D(x_D) \tag{15.5}$$

2. When we evaluate the PDE on a function of type (15.5), we may find that the PDE decomposes into D separate, *ordinary* differential equations for each of the D functions $u_1, \ldots, u_D$. Thus, we can solve these ODEs independently, and combine the resulting solutions to get a solution for u.

Example 15.3: *Laplace's Equation in* $\mathbb{R}^D$*:*

Suppose we want to find a function $u : \mathbb{R}^D \longrightarrow \mathbb{R}$ such that $\triangle u \equiv 0$. As in the two-dimensional case (Example 15.2), we reason:

$$\text{If } u(\mathbf{x}) \;=\; X_1(x_1) \cdot X_2(x_2) \ldots X_D(x_D), \qquad \text{then } \triangle u \;=\; \left(\frac{X_1''}{X_1} + \frac{X_2''}{X_2} + \ldots + \frac{X_D''}{X_D} \right) \cdot u.$$

Thus, Laplace's equation is equivalent to:

$$0 \;=\; \frac{\triangle u}{u}(\mathbf{x}) \;=\; \frac{X_1''}{X_1}(x_1) \;+\; \frac{X_2''}{X_2}(x_2) \;+\; \ldots \;+\; \frac{X_D''}{x_D}(x_D).$$

This is a sum of D distinct functions, each of which depends on a different variable. The only way the sum can be identically zero is if each of the component functions is constant:

$$\frac{X_1''}{X_1} \equiv \lambda_1, \qquad \frac{X_2''}{X_2} \equiv \lambda_2, \qquad \ldots, \qquad \frac{X_D''}{X_D} \equiv \lambda_D, \tag{15.6}$$

such that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_D = 0. \tag{15.7}$$

So, pick some **separation constant** $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_D) \in \mathbb{R}^D$ satisfying (15.7), and then solve the ODEs:

$$X_d'' = \lambda_d \cdot X_d \qquad \text{for d=1,2,\ldots,D} \tag{15.8}$$

The (real-valued) solutions to (15.8) depends on the sign of λ (and clearly, if (15.7) is going to be true, either all λ_d are zero, or some are negative and some are positive). Let $\mu = \sqrt{|\lambda|}$. Then the solutions of (15.8) have the form:

$$X(x) \;=\; \begin{cases} A\exp(\mu x) + B\exp(-\mu x) & \text{if } \lambda > 0 \\ Ax + B & \text{if } \lambda = 0 \\ A\sin(\mu x) + B\cos(\mu x) & \text{if } \lambda < 0 \end{cases}$$

where A and B are arbitrary constants. We then combine these as in Example 15.2. ♢

15.3 Separation in polar coordinates: Bessel's Equation

Prerequisites: §1.6(b), §2.3 **Recommended:** §14.3, §15.1

In §14.3-§14.6, we explained how to use solutions of Bessel's equation to solve the Heat Equation or Wave equation in polar coordinates. In this section, we will see how Bessel derived his equation in the first place: it arises naturally when one uses 'separation of variables' to find eigenfunctions of the Laplacian in polar coordinates. First, a technical lemma from the theory of ordinary differential equations:

Lemma 15.4: *Let $\Theta : [-\pi, \pi] \longrightarrow \mathbb{R}$ be a function satisfying periodic boundary conditions [ie. $\Theta(-\pi) = \Theta(\pi)$ and $\Theta'(-\pi) = \Theta'(\pi)$]. Let $\mu > 0$ be some constant, and suppose Θ satisfies the linear ordinary differential equation:*

$$\Theta''(\theta) \ = \ -\mu \cdot \Theta(\theta), \qquad \text{for all } \theta \in [-\pi, \pi]. \tag{15.9}$$

Then $\mu = m^2$ for some $m \in \mathbb{N}$, and Θ must be a function of the form:

$$\Theta(\theta) \quad = \quad A\cos(m\theta) + B\sin(m\theta), \qquad \text{(for constants } A, B \in \mathbb{C}.\text{)}$$

Proof: Eqn.(15.9) is a second-order linear ODE, so the set of all solutions to eqn.(15.9) is a two-dimensional vector space. This vector space is spanned by functions of the form $\Theta(\theta) = e^{r\theta}$, where r is any root of the characteristic polynomial $p(x) = x^2 + \mu$. The two roots of this polynomial are of course $r = \pm\sqrt{\mu}\mathbf{i}$. Let $m = \sqrt{\mu}$ (it will turn out that m is an integer, although we don't know this yet). Hence the general solution to (15.9) is

$$\Theta(\theta) \ = \ C_1 e^{m\mathbf{i}\theta} \ + \ C_2 e^{-m\mathbf{i}\theta},$$

where C_1 and C_2 are any two constants. The periodic boundary conditions mean that

$$\Theta(-\pi) = \Theta(\pi) \qquad \text{and} \qquad \Theta'(-\pi) = \Theta'(\pi),$$

which means

$$C_1 e^{-m\mathbf{i}\pi} \ + \ C_2 e^{m\mathbf{i}\pi} \ = \ C_1 e^{m\mathbf{i}\pi} \ + \ C_2 e^{-m\mathbf{i}\pi}, \tag{15.10}$$

$$\text{and } \ m\mathbf{i}C_1 e^{-m\mathbf{i}\pi} \ - \ m\mathbf{i}C_2 e^{m\mathbf{i}\pi} \ = \ m\mathbf{i}C_1 e^{m\mathbf{i}\pi} \ - \ m\mathbf{i}C_2 e^{-m\mathbf{i}\pi}. \tag{15.11}$$

If we divide both sides of the eqn.(15.11) by $m\mathbf{i}$, we get

$$C_1 e^{-m\mathbf{i}\pi} \ - \ C_2 e^{m\mathbf{i}\pi} \quad = \quad C_1 e^{m\mathbf{i}\pi} \ - \ C_2 e^{-m\mathbf{i}\pi}.$$

If we add this to eqn.(15.10), we get

$$2C_1 e^{-m\mathbf{i}\pi} \quad = \quad 2C_1 e^{m\mathbf{i}\pi},$$

which is equivalent to $e^{2m\mathbf{i}\pi} = 1$. Hence, m must be some integer, and $\mu = m^2$.

Now, let $A := C_1 + C_2$ and $B' := C_1 - C_2$. Then $C_1 = \frac{1}{2}(A+B')$ and $C_2 = \frac{1}{2}(A-B')$. Thus,

$$\begin{aligned}\Theta(\theta) &= C_1 e^{m\mathbf{i}\theta} + C_2 e^{-m\mathbf{i}\theta} = (A+B')e^{m\mathbf{i}\theta} + (A-B')e^{-m\mathbf{i}\theta} \\ &= \frac{A}{2}\left(e^{m\mathbf{i}\theta} + e^{-m\mathbf{i}\theta}\right) + \frac{B'\mathbf{i}}{2\mathbf{i}}\left(e^{m\mathbf{i}\theta} - e^{-m\mathbf{i}\theta}\right) = A\cos(m\theta) + B'\mathbf{i}\sin(m\theta)\end{aligned}$$

because of the Euler formulas: $\cos(x) = \frac{1}{2}(e^{\mathbf{i}x} + e^{-\mathbf{i}x})$ and $\sin(x) = \frac{1}{2\mathbf{i}}(e^{\mathbf{i}x} - e^{-\mathbf{i}x})$.

Now let $B = B'\mathbf{i}$; then $\Theta(\theta) = A\cos(m\theta) + B\sin(m\theta)$, as desired. □

Proposition 15.5: *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be an eigenfunction of the Laplacian* [ie. $\triangle f = -\lambda^2 \cdot f$ for some constant $\lambda \in \mathbb{R}$]. *Suppose f separates in polar coordinates, meaning that there is a function $\Theta : [-\pi, \pi] \longrightarrow \mathbb{R}$* (satisfying periodic boundary conditions) *and a function $\mathcal{R} : [0, \infty) \longrightarrow \mathbb{R}$ such that*

$$f(r,\theta) = \mathcal{R}(r) \cdot \Theta(\theta), \quad \text{for all } r \geq 0 \text{ and } \theta \in [-\pi, \pi].$$

Then there is some $m \in \mathbb{N}$ so that

$$\Theta(\theta) = A\cos(m\theta) + B\sin(m\theta), \qquad \text{(for constants } A, B \in \mathbb{R}.)$$

and $\mathcal{R}$ is a solution to the (mth order) **Bessel Equation:**

$$r^2\mathcal{R}''(r) + r \cdot \mathcal{R}'(r) + (\lambda^2 r^2 - m^2) \cdot \mathcal{R}(r) = 0, \quad \text{for all } r > 0. \tag{15.12}$$

Proof: Recall that, in polar coordinates, $\triangle f = \partial_r^2 f + \frac{1}{r}\partial_r f + \frac{1}{r^2}\partial_\theta^2 f$. Thus, if $f(r,\theta) = \mathcal{R}(r) \cdot \Theta(\theta)$, then the eigenvector equation $\triangle f = -\lambda^2 \cdot f$ becomes

$$\begin{aligned}-\lambda^2 \cdot \mathcal{R}(r) \cdot \Theta(\theta) &= \triangle \mathcal{R}(r) \cdot \Theta(\theta) \\ &= \partial_r^2 \mathcal{R}(r) \cdot \Theta(\theta) + \frac{1}{r}\partial_r \mathcal{R}(r) \cdot \Theta(\theta) + \frac{1}{r^2}\partial_\theta^2 \mathcal{R}(r) \cdot \Theta(\theta) \\ &= \mathcal{R}''(r)\Theta(\theta) + \frac{1}{r}\mathcal{R}'(r)\Theta(\theta) + \frac{1}{r^2}\mathcal{R}(r)\Theta''(\theta),\end{aligned}$$

which is equivalent to

$$\begin{aligned}-\lambda^2 &= \frac{\mathcal{R}''(r)\Theta(\theta) + \frac{1}{r}\mathcal{R}'(r)\Theta(\theta) + \frac{1}{r^2}\mathcal{R}(r)\Theta''(\theta)}{\mathcal{R}(r) \cdot \Theta(\theta)} \\ &= \frac{\mathcal{R}''(r)}{\mathcal{R}(r)} + \frac{\mathcal{R}'(r)}{r\mathcal{R}(r)} + \frac{\Theta''(\theta)}{r^2\Theta(\theta)},\end{aligned} \tag{15.13}$$

If we multiply both sides of (15.13) by r^2 and isolate the Θ'' term, we get:

$$-\lambda^2 r^2 - \frac{r^2 \mathcal{R}''(r)}{\mathcal{R}(r)} + \frac{r\mathcal{R}'(r)}{\mathcal{R}(r)} = \frac{\Theta''(\theta)}{\Theta(\theta)}. \tag{15.14}$$

Abstractly, equation (15.14) has the form: $F(r) = G(\theta)$, where F is a function depending only on r and G is a function depending only on θ. The only way this can be true is if there is some constant $\mu \in \mathbb{R}$ so that $F(r) = -\mu$ for all $r > 0$ and $G(\theta) = -\mu$ for all $\theta \in [-\pi, \pi)$. In other words,

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -\mu, \quad \text{for all } \theta \in [-\pi, \pi), \tag{15.15}$$

$$\text{and } \lambda^2 r^2 + \frac{r^2 \mathcal{R}''(r)}{\mathcal{R}(r)} + \frac{r\mathcal{R}'(r)}{\mathcal{R}(r)} = \mu, \quad \text{for all } r \geq 0, \tag{15.16}$$

Multiply both sides of equation (15.15) by $\Theta^2(\theta)$ to get:

$$\Theta''(\theta) = -\mu \cdot \Theta(\theta), \quad \text{for all } \theta \in [-\pi, \pi). \tag{15.17}$$

Multiply both sides of equation (15.16) by $\mathcal{R}^2(r)$ to get:

$$r^2\mathcal{R}''(r) + r \cdot \mathcal{R}'(r) + \lambda^2 r^2 \mathcal{R}(r) = \mu\mathcal{R}(r), \quad \text{for all } r > 0. \tag{15.18}$$

Apply Lemma 15.4 to to eqn.(15.17) to deduce that $\mu = m^2$ for some $m \in \mathbb{N}$, and that $\Theta(\theta) = A\cos(m\theta) + B\sin(m\theta)$. Substitute $\mu = m^2$ into eqn.(15.18) to get

$$r^2\mathcal{R}''(r) + r \cdot \mathcal{R}'(r) + \lambda^2 r^2 \mathcal{R}(r) = m^2\mathcal{R}(r),$$

Now subtract $m^2\mathcal{R}(r)$ from both sides to get Bessel's equation (15.12). □

15.4 Separation in spherical coordinates: Legendre's Equation

Prerequisites: §1.6(d), §2.3, §6.5(a), §7.5 **Recommended:** §15.3

Recall that *spherical coordinates* (r, θ, ϕ) on $\mathbb{R}^3$ are defined by the transformation:

$$x = r \cdot \sin(\phi) \cdot \cos(\theta), \qquad y = r \cdot \sin(\phi) \cdot \sin(\theta) \qquad \text{and} \qquad z = r \cdot \cos(\phi).$$

where $r \in [0, \infty)$, $\theta \in [-\pi, \pi)$, and $\phi \in [0, \pi]$. The reverse transformation is defined:

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \arctan\left(\frac{y}{x}\right) \qquad \text{and } \phi = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right).$$

Adrien-Marie Legendre
Born: September 18, 1752 in Paris
Died: January 10, 1833 in Paris

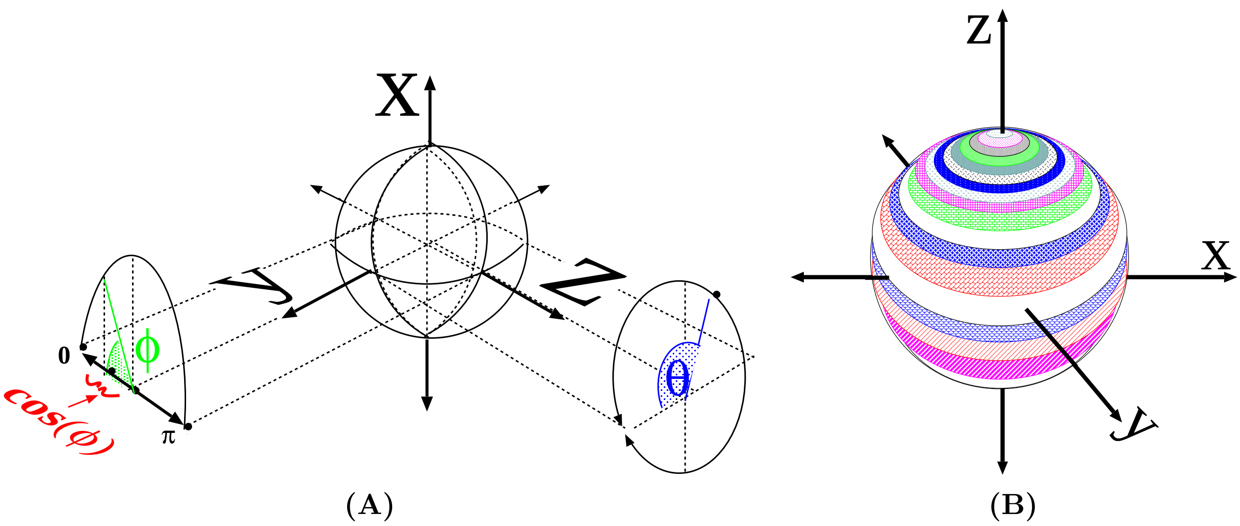

Figure 15.1: **(A)** Spherical coordinates. **(B)** Zonal functions.

[See Figure 15.1(A)]. Geometrically, r is the radial distance from the origin. If we fix $r = 1$, then we get a sphere of radius 1. On the surface of this sphere, θ is *longitude* and ϕ is *latitude.* In terms of these coordinates, the Laplacian is written:

$$\triangle f(r,\theta,\phi) \quad = \quad \partial_r^2 f \ + \ \frac{2}{r}\partial_r f \ + \ \frac{1}{r^2\sin(\phi)}\partial_\phi^2 f \ + \ \frac{\cot(\phi)}{r^2}\partial_\phi f \ + \ \frac{1}{r^2\sin(\phi)^2}\partial_\theta^2 f.$$

(**Exercise 15.1**)

A function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is called **zonal** if $f(r,\theta,\phi)$ depends only on on r and ϕ –in other words, $f(r,\theta,\phi) = F(r,\phi)$, where $F : [0,\infty) \times [0,\pi] \longrightarrow \mathbb{R}$ is some other function. If we restrict f to the aforementioned sphere of radius 1, then f is invariant under rotations around the 'north-south axis' of the sphere. Thus, f is constant along lines of equal latitude around the sphere, so it divides the sphere into 'zones' from north to south [Figure 15.1(B)].

Proposition 15.6: *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be zonal. Suppose f is a harmonic function (ie. $\triangle f = 0$). Suppose f separates in spherical coordinates, meaning that there are (bounded) functions $\Phi : [0,\pi] \longrightarrow \mathbb{R}$ and $\mathcal{R} : [0,\infty) \longrightarrow \mathbb{R}$ such that*

$$f(r,\theta,\phi) \quad = \quad \mathcal{R}(r)\cdot\Phi(\phi), \qquad \text{for all } r \geq 0,\ \ \phi \in [0,\pi], \text{ and } \theta \in [-\pi,\pi].$$

Then there is some $\mu \in \mathbb{R}$ so that $\Phi(\phi) = \mathcal{L}[\cos(\phi)]$, where $\mathcal{L} : [-1,1] \longrightarrow \mathbb{R}$ is a (bounded) solution of the **Legendre Equation:**

$$(1-x^2)\mathcal{L}''(x) - 2x\mathcal{L}'(x) + \mu\mathcal{L}(x) \quad = \quad 0 \qquad (15.19)$$

and $\mathcal{R}$ is a (bounded) solution to the **Cauchy-Euler Equation:**

$$r^2\mathcal{R}''(r) + 2r\cdot\mathcal{R}'(r) - \mu\cdot\mathcal{R}(r) \quad = \quad 0, \quad \text{for all } r > 0. \qquad (15.20)$$

Proof: By hypothesis

$$\begin{aligned} 0 \ = \ \triangle f(r,\theta,\phi) \quad &= \quad \partial_r^2 f \ + \ \frac{2}{r}\partial_r f \ + \ \frac{1}{r^2\sin(\phi)}\partial_\phi^2 f \ + \ \frac{\cot(\phi)}{r^2}\partial_\phi f \ + \ \frac{1}{r^2\sin(\phi)^2}\partial_\theta^2 f \\ \underset{(*)}{=} \quad & \mathcal{R}''(r)\cdot\Phi(\phi) \ + \ \frac{2}{r}\mathcal{R}'(r)\cdot\Phi(\phi) \ + \ \frac{1}{r^2\sin(\phi)}\mathcal{R}(r)\cdot\Phi''(\phi) \ + \ \frac{\cot(\phi)}{r^2}\mathcal{R}(r)\cdot\Phi'(\phi) \ + \ 0. \end{aligned}$$

[where $(*)$ is because $f(r,\theta,\phi) = \mathcal{R}(r)\cdot\Phi(\phi)$.] Hence, multiplying both sides by $\dfrac{r^2}{\mathcal{R}(r)\cdot\Phi(\phi)}$, we obtain

$$0 \quad = \quad \frac{r^2\mathcal{R}''(r)}{\mathcal{R}(r)} \ + \ \frac{2r\mathcal{R}'(r)}{\mathcal{R}(r)} \ + \ \frac{1}{\sin(\phi)}\frac{\Phi''(\phi)}{\Phi(\phi)} \ + \ \frac{\cot(\phi)\Phi'(\phi)}{\Phi(\phi)},$$

Or, equivalently,

$$\frac{r^2\mathcal{R}''(r)}{\mathcal{R}(r)} \ + \ \frac{2r\mathcal{R}'(r)}{\mathcal{R}(r)} \quad = \quad \frac{-1}{\sin(\phi)}\frac{\Phi''(\phi)}{\Phi(\phi)} \ - \ \frac{\cot(\phi)\Phi'(\phi)}{\Phi(\phi)}. \qquad (15.21)$$

Now, the left hand side of (15.21) depends only on the variable r, whereas the right hand side depends only on ϕ. The only way that these two expressions can be equal for *all* values of r and ϕ is if both expressions are constants. In other words, there is some constant $\mu \in \mathbb{R}$ (called a *separation constant*) such that

$$\begin{aligned} \frac{r^2\mathcal{R}''(r)}{\mathcal{R}(r)} + \frac{2r\mathcal{R}'(r)}{\mathcal{R}(r)} &= \mu, \qquad \text{for all } r \geq 0, \\ \text{and} \quad \frac{1}{\sin(\phi)}\frac{\Phi''(\phi)}{\Phi(\phi)} + \frac{\cot(\phi)\Phi'(\phi)}{\Phi(\phi)} &= -\mu, \qquad \text{for all } \phi \in [0,\pi]. \end{aligned}$$

Or, equivalently,

$$\begin{aligned} r^2\mathcal{R}''(r) + 2r\mathcal{R}'(r) &= \mu\mathcal{R}(r), \qquad \text{for all } r \geq 0, && (15.22) \\ \text{and} \quad \frac{\Phi''(\phi)}{\sin(\phi)} + \cot(\phi)\Phi'(\phi) &= -\mu\Phi(\phi), \qquad \text{for all } \phi \in [0,\pi]. && (15.23) \end{aligned}$$

If we make the change of variables $x = \cos(\phi)$ (so that $\phi = \arccos(x)$, where $x \in [-1,1]$), then $\Phi(\phi) = \mathcal{L}(\cos(\phi)) = \mathcal{L}(x)$, where $\mathcal{L}$ is some other (unknown) function.

Claim 1: *The function Φ satisfies the ODE* (15.23) *if and only if $\mathcal{L}$ satisfies the Legendre equation* (15.19).

Proof: <u>**Exercise 15.2**</u> (Hint: This is a straightforward application of the Chain Rule.) $\diamond_{\text{Claim 1}}$

Finally, observe that that the ODE (15.22) is equivalent to the Cauchy-Euler equation (15.20). □

For all $n \in \mathbb{N}$, we define the nth **Legendre Polynomial** by

$$\mathcal{P}_n(x) \quad := \quad \frac{1}{n!\,2^n}\partial_x^n\left[(x^2-1)\right]^n. \qquad (15.24)$$

For example:

$$\begin{aligned} \mathcal{P}_0(x) &= 1 & \mathcal{P}_3(x) &= \tfrac{1}{2}(5x^3 - 3x) \\ \mathcal{P}_1(x) &= x & \mathcal{P}_4(x) &= \tfrac{1}{8}(35x^4 - 30x^2 + 3) \qquad \text{(see Figure 15.2)} \\ \mathcal{P}_2(x) &= \tfrac{1}{2}(3x^2 - 1) & \mathcal{P}_5(x) &= \tfrac{1}{8}(63x^5 - 70x^3 + 15x). \end{aligned}$$

Lemma 15.7: *Let $n \in \mathbb{N}$. Then the Legendre Polynomial $\mathcal{P}_n$ is a solution to the Legendre Equation* (15.19) *with $\mu = n(n+1)$.*

Proof: <u>**Exercise 15.3**</u> (Direct computation) □

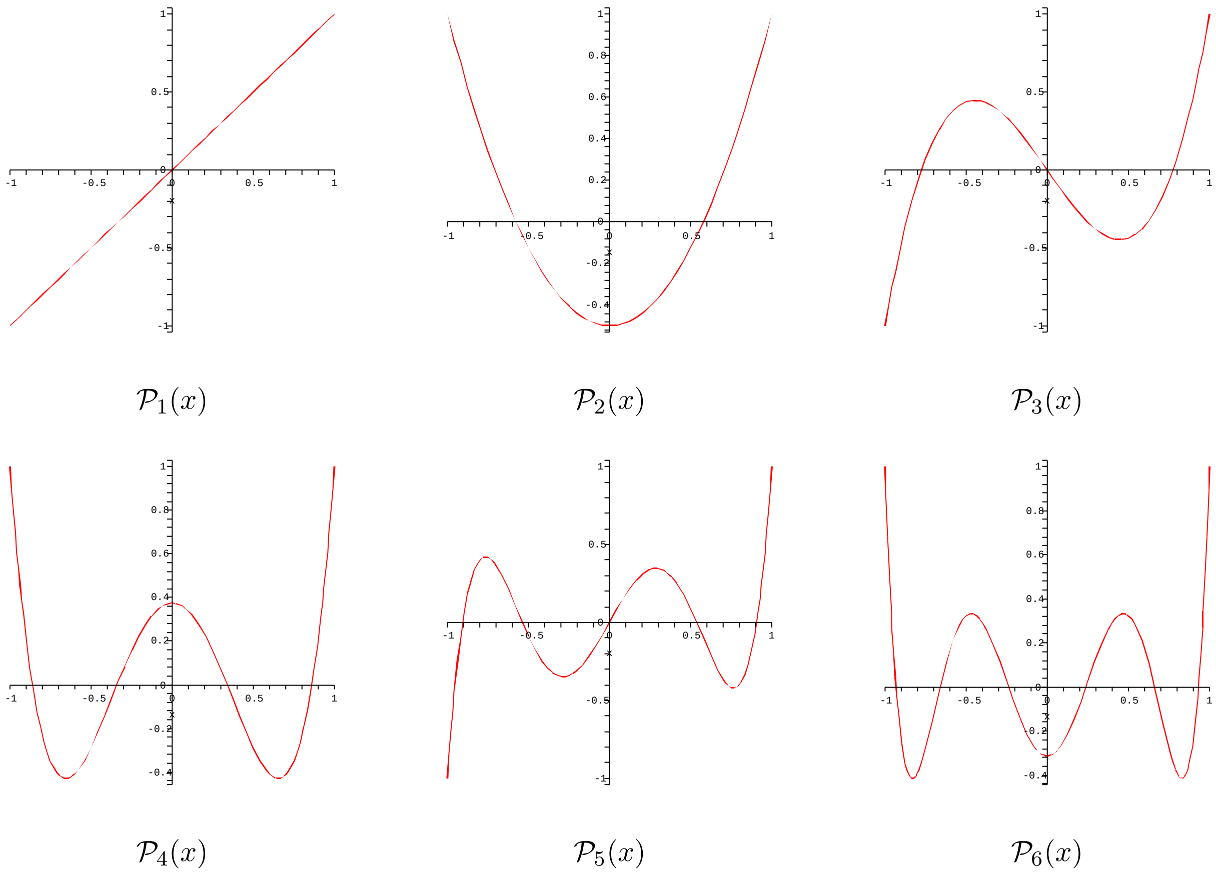

Figure 15.2: The Legendre polynomials $\mathcal{P}_1(x)$ to $\mathcal{P}_6(x)$, plotted for $x \in [-1, 1]$.

Is $\mathcal{P}_n$ the *only* solution to the Legendre Equation (15.19)? No, because the Legendre Equation is an order-two linear ODE, so the set of solutions forms a two-dimensional vector space $\mathcal{V}$. The scalar multiples of $\mathcal{P}_n$ form a one-dimensional subspace of $\mathcal{V}$. However, recall that, to be physically meaningful, we need the solutions to be bounded at $x = \pm 1$. So instead we ask: is $\mathcal{P}_n$ the only *bounded* solution to the Legendre Equation (15.19)? Also, what happens if $\mu \neq n(n+1)$ for any $n \in \mathbb{N}$?

Lemma 15.8:

(a) *If* $\mu = n(n+1)$ *for some* $n \in \mathbb{N}$, *then* (up to multiplication by a scalar), *the Legendre polynomial* $\mathcal{P}_n(x)$ *is the unique solution to the Legendre Equation* (15.19) *which is bounded on* $[-1, 1]$.

(b) *If* $\mu \neq n(n+1)$ *for any* $n \in \mathbb{N}$, *then all solutions to the Legendre Equation* (15.19) *are infinite power series which diverge at* $x = \pm 1$ (and thus, are unsuitable for Proposition 15.6).

Proof: We apply the *Method of Frobenius.* Suppose $\mathcal{L}(x) = \sum_{n=0}^{\infty} a_n x^n$ is some analytic function defined on $[-1, 1]$ (where the coefficients $\{a_n\}_{n=1}^{\infty}$ are as yet unknown).

Claim 1: $\mathcal{L}(x)$ *satisfies the Legendre Equation* (15.19) *if and only if the coefficients* $\{a_0, a_1, a_2, \ldots\}$ *satisfy the recurrence relation*

$$a_{k+2} \quad = \quad \frac{k(k+1) - \mu}{(k+2)(k+1)} \, a_k, \qquad \text{for all } k \in \mathbb{N}. \tag{15.25}$$

In particular, $a_2 = \dfrac{-\mu}{2} a_0$ *and* $a_3 = \dfrac{2-\mu}{6} a_1$.

Proof: We will substitute the power series $\sum_{n=0}^{\infty} a_n x^n$ into the Legendre Equation (15.19).

If

$$\begin{array}{rcrcrcrcrc}
\mathcal{L}(x) & = & a_0 & + & a_1 x & + & a_2 x^2 & +\cdots+ & a_k x^k & +\cdots \\
\text{Then} & & & & & & & & & \\
\mu\mathcal{L}(x) & = & \mu a_0 & + & \mu a_1 x & + & \mu a_2 x^2 & +\cdots+ & \mu a_k x^k & +\cdots \\
-2x\mathcal{L}'(x) & = & & - & 2a_1 x & - & 4a_2 x^2 & -\cdots- & 2k a_k x^k & +\cdots \\
\mathcal{L}''(x) & = & 2a_2 & + & 6a_3 x & + & 12 a_4 x^2 & +\cdots+ & (k+2)(k+1)a_{k+2} x^k & +\cdots \\
-x^2\mathcal{L}''(x) & = & & & & - & 2a_2 x^2 & -\cdots- & (k-1)k a_k x^k & +\cdots \\
\text{Thus } 0 & = & \multicolumn{8}{l}{(1-x^2)\mathcal{L}''(x) - 2x\mathcal{L}'(x) + \mu\mathcal{L}(x)} \\
 & = & \begin{pmatrix} \mu a_0 \\ -2a_2 \end{pmatrix} & + & \begin{pmatrix} (\mu-2)a_1 \\ +6a_3 \end{pmatrix} x & + & \begin{pmatrix} (\mu-6)a_2 \\ +12a_4 \end{pmatrix} x^2 & +\cdots+ & b_k x^k & +\cdots
\end{array}$$

where $b_k = (k+2)(k+1)a_{k+2} + \Big[\mu - k(k+1)\Big] a_k$ for all $k \in \mathbb{N}$. Since the last power series must equal zero, we conclude that $b_k = 0$ for all $k \in \mathbb{N}$; in other words, that

$$(k+2)(k+1)a_{k+2} + \Big[\mu - k(k+1)\Big] a_k \quad = \quad 0, \qquad \text{for all } k \in \mathbb{N}.$$

Rearranging this equation produces the desired recurrence relation (15.25). $\diamond_{\text{Claim 1}}$

The space of all solutions to the Legendre Equation (15.19) is a two-dimensional vector space, because the Legendre equation is a *linear* differential equation of order 2. We will now find a basis for this space. Recall that $\mathcal{L}$ is *even* if $\mathcal{L}(-x) = \mathcal{L}(x)$ for all $x \in [-1,1]$, and $\mathcal{L}$ is *odd* if $\mathcal{L}(-x) = -\mathcal{L}(x)$ for all $x \in [-1,1]$.

Claim 2: *There is a unique even analytic function $\mathcal{E}(x)$ and a unique odd analytic function $\mathcal{O}(x)$ which satisfy the Legendre Equation* (15.19)*, so that $\mathcal{E}(1) = 1 = \mathcal{O}(1)$, and so that any other solution $\mathcal{L}(x)$ can be written as a linear combination $\mathcal{L}(x) = a\,\mathcal{E}(x) + b\,\mathcal{O}(x)$, for some constants $a, b \in \mathbb{R}$.*

Proof: Claim 1 implies that the power series $\mathcal{L}(x) = \sum_{n=0}^{\infty} a_n x^n$ is entirely determined by the coefficients a_0 and a_1. To be precise, $\mathcal{L}(x) = \mathcal{E}(x) + \mathcal{O}(x)$, where $\mathcal{E}(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n}$ and $\mathcal{O}(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$ both satisfy the recurrence relation (15.25), and thus, are solutions to the Legendre Equation (15.19). $\diamond_{\text{Claim 2}}$

Claim 3: *Suppose $\mu = n(n+1)$ for some $n \in \mathbb{N}$. Then the Legendre equation* (15.19) *has a degree n polynomial of degree as one of its solutions. To be precise:*

(a) *If n is even, then then $a_k = 0$ for all even $k > n$. Hence, $\mathcal{E}(x)$ is a degree n polynomial.*

(b) *If n is odd, then then $a_k = 0$ for all odd $k > n$. Hence, $\mathcal{O}(x)$ is a degree n polynomial.*

Proof: **Exercise 15.4** $\diamond_{\text{Claim 3}}$

Thus, there is a one-dimensional space of *polynomial* solutions to the Legendre equation –namely all scalar multiples of $\mathcal{E}(x)$ (if n is even) or $\mathcal{O}(x)$ (if n is odd).

Claim 4: *If $\mu \neq n(n+1)$ for any $n \in \mathbb{N}$, the series $\mathcal{E}(x)$ and $\mathcal{O}(x)$ both diverge at $x = \pm 1$.*

Proof: **Exercise 15.5** (a) First note that that an infinite number of coefficients $\{a_n\}_{n=0}^{\infty}$ are nonzero.

(b) Show that $\lim_{n\to\infty} |a_n| = 1$.

(c) Conclude that the series $\mathcal{E}(x)$ and $\mathcal{O}(x)$ diverge when $x = \pm 1$. $\diamond_{\text{Claim 4}}$

So, there exist solutions to the Legendre equation (15.19) that are bounded on $[-1,1]$ if *and only if* $\mu = n(n+1)$ for some $n \in \mathbb{N}$, and in this case, the bounded solutions are all scalar multiples of a polynomial of degree n [either $\mathcal{E}(x)$ or $\mathcal{O}(x)$]. But Lemma 15.7 says that the Legendre polynomial $\mathcal{P}_n(x)$ is a solution to the Legendre equation (15.19). Thus, (up to multiplication by a constant), $\mathcal{P}_n(x)$ must be equal to $\mathcal{E}(x)$ (if n is even) or $\mathcal{O}(x)$ (if n is odd). □

Remark: Sometimes the Legendre polynomials are *defined* as the (unique) polynomial solutions to Legendre's equation; the definition we have given in eqn.(15.24) is then *derived* from this definition, and is called *Rodrigues Formula*.

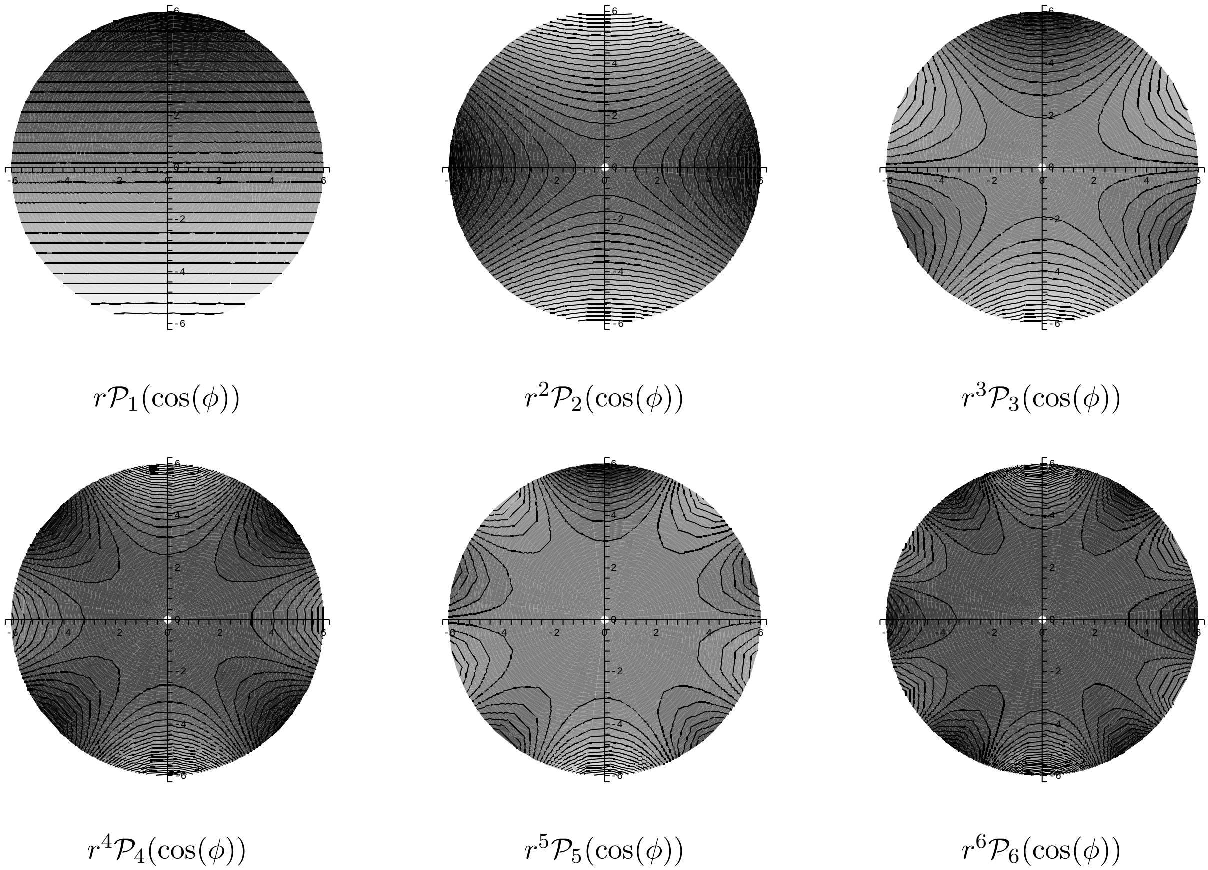

Figure 15.3: Planar cross-sections of the zonal harmonic functions $r\mathcal{P}_1(\cos(\phi))$ to $r^6\mathcal{P}_6(\cos(\phi))$, plotted for $r \in [0, 6]$; see Corollary 15.10. Remember that these are functions in $\mathbb{R}^3$. To visualize these functions in three dimensions, take the above contour plots and mentally rotate them around the vertical axis.

Lemma 15.9: *Let $\mathcal{R} : [0, \infty) \longrightarrow \mathbb{R}$ be a solution to the Cauchy-Euler equation*

$$r^2\mathcal{R}''(r) + 2r \cdot \mathcal{R}'(r) - n(n+1) \cdot \mathcal{R}(r) \quad = \quad 0, \quad \textit{for all } r > 0. \tag{15.26}$$

Then $\mathcal{R}(r) = Ar^n + \frac{B}{r^{n+1}}$ for some constants A and B.

If $\mathcal{R}$ is bounded at zero, then $B = 0$, so $\mathcal{R}(r) = Ar^n$.

Proof: Check that $f(r) = r^n$ and $g(r) = r^{-n-1}$ are solutions to eqn.(15.26). But (15.26) is a second-order linear ODE, so the solutions form a 2-dimensional vector space. Since f and g are linearly independent, they span this vector space. □

Corollary 15.10: *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a zonal harmonic function that separates in spherical coordinates* (as in Proposition 15.6). *Then there is some $m \in \mathbb{N}$ so that* $f(r, \phi, \theta) \; = \; Cr^n \cdot$

$\mathcal{P}_n[\cos(\phi)]$, *where* $\mathcal{P}_n$ *is the* n*th Legendre Polynomial, and* $C \in \mathbb{R}$ *is some constant.* (See Figure 15.3.)

Proof: Combine Proposition 15.6 with Lemmas 15.8 and 15.9 □

Thus, the Legendre polynomials are important when solving the Laplace equation on spherical domains. We now describe some of their important properties

Proposition 15.11: *Legendre polynomials satisfy the following recurrence relations:*

(a) $(2n+1)\mathcal{P}_n(x) \;=\; \mathcal{P}'_{n+1}(x) - \mathcal{P}'_{n-1}(x).$

(b) $(2n+1)x\mathcal{P}_n(x) \;=\; (n+1)\mathcal{P}_{n+1}(x) + n\mathcal{P}'_{n-1}(x).$

Proof: **Exercise 15.6** □

Proposition 15.12: *The Legendre polynomials form an orthogonal set for* $\mathbf{L}^2[-1,1]$. *That is:*

(a) *For any* $n \neq m$, $\langle \mathcal{P}_n, \mathcal{P}_m \rangle \;=\; \frac{1}{2}\int_{-1}^{1} \mathcal{P}_n(x)\mathcal{P}_m(x)\,dx \;=\; 0.$

(b) *For any* $n \in \mathbb{N}$, $\|\mathcal{P}_n\|_2^2 \;=\; \frac{1}{2}\int_{-1}^{1} \mathcal{P}_n^2(x)dx \;=\; \frac{1}{2n+1}.$

Proof: **(a) Exercise 15.7** (Hint: Start with the Rodrigues formula (15.24). Apply integration by parts n times.)

(b) Exercise 15.8 (Hint: Use Proposition 15.11(b).) □

Because of Proposition 15.12, we can try to represent an arbitrary function $f \in \mathbf{L}^2[-1,1]$ in terms of Legendre polynomials, to obtain a **Legendre Series:**

$$f(x) \;\underset{\text{L2}}{\approx}\; \sum_{n=0}^{\infty} a_n \mathcal{P}_n(x), \tag{15.27}$$

where $a_n \;:=\; \frac{\langle f, \mathcal{P}_n \rangle}{\|\mathcal{P}_n\|_2^2} \;=\; \frac{2n+1}{2}\int_{-1}^{1} f(x)\mathcal{P}_n(x)\,dx$ is the nth **Legendre coefficient** of f.

Theorem 15.13: *The Legendre polynomials form an* orthogonal basis *for* $\mathbf{L}^2[-1,1]$. *Thus, if* $f \in \mathbf{L}^2[-1,1]$, *then the Legendre series* (15.27) *converges to* f *in* $\mathbf{L}^2$.

Proof: See [Bro89, Thm 3.2.4, p.50] □

Let $\mathbb{B} = \{(r,\theta,\phi)\ ;\ r \leq 1,\ \theta \in [-\pi,\pi],\ \phi \in [0,\pi]\}$ be the unit ball in spherical coordinates. Thus, $\partial\mathbb{B} = \{(1,\theta,\phi)\ ;\ \theta \in [-\pi,\pi],\ \phi \in [0,\pi]\}$ is the unit sphere. Recall that a *zonal* function on $\partial\mathbb{B}$ is a function which depends only on the 'latitude' coordinate ϕ, and not on the 'longitude' coordinate θ.

Theorem 15.14: Dirichlet problem on a ball

Let $f : \partial\mathbb{B} \longrightarrow \mathbb{R}$ *be some function describing a heat distribution on the surface of the ball. Suppose* f *is zonal –ie.* $f(1,\theta,\phi) = F(\cos(\phi))$, *where* $F \in \mathbf{L}^2[-1,1]$, *and* F *has Legendre series*

$$F(x) \underset{\text{L2}}{\approx} \sum_{n=0}^{\infty} a_n \mathcal{P}_n(x).$$

Define $u : \mathbb{B} \longrightarrow \mathbb{R}$ *by* $u(r,\phi,\theta) = \sum_{n=0}^{\infty} a_n r^n \mathcal{P}_n(\cos(\phi))$. *Then* u *is the uniqe solution to the Laplace equation, satisfying the nonhomogeneous Dirichlet boundary conditions*

$$u(1,\theta,\phi) \underset{\text{L2}}{\approx} f(\theta,\phi), \qquad \text{for all } (1,\theta,\phi) \in \partial\mathbb{B}.$$

Proof: Exercise 15.9 □

15.5 Separated vs. Quasiseparated

Prerequisites: 15.2

If we use functions of type (15.4) as the components of the separated solution (15.5) we will still get mathematically valid solutions to Laplace's equation (as long as (15.7) is true). However, these solutions are not physically meaningful —what does a *complex*-valued heat distribution feel like? This is not a problem, because we can extract *real*-valued solutions from the complex solution as follows.

Proposition 15.15: *Suppose* $\mathcal{L}$ *is a linear differential operator with real-valued coefficients, and* $g : \mathbb{R}^D \longrightarrow \mathbb{R}$, *and consider the nonhomogeneous PDE "*$\mathcal{L}\, u = g$*".*

If $u : \mathbb{R}^D \longrightarrow \mathbb{C}$ *is a (complex-valued) solution to this PDE, and we define* $u_R(\mathbf{x}) = \mathbf{re}\,[u(\mathbf{x})]$ *and* $u_I(\mathbf{x}) = \mathbf{im}\,[u(\mathbf{x})]$, *then* $\mathcal{L}\, u_R = g$ *and* $\mathcal{L}\, u_I = 0$.

Proof: Exercise 15.10 □

In this case, the solutions u_R and u_I are not themselves generally going to be in separated form. Since they arise as the real and imaginary components of a complex separated solution, we call u_R and u_I **quasiseparated** solutions.

Example Recall the separated solutions to the two-dimensional Laplace equation from Example 15.2. Here, $\mathcal{L} = \triangle$ and $g \equiv 0$, and, for any fixed $\mu \in \mathbb{R}$, the function

$$u(x, y) \ = \ X(x) \cdot Y(y) \ = \ \exp(\mu y) \cdot \exp(\mu \mathbf{i} y)$$

is a complex solution to Laplace's equation. Thus,

$$u_R(x, y) \ = \ \exp(\mu x)\cos(\mu y) \qquad \text{and} \qquad u_I(x, y) \ = \ \exp(\mu x)\sin(\mu y)$$

are real-valued solutions of the form obtained earlier.

15.6 The Polynomial Formalism

Prerequisites: §15.2, §5.2

Separation of variables seems like a bit of a miracle. Just how generally applicable is it? To answer this, it is convenient to adopt a **polynomial formalism** for differential operators. If $\mathcal{L}$ is a differential operator with *constant*[1] coefficients, we will formally represent $\mathcal{L}$ as a "polynomial" in the "variables" $\partial_1, \partial_2, \ldots, \partial_D$. For example, we can write the Laplacian:

$$\triangle = \partial_1^2 + \partial_2^2 + \ldots + \partial_D^2 = \mathcal{P}(\partial_1, \partial_2, \ldots, \partial_D),$$

where $\mathcal{P}(x_1, x_2, \ldots, x_D) = x_1^2 + x_2^2 + \ldots + x_D^2$.

In another example, the general second-order linear PDE

$$A\partial_x^2 u \ + \ B\partial_x\partial_y u \ + \ C\partial_y^2 u \ + \ D\partial_x u \ + \ E\partial_y u \ + \ Fu \ = \ G$$

(where $A, B, C, \ldots, F$ are constants) can be rewritten:

$$\mathcal{P}(\partial_x, \partial_y)u \ = \ g$$

where $\mathcal{P}(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$.

The polynomial $\mathcal{P}$ is called the **polynomial symbol** of $\mathcal{L}$, and provides a convenient method for generating separated solutions

Proposition 15.16: *Suppose that $\mathcal{L}$ is a linear differential operator on $\mathbb{R}^D$ with polynomial symbol $\mathcal{P}$. Regard $\mathcal{P} : \mathbb{C}^D \longrightarrow \mathbb{C}$ as a function.*

If $\mathbf{z} = (z_1, \ldots, x_D) \in \mathbb{C}$, and $u_{\mathbf{z}} : \mathbb{R}^D \longrightarrow \mathbb{R}$ is defined

$$u_{\mathbf{z}}(x_1, \ldots, x_D) \ = \ \exp(z_1 x_1) \cdot \exp(z_2 x_2) \ldots \exp(z_D x_D) \ = \ \exp\langle \mathbf{z}, \mathbf{x} \rangle,$$

Then $\mathcal{L}u_{\mathbf{z}}(\mathbf{x}) \ = \ \mathcal{P}(\mathbf{z}) \cdot u_{\mathbf{z}}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^D$.

In particular, if $\mathbf{z}$ is a **root** *of $\mathcal{P}$ (that is, $\mathcal{P}(z_1, \ldots, z_D) = 0$), then $\mathcal{L}u = 0$.*

[1] This is important.

Proof: **Exercise 15.11** Hint: First, use formula (1.1) on page 9 to show that $\partial_d u_{\mathbf{z}} = z_d \cdot u_{\mathbf{z}}$, and, more generally, $\partial_d^n u_{\mathbf{z}} = z_d^n \cdot u_{\mathbf{z}}$. □

Thus, many[2] separated solutions of the *differential* equation "$\mathcal{L}u = 0$" are defined by the the complex-valued solutions of the *algebraic* equation "$\mathcal{P}(\mathbf{z}) = 0$".

Example 15.17: Consider again the two-dimensional Laplace equation

$$\partial_x^2 u + \partial_y^2 u = 0$$

The corresponding polynomial is $\mathcal{P}(x, y) = x^2 + y^2$. Thus, if $z_1, z_2 \in \mathbb{C}$ are any complex numbers so that $z_1^2 + z_2^2 = 0$, then

$$u(x, y) = \exp(z_1 x + z_2 y) = \exp(z_1 x) \cdot \exp(z_2 y)$$

is a solution to Laplace's equation. In particular, if $z_1 = 1$, then we must have $z_2 = \pm\mathbf{i}$. Say we pick $z_2 = \mathbf{i}$; then the solution becomes

$$u(x, y) \quad = \quad \exp(x) \cdot \exp(\mathbf{i}y) \quad = \quad e^{\mu x} \cdot \Big(\cos(y) + \mathbf{i}\sin(y)\Big).$$

More generally, if we choose $z_1 = \mu \in \mathbb{R}$ to be a real number, we must choose $z_2 = \pm\mu\mathbf{i}$ to be purely imaginary, and the solution becomes

$$u(x, y) \quad = \quad \exp(\mu x) \cdot \exp(\pm\mu\mathbf{i}y) \quad = \quad e^{\mu x} \cdot \Big(\cos(\pm\mu y) + \mathbf{i}\sin(\pm\mu y)\Big).$$

Compare this with the separated solutions obtained from Example 15.2 on page 278. ♢

Example 15.18: Consider the one-dimensional **Telegraph equation:**

$$\partial_t^2 u + 2\partial_t u + u = \triangle u \tag{15.28}$$

We can rewrite this as

$$\partial_t^2 u + 2\partial_t u + u - \partial_x^2 u = 0$$

which is equivalent to "$\mathcal{L}\,u = 0$", where $\mathcal{L}$ is the linear differential operator

$$\mathcal{L} \; = \; \partial_t^2 + 2\partial_t + u - \partial_x^2$$

with polynomial symbol

$$\mathcal{P}(x, t) \; = \; t^2 + 2t + 1 - x^2 \; = \; (t + 1 + x)(t + 1 - x)$$

Thus, the equation "$\mathcal{P}(\alpha, \beta) = 0$" has solutions:

$$\alpha = \pm(\beta + 1)$$

[2]But not all.

So, if we define $u(x,t) = \exp(\alpha \cdot x)\exp(\beta \cdot t)$, then u is a separated solution to equation (15.28). (**Exercise 15.12** Check this.). In particular, suppose we choose $\alpha = -\beta - 1$. Then the separated solution is $u(x,t) = \exp(\beta(t-x) - x)$. If $\beta = \beta_R + \beta_I \mathbf{i}$ is a complex number, then the quasiseparated solutions are:

$$\begin{aligned} u_R &= \exp\left(\beta_R(x+t) - x\right) \cdot \cos\left(\beta_I(x+t)\right) \\ u_I &= \exp\left(\beta_R(x+t) - x\right) \cdot \sin\left(\beta_I(x+t)\right) \quad . \end{aligned}$$ ♢

Remark: This provides part of the motivation for the classification of PDEs as *elliptic*, *hyperbolic*[3], etc. Notice that, if $\mathcal{L}$ is an elliptic differential operator on $\mathbb{R}^2$, then the real-valued solutions to $\mathcal{P}(z_1, z_2) = 0$ (if any) form an *ellipse* in $\mathbb{R}^2$. In $\mathbb{R}^D$, the solutions form an *ellipsoid*.

Similarly, if we consider the parabolic PDE "$\partial_t u = \mathcal{L}u$", the the corresponding differential operator $\mathcal{L} - \partial_t$ has polynomial symbol $\mathcal{Q}(\mathbf{x}; t) = \mathcal{P}(\mathbf{x}) - t$. The real-valued solutions to $\mathcal{Q}(\mathbf{x}; t) = 0$ form a *paraboloid* in $\mathbb{R}^D \times \mathbb{R}$. For example, the 1-dimensional Heat Equation "$\partial_x^2 u - \partial_t u = 0$" yields the classic equation "$t = x^2$" for a parabola in the (x,t)-plane. Similarly, with a hyperbolic PDE, the differential operator $\mathcal{L} - \partial_t^2$ has polynomial symbol $\mathcal{Q}(\mathbf{x}; t) = \mathcal{P}(\mathbf{x}) - t^2$, and the roots form a *hyperboloid*.

15.7 Constraints

Prerequisites: §15.6

Normally, we are not interested in just *any* solution to a PDE; we want a solution which satisfies certain constraints. The most common constraints are:

- **Boundary Conditions:** If the PDE is defined on some bounded domain $\mathbb{X} \subset \mathbb{R}^D$, then we may want the solution function u (or its derivatives) to have certain values on the boundary of this domain.

- **Boundedness:** If the domain $\mathbb{X}$ is unbounded (eg. $\mathbb{X} = \mathbb{R}^D$), then we may want the solution u to be *bounded*; in other words, we want some finite $M > 0$ so that $|u(\mathbf{x})| < M$ for all values of some coordinate x_d.

15.7(a) Boundedness

The solution obtained through Proposition 15.16 is not generally going to be bounded, because the exponential function $f(x) = \exp(\lambda x)$ is not bounded as a function of x, unless λ is a purely imaginary number. More generally:

Proposition 15.19: *If* $\mathbf{z} = (z_1, \ldots, x_D) \in \mathbb{C}$*, and* $u_{\mathbf{z}} : \mathbb{R}^D \longrightarrow \mathbb{R}$ *is defined as in Proposition 15.16:*

$$u_{\mathbf{z}}(x_1, \ldots, x_D) \;=\; \exp(z_1 x_1) \cdot \exp(z_2 x_2) \ldots \exp(z_D x_D) \;=\; \exp\langle \mathbf{z}, \mathbf{x} \rangle$$

then:

[3] See §6.2

1. *$u(\mathbf{x})$ is bounded for all values of the variable $x_d \in \mathbb{R}$ if and only if $z_d = \lambda\mathbf{i}$ for some $\lambda \in \mathbb{R}$.*

2. *$u(\mathbf{x})$ is bounded for all $x_d > 0$ if and only if $z_d = \rho + \lambda\mathbf{i}$ for some $\rho \leq 0$.*

3. *$u(\mathbf{x})$ is bounded for all $x_d < 0$ if and only if $z_d = \rho + \lambda\mathbf{i}$ for some $\rho \geq 0$.*

Proof: **Exercise 15.13** □

Example 15.20: Recall the one-dimensional Telegraph equation of Example 15.18:

$$\partial_t^2 u + 2\partial_t u + u = \triangle u$$

We constructed a separated solution of the form: $u(x,t) = \exp(\alpha x + \beta t)$, where $\alpha = \pm(\beta+1)$. This solution will be bounded in time if and only if β is a purely imaginary number; ie. $\beta = \beta_I \cdot \mathbf{i}$. Then $\alpha = \pm(\beta_I \cdot \mathbf{i} + 1)$, so that $u(x,t) = \exp(\pm x) \cdot \exp\Big(\beta_I \cdot (t \pm x) \cdot \mathbf{i}\Big)$; thus, the quasiseparated solutions are:

$$u_R = \exp(\pm x) \cdot \cos\Big(\beta_I \cdot (t \pm x)\Big) \qquad \text{and} \qquad u_I = \exp(\pm x) \cdot \sin\Big(\beta_I \cdot (t \pm x)\Big).$$

Unfortunately, this solution is *unbounded* in *space*, which is probably not what we want. An alternative is to set $\beta = \beta_I\mathbf{i} - 1$, and then set $\alpha = \beta + 1 = \beta_I\mathbf{i}$. Then the solution becomes $u(x,t) = \exp(\beta_I\mathbf{i}(x+t) - t) = e^{-t}\exp(\beta_I\mathbf{i}(x+t))$, and the quasiseparated solutions are:

$$u_R = e^{-t} \cdot \cos(\beta_I(x+t)) \qquad \text{and} \qquad u_I = e^{-t} \cdot \sin(\beta_I(x+t)).$$

These solutions are exponentially decaying as $t \to \infty$, and thus, bounded in "forwards time". For any fixed time t, they are also bounded (and actually periodic) functions of the space variable x. They are exponentially growing as $t \to -\infty$, but if we insist that $t > 0$, this isn't a problem. ♢

15.7(b) Boundary Conditions

Prerequisites: §6.5

There is no cureall like Proposition 15.19 for satisfying boundary conditions, since generally they are different in each problem. Generally, a single separated solution (say, from Proposition 15.18) will *not* be able to satisfy the conditions; what we need to do is sum together several solutions, so that they "cancel out" in suitable ways along the boundaries. For these purposes, the following *de Moivre identities* are often useful:

$$\sin(x) = \frac{e^{x\mathbf{i}} - e^{-x\mathbf{i}}}{2\mathbf{i}}; \qquad \cos(x) = \frac{e^{x\mathbf{i}} + e^{-x\mathbf{i}}}{2\mathbf{i}};$$
$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}; \qquad \cosh(x) = \frac{e^{x} + e^{-x}}{2}.$$

which we can utilize along with the following boundary information:

$$\begin{array}{rcccll} -\cos'(n\pi) & = & \sin(n\pi) & = & 0, & \text{for all } n \in \mathbb{Z}; \\ \sin'\left(\left(n+\frac{1}{2}\right)\pi\right) & = & \cos\left(\left(n+\frac{1}{2}\right)\pi\right) & = & 0, & \text{for all } n \in \mathbb{Z}; \\ \cosh'(0) & = & \sinh(0) & = & 0. & \end{array}$$

For "rectangular" domains, the boundaries are obtained by fixing a particular coordinate at a particular value; ie. they are each of the form form $\left\{\mathbf{x} \in \mathbb{R}^D \; ; \; x_d = K\right\}$ for some constant K and some dimension d. The convenient thing about a separated solution is that it is a product of D functions, and only *one* of them is involved in satisfying this boundary condition.

For example, recall Example 15.17 on page 294, which gave the separated solution $u(x,y) = e^{\mu x} \cdot \Big(\cos(\pm\mu y) + \mathbf{i}\sin(\pm\mu y)\Big)$ for the two-dimensional Laplace equation, where $\mu \in \mathbb{R}$. Suppose we want the solution to satisfy *homogeneous Dirichlet boundary conditions*:

$$u(x,y) = 0 \quad \text{if} \quad x = 0, \; y = 0, \quad \text{or} \quad y = \pi.$$

To satisfy these three conditions, we proceed as follows:

$$\begin{array}{rcl} \text{First, let } \; u_1(x,y) & = & e^{\mu x} \cdot \Big(\cos(\mu y) + \mathbf{i}\sin(\mu y)\Big), \\ \text{and } \; u_2(x,y) & = & e^{\mu x} \cdot \Big(\cos(-\mu y) + \mathbf{i}\sin(-\mu y)\Big) \;=\; e^{\mu x} \cdot \Big(\cos(\mu y) - \mathbf{i}\sin(\mu y)\Big). \end{array}$$

If we define $v(x,y) \; = \; u_1(x,y) - u_2(x,y)$, then

$$v(x,y) \; = \; 2e^{\mu x} \cdot \mathbf{i}\sin(\mu y).$$

At this point, $v(x,y)$ already satisfies the boundary conditions for $\{y = 0\}$ and $\{y = \pi\}$. To satisfy the remaining condition:

$$\begin{array}{rcl} \text{Let } \; v_1(x,y) & = & 2e^{\mu x} \cdot \; \mathbf{i}\sin(\mu y), \\ \text{and } \; v_1(x,y) & = & 2e^{-\mu x} \cdot \mathbf{i}\sin(\mu y). \end{array}$$

If we define $w(x,y) = v_1(x,y) - v_2(x,y)$, then

$$w(x,y) \; = \; 4\sinh(\mu x) \cdot \mathbf{i}\sin(\mu y)$$

also satisfies the boundary condition at $\{x = 0\}$.

Notes: ....
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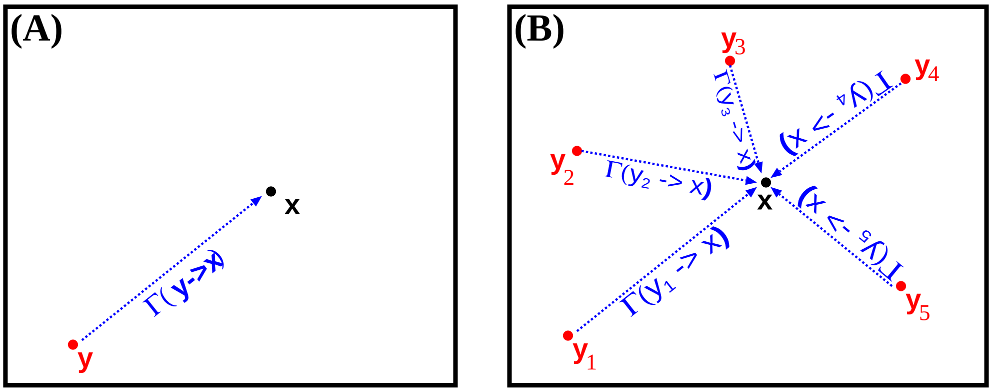


Figure 16.1: **(A)** $\Gamma(\mathbf{y} \to \mathbf{x})$ describes the 'response' at $\mathbf{x}$ to an 'impulse' at $\mathbf{y}$. **(B)** The state at $\mathbf{x}$ is a sum of its responses to the impulses at $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_5$.

16 Impulse-Response Methods

16.1 Introduction

A fundamental concept in science is *causality*: an initial event (an *impulse*) at some location $\mathbf{y}$ causes a later event (a *response*) at another location $\mathbf{x}$ (Figure 16.1A). In an evolving, spatially distributed system (eg. a temperature distribution, a rippling pond, etc.), the system state at each location results from a *combination* of the responses to the impulses from all other locations (as in Figure 16.1B).

If the system is described by a linear PDE, then we expect some sort of 'superposition principle' to apply (Theorem 5.10 on page 83). Hence, we can replace the word 'combination' with 'sum', and say:

The state of the system at $\mathbf{x}$ *is a* sum *of the responses to the impulses from all other locations.* (see Figure 16.1B) (16.1)

However, there are an infinite number —indeed, a continuum —of 'other locations', so we are 'summing' over a *continuum* of responses. But a 'sum' over a continuum is just an *integral*. Hence, statement (16.1) becomes:

In a linear partial differential equation, the solution at $\mathbf{x}$ *is an* <u>integral</u> *of the responses to the impulses from all other locations.* (16.2)

The relation between impulse and response (ie. between cause and effect) is described by *impulse-response function*, $\Gamma(\mathbf{y} \to \mathbf{x})$, which measures the degree of 'influence' which point $\mathbf{y}$ has on point $\mathbf{x}$. In other words, $\Gamma(\mathbf{y} \to \mathbf{x})$ measures the strength of the response at $\mathbf{x}$ to an impulse at $\mathbf{y}$. In a system which evolves in time, Γ may also depend on time (since it takes time for the effect from $\mathbf{y}$ to propagate to $\mathbf{x}$), so Γ also depends on time, and is written $\Gamma_t(\mathbf{y} \to \mathbf{x})$.

Intuitively, Γ should have four properties:

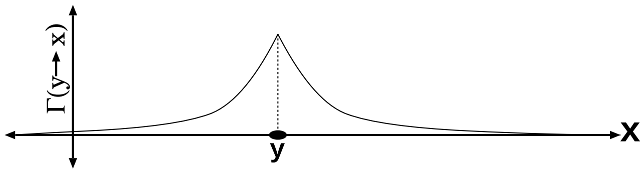


Figure 16.2: The influence of $\mathbf{y}$ on $\mathbf{x}$ becomes small as the distance from $\mathbf{y}$ to $\mathbf{x}$ grows large.

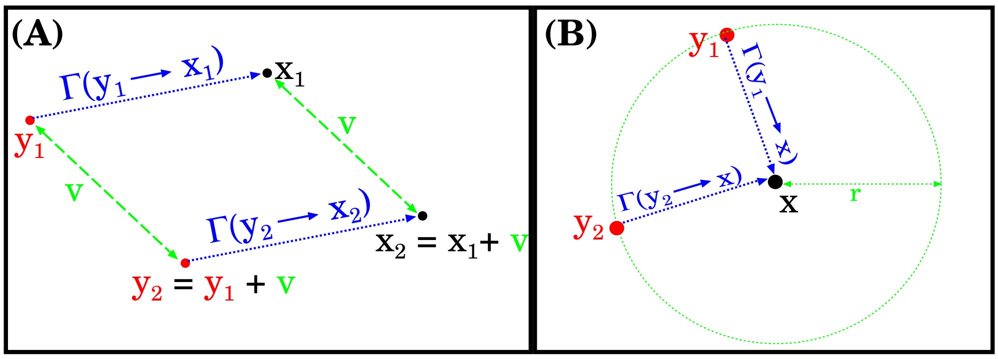


Figure 16.3: **(A)** *Translation invariance*: If $\mathbf{y}_2 = \mathbf{y}_1 + \mathbf{v}$ and $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{v}$, then $\Gamma(\mathbf{y}_2 \to \mathbf{x}_2) = \Gamma(\mathbf{y}_1 \to \mathbf{x}_1)$. **(B)** *Rotation invariance*: If $\mathbf{y}_1$ and $\mathbf{y}_2$ are both the same distance from $\mathbf{x}$ (ie. they lie on the circle of radius r around $\mathbf{x}$), then $\Gamma(\mathbf{y}_2 \to \mathbf{x}) = \Gamma(\mathbf{y}_1 \to \mathbf{x})$.

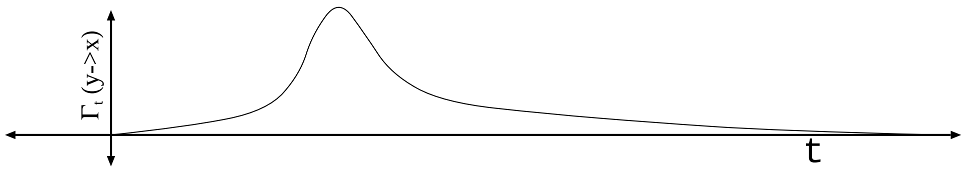


Figure 16.4: The time-dependent impulse-response function first grows large, and then decays to zero.

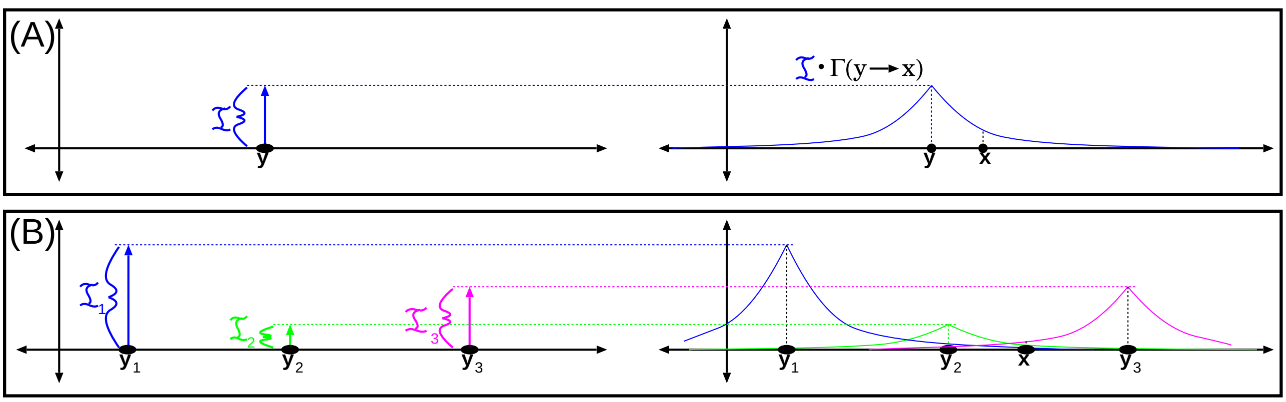


Figure 16.5: **(A)** An 'impulse' of magnitude $\mathcal{I}$ at $\mathbf{y}$ triggers a 'response' of magnitude $\mathcal{I} \cdot \Gamma(\mathbf{y} \to \mathbf{x})$ at $\mathbf{x}$. **(B)** Multiple 'impulses' of magnitude $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ at $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$, respectively, triggers a 'response' at $\mathbf{x}$ of magnitude $\mathcal{I}_1 \cdot \Gamma(\mathbf{y}_1 \to \mathbf{x}) + \mathcal{I}_2 \cdot \Gamma(\mathbf{y}_2 \to \mathbf{x}) + \mathcal{I}_3 \cdot \Gamma(\mathbf{y}_3 \to \mathbf{x})$.

(i) Influence should *decay with distance*. In other words, if $\mathbf{y}$ and $\mathbf{x}$ are close together, then $\Gamma(\mathbf{y} \to \mathbf{x})$ should be large; if $\mathbf{y}$ and $\mathbf{x}$ are far apart, then $\Gamma(\mathbf{y} \to \mathbf{x})$ should be small (Figure 16.2).

(ii) In a *spatially homogeneous* or *translation invariant* system (Figure 16.3(A)), Γ should only depend on the *displacement* from $\mathbf{y}$ to $\mathbf{x}$, so that we can write $\Gamma(\mathbf{y} \to \mathbf{x}) = \gamma(\mathbf{x}-\mathbf{y})$, where γ is some other function.

(iii) In an *isotropic* or *rotation invariant* system system (Figure 16.3(B)), Γ should only depend on the *distance* between $\mathbf{y}$ and $\mathbf{x}$, so that we can write $\Gamma(\mathbf{y} \to \mathbf{x}) = \psi\Big(|\mathbf{x}-\mathbf{y}|\Big)$, where ψ is a function of one real variable, and $\lim_{r\to\infty} \psi(r) = 0$.

(iv) In a *time-evolving* system, the value of $\Gamma_t(\mathbf{y} \to \mathbf{x})$ should first grow as t increases (as the effect 'propagates' from $\mathbf{y}$ to $\mathbf{x}$), reach a maximum value, and then decrease to zero as t grows large (as the effect 'dissipates' through space) (see Figure 16.4).

Thus, if there is an 'impulse' of magnitude $\mathcal{I}$ at $\mathbf{y}$, and $\mathcal{R}(\mathbf{x})$ is the 'response' at $\mathbf{x}$, then

$$\mathcal{R}(\mathbf{x}) \quad = \quad \mathcal{I} \cdot \Gamma(\mathbf{y} \to \mathbf{x}) \qquad \text{(see Figure 16.5A)}$$

What if there is an impulse $\mathcal{I}(\mathbf{y}_1)$ at $\mathbf{y}_1$, an impulse $\mathcal{I}(\mathbf{y}_2)$ at $\mathbf{y}_2$, and an impulse $\mathcal{I}(\mathbf{y}_3)$ at $\mathbf{y}_3$? Then statement (16.1) implies:

$$\mathcal{R}(\mathbf{x}) \quad = \quad \mathcal{I}(\mathbf{y}_1)\cdot\Gamma(\mathbf{y}_1 \to \mathbf{x}) \; + \; \mathcal{I}(\mathbf{y}_2)\cdot\Gamma(\mathbf{y}_2 \to \mathbf{x}) \; + \; \mathcal{I}(\mathbf{y}_3)\cdot\Gamma(\mathbf{y}_3 \to \mathbf{x}). \qquad \text{(Figure 16.5B)}$$

If $\mathbb{X}$ is the domain of the PDE, then suppose, for every $\mathbf{y}$ in $\mathbb{X}$, that $\mathcal{I}(\mathbf{y})$ is the impulse at $\mathbf{y}$. Then statement (16.1) takes the form:

$$\mathcal{R}(\mathbf{x}) \quad = \quad \sum_{\mathbf{y}\in\mathbb{X}} \mathcal{I}(\mathbf{y}) \cdot \Gamma(\mathbf{y} \to \mathbf{x}) \qquad (16.3)$$

But now we are summing over all $\mathbf{y}$ in $\mathbb{X}$, and usually, $\mathbb{X} = \mathbb{R}^D$ or some subset, so the 'summation' in (16.3) doesn't make mathematical sense. We must replace the sum with an *integral*, as in statement (16.2), to obtain:

$$\mathcal{R}(\mathbf{x}) \quad = \quad \int_{\mathbb{X}} \mathcal{I}(\mathbf{y}) \cdot \Gamma(\mathbf{y} \to \mathbf{x}) \; d\mathbf{y} \qquad (16.4)$$

If the system is spatially homogeneous, then according to **(ii)**, this becomes

$$\mathcal{R}(\mathbf{x}) \quad = \quad \int \mathcal{I}(\mathbf{y}) \cdot \gamma(\mathbf{x}-\mathbf{y}) \; d\mathbf{y}$$

This integral is called a **convolution**, and is usually written as $\mathcal{I} * \gamma$. In other words,

$$\mathcal{R}(\mathbf{x}) \quad = \quad \mathcal{I} * \gamma(\mathbf{x}), \qquad \text{where} \qquad \mathcal{I} * \gamma(\mathbf{x}) \quad = \quad \int \mathcal{I}(\mathbf{y}) \cdot \gamma(\mathbf{x}-\mathbf{y}) \; d\mathbf{y}. \qquad (16.5)$$

Note that $\mathcal{I} * \gamma$ is a function of $\mathbf{x}$. The variable $\mathbf{y}$ appears on the right hand side, but as only an *integration* variable.

In a time-dependent system, (16.4) becomes:

$$\mathcal{R}(\mathbf{x};\ t) \quad = \quad \int_{\mathbb{X}} \mathcal{I}(\mathbf{y}) \cdot \Gamma_t(\mathbf{y} \to \mathbf{x})\ d\mathbf{y}.$$

while (16.5) becomes:

$$\mathcal{R}(\mathbf{x};\ t) \quad = \quad \mathcal{I} * \gamma_t(\mathbf{x}), \qquad \text{where} \qquad \mathcal{I} * \gamma_t(\mathbf{x}) \quad = \quad \int \mathcal{I}(\mathbf{y}) \cdot \gamma_t(\mathbf{x} - \mathbf{y})\ d\mathbf{y}. \tag{16.6}$$

The following surprising property is often useful:

Proposition 16.1: *If $f, g : \mathbb{R}^D \longrightarrow \mathbb{R}$ are integrable functions, then $g * f \ = \ f * g$.*

Proof: (Case $D = 1$) Fix $x \in \mathbb{R}$. Then

$$\begin{aligned}(g * f)(x) \quad &= \quad \int_{-\infty}^{\infty} g(y) \cdot f(x - y)\ dy \ \underset{\text{(s)}}{=} \ \int_{\infty}^{-\infty} g(x - z) \cdot f(z) \cdot (-1)\ dz \\ &= \quad \int_{-\infty}^{\infty} f(z) \cdot g(x - z)\ dz \quad = \quad (f * g)(x).\end{aligned}$$

Here, step **(s)** was the substitution $z = x - y$, so that $y = x - z$ and $dy \ = \ -dz$.

<u>**Exercise 16.1**</u> Generalize this proof to the case $D \geq 2$. □

Remark: Impulse-response functions are sometimes called *solution kernels*, or *Green's functions* or *impulse functions*.

16.2 Approximations of Identity

16.2(a) ...in one dimension

Prerequisites: §16.1

Suppose $\gamma : \mathbb{R} \times (0, \infty) \longrightarrow \mathbb{R}$ was a one-dimensional *impulse response function*, as in equation (16.6) of §16.1. Thus, if $\mathcal{I} : \mathbb{R} \longrightarrow \mathbb{R}$ is a function describing the initial 'impulse', then for any time $t > 0$, the 'response' is given by the function $\mathcal{R}_t$ defined:

$$\mathcal{R}_t(x) \quad = \quad \mathcal{I} * \gamma_t(x) \quad = \quad \int_{-\infty}^{\infty} \mathcal{I}(y) \cdot \gamma_t(x - y)\ dy. \tag{16.7}$$

Intuitively, if t is close to zero, then the response $\mathcal{R}_t$ should be concentrated near the locations where the impulse $\mathcal{I}$ is concentrated (because the energy has not yet been able to propagate very far). By inspecting eqn.(16.7), we see that this means that the mass of γ_t should be 'concentrated' near zero. Formally, we say that γ is an **approximation of the identity** if it has the following properties (Figure 16.6):

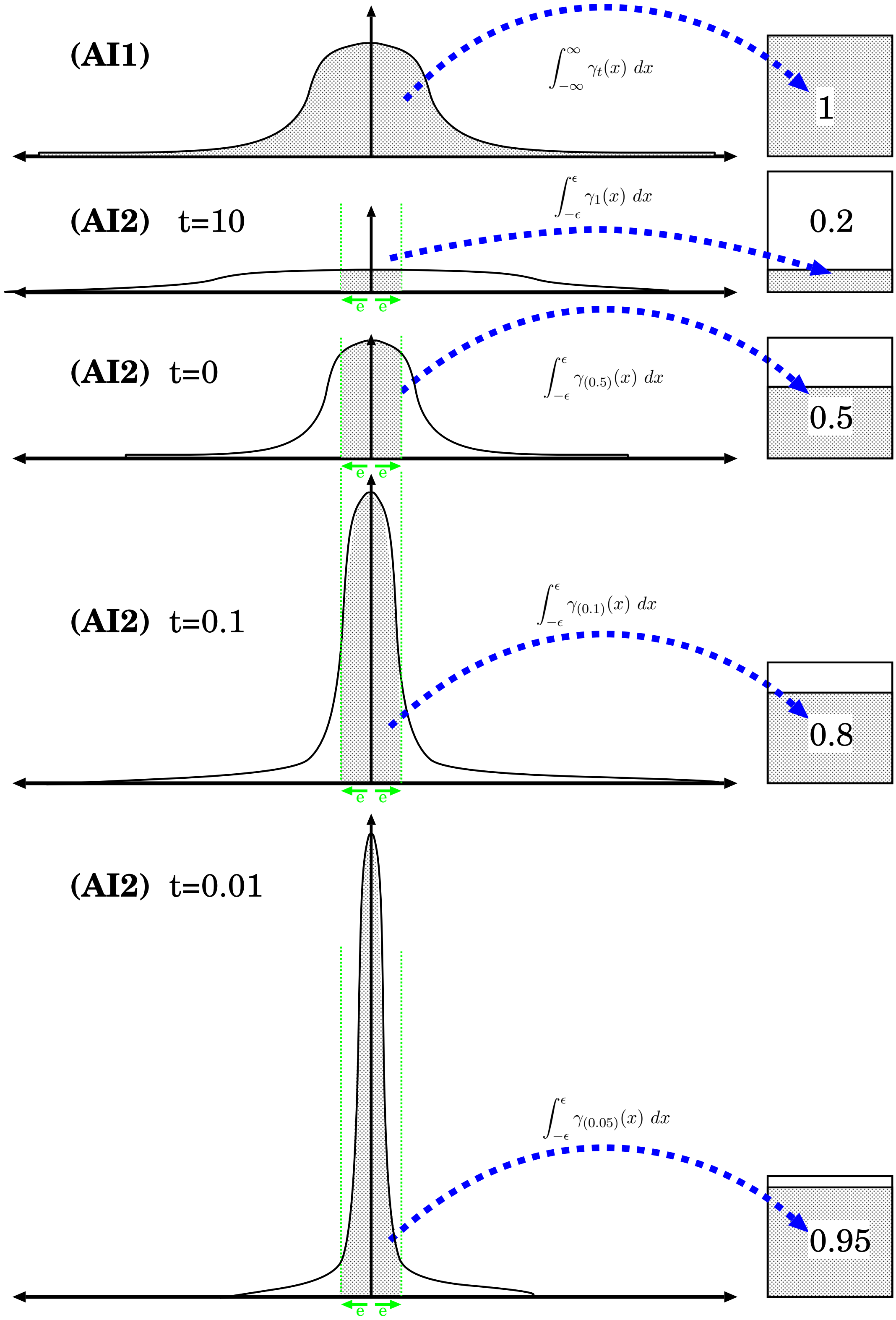


Figure 16.6: γ is an approximation of the identity.

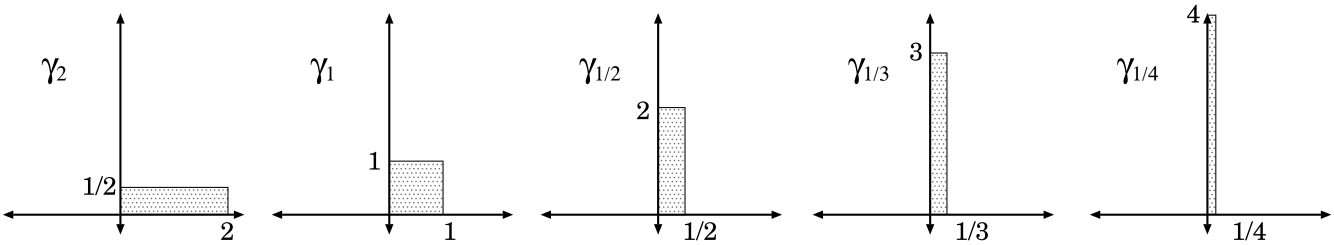


Figure 16.7: Example 16.2(a)

(AI1) $\gamma_t(x) \geq 0$ everywhere, and $\displaystyle\int_{-\infty}^{\infty} \gamma_t(x)\, dx = 1$ for any fixed $t > 0$.

(AI2) For any $\epsilon > 0$, $\displaystyle\lim_{t \to 0} \int_{-\epsilon}^{\epsilon} \gamma_t(x)\, dx = 1$.

Property **(AI1)** says that γ_t is a probability density. **(AI2)** says that γ_t concentrates all of its "mass" at zero as $t \to 0$. (Heuristically speaking, the function γ_t is converging to the 'Dirac delta function' δ_0 as $t \to 0$.)

Example 16.2:

(a) Let $\gamma_t(x) = \begin{cases} \frac{1}{t} & \text{if} \quad 0 \leq x \leq t; \\ 0 & \text{if} \quad x < 0 \text{ or } t < x. \end{cases}$ (Figure 16.7)

Thus, for any $t > 0$, the graph of γ_t is a 'box' of width t and height $1/t$. Then γ is an approximation of identity. (See Practice Problem # 11 on page 334 of §16.8.)

(b) Let $\gamma_t(x) = \begin{cases} \frac{1}{t} & \text{if} \quad |x| \leq t \\ 0 & \text{if} \quad t < |x| \end{cases}$.

Thus, for any $t > 0$, the graph of γ_t is a 'box' of width $2t$ and height $1/2t$. Then γ is an approximation of identity. (See Practice Problem # 12 on page 334 of §16.8.) ♢

A function satisfying properties **(AI1)** and **(AI2)** is called an *approximation of the identity* because of the following theorem:

Proposition 16.3: *Let $\gamma : \mathbb{R} \times (0, \infty) \longrightarrow \mathbb{R}$ be an approximation of identity.*

(a) *Let $\mathcal{I} : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded continuous function. Then for all $x \in \mathbb{R}$, $\displaystyle\lim_{t \to 0} \mathcal{I} * \gamma_t(x) = \mathcal{I}(x)$.*

(b) *Let $\mathcal{I} : \mathbb{R} \longrightarrow \mathbb{R}$ be any bounded integrable function. If $x \in \mathbb{R}$ is any continuity-point of $\mathcal{I}$, then $\displaystyle\lim_{t \to 0} \mathcal{I} * \gamma_t(x) = \mathcal{I}(x)$.*

Proof: **(a)** Fix $x \in \mathbb{R}$. Given any $\epsilon > 0$, find $\delta > 0$ so that,

$$\text{For all } y \in \mathbb{R}, \quad \Big(|y - x| < \delta \Big) \quad \Longrightarrow \quad \Big(\Big|\mathcal{I}(y) - \mathcal{I}(x)\Big| \ < \ \tfrac{\epsilon}{3} \Big).$$

(You can do this because $\mathcal{I}$ is *continuous*). Thus,

$$\begin{aligned}
&\left|\mathcal{I}(x) \cdot \int_{x-\delta}^{x+\delta} \gamma_t(x-y)\, dy \quad - \quad \int_{x-\delta}^{x+\delta} \mathcal{I}(y) \cdot \gamma_t(x-y)\, dy\right| \\
&= \quad \left|\int_{x-\delta}^{x+\delta} \Big(\mathcal{I}(x) - \mathcal{I}(y)\Big) \cdot \gamma_t(x-y)\, dy\right| \quad \leq \quad \int_{x-\delta}^{x+\delta} \Big|\mathcal{I}(x) - \mathcal{I}(y)\Big| \cdot \gamma_t(x-y)\, dy \\
&< \quad \frac{\epsilon}{3} \int_{x-\delta}^{x+\delta} \gamma_t(x-y)\, dy \quad \underset{\text{(AI1)}}{<} \quad \frac{\epsilon}{3}. \qquad (16.8)
\end{aligned}$$

(Here **(AI1)** is by property **(AI1)** of γ_t.)

Recall that $\mathcal{I}$ is *bounded*. Suppose $|\mathcal{I}(y)| < M$ for all $y \in \mathbb{R}$; using **(AI2)**, find some small $\tau > 0$ so that, if $t < \tau$, then $\displaystyle\int_{x-\delta}^{x+\delta} \gamma_t(y)\, dy \ > \ 1 - \frac{\epsilon}{3M}$; hence

$$\begin{aligned}
\int_{-\infty}^{x-\delta} \gamma_t(y)\, dy \quad + \quad \int_{x+\delta}^{\infty} \gamma_t(y)\, dy \quad &= \quad \int_{-\infty}^{\infty} \gamma_t(y)\, dy \quad - \quad \int_{x-\delta}^{x+\delta} \gamma_t(y)\, dy \\
&\underset{\text{(AI1)}}{<} \quad 1 \ - \ \left(1 - \frac{\epsilon}{3M}\right) \quad = \quad \frac{\epsilon}{3M}. \qquad (16.9)
\end{aligned}$$

(Here **(AI1)** is by property **(AI1)** of γ_t.) Thus,

$$\begin{aligned}
&\left|\mathcal{I} * \gamma_t(x) \quad - \quad \int_{x-\delta}^{x+\delta} \mathcal{I}(y) \cdot \gamma_t(x-y)\, dy\right| \\
&\leq \quad \left|\int_{-\infty}^{\infty} \mathcal{I}(y) \cdot \gamma_t(x-y)\, dy \quad - \quad \int_{x-\delta}^{x+\delta} \mathcal{I}(y) \cdot \gamma_t(x-y)\, dy\right| \\
&= \quad \left|\int_{-\infty}^{x-\delta} \mathcal{I}(y) \cdot \gamma_t(x-y)\, dy \quad + \quad \int_{x+\delta}^{\infty} \mathcal{I}(y) \cdot \gamma_t(x-y)\, dy\right| \\
&< \quad \int_{-\infty}^{x-\delta} \Big|\mathcal{I}(y) \cdot \gamma_t(x-y)\Big|\ dy \quad + \quad \int_{x+\delta}^{\infty} \Big|\mathcal{I}(y) \cdot \gamma_t(x-y)\Big|\ dy \\
&< \quad \int_{-\infty}^{x-\delta} M \cdot \gamma_t(x-y)\, dy \quad + \quad \int_{x+\delta}^{\infty} M \cdot \gamma_t(x-y)\, dy \\
&< \quad M \cdot \left(\int_{-\infty}^{x-\delta} \gamma_t(x-y)\, dy \quad + \quad \int_{x+\delta}^{\infty} \gamma_t(x-y)\, dy\right) \quad \underset{(16.9)}{\leq} \quad M \cdot \frac{\epsilon}{3M} \quad = \quad \frac{\epsilon}{3}. \qquad (16.10)
\end{aligned}$$

(Here, **(16.9)** is by eqn.(16.9).) Combining equations (16.8) and (16.10) we have:

$$\left|\mathcal{I}(x) \cdot \int_{x-\delta}^{x+\delta} \gamma_t(x-y)\, dy \quad - \quad \mathcal{I} * \gamma_t(x)\right|$$

$$\leq \left| \mathcal{I}(x) \cdot \int_{x-\delta}^{x+\delta} \gamma_t(x-y)\ dy \ - \ \int_{x-\delta}^{x+\delta} \mathcal{I}(y) \cdot \gamma_t(x-y)\ dy \right|$$
$$+ \left| \int_{x-\delta}^{x+\delta} \mathcal{I}(y) \cdot \gamma_t(x-y)\ dy \ - \ \mathcal{I} * \gamma_t(x) \right| \ \leq \ \frac{\epsilon}{3} + \frac{\epsilon}{3} \ = \ \frac{2\epsilon}{3}. \quad (16.11)$$

But if $t < \tau$,then $\left| 1 - \int_{x-\delta}^{x+\delta} \gamma_t(x-y)\ dy \right| \ < \ \frac{\epsilon}{3M}$. Thus,

$$\left| \mathcal{I}(x) \ - \ \mathcal{I}(x) \cdot \int_{x-\delta}^{x+\delta} \gamma_t(x-y)\ dy \right| \ \leq \ |\mathcal{I}(x)| \cdot \left| 1 \ - \ \int_{x-\delta}^{x+\delta} \gamma_t(x-y)\ dy \right|$$
$$< \ |\mathcal{I}(x)| \cdot \frac{\epsilon}{3M} \ \leq \ M \cdot \frac{\epsilon}{3M} \ = \ \frac{\epsilon}{3}. \quad (16.12)$$

Combining equations (16.11) and (16.12) we have:

$$|\mathcal{I}(x) \ - \ \mathcal{I} * \gamma_t(x)|$$
$$\leq \left| \mathcal{I}(x) \ - \ \mathcal{I}(x) \cdot \int_{x-\delta}^{x+\delta} \gamma_t(x-y)\ dy \right| \ + \ \left| \mathcal{I}(x) \cdot \int_{x-\delta}^{x+\delta} \gamma_t(x-y)\ dy \ - \ \mathcal{I} * \gamma_t(x) \right|$$
$$\leq \frac{\epsilon}{3} + \frac{2\epsilon}{3}. \ = \ \epsilon.$$

Since ϵ can be made arbitrarily small, we're done.

(b) <u>**Exercise 16.2**</u> (Hint: imitate part **(a)**). □

In other words, as $t \to 0$, the convolution $\mathcal{I} * \gamma_t$ resembles $\mathcal{I}$ with arbitrarily high accuracy. Similar convergence results can be proved in other norms (eg. $\mathbf{L}^2$ convergence, uniform convergence).

Example 16.4: Let $\gamma_t(x) \ = \ \begin{cases} \frac{1}{t} & \text{if} \quad 0 \leq x \leq t \\ 0 & \text{if} \quad x < 0 \text{ or } t < x \end{cases}$, as in Example 16.2(a). Suppose $\mathcal{I} : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. Then for any $x \in \mathbb{R}$,

$$\mathcal{I} * \gamma_t(x) \ = \ \int_{-\infty}^{\infty} \mathcal{I}(y) \cdot \gamma_t(x-y)\ dy \ = \ \frac{1}{t} \int_{x-t}^{x} \mathcal{I}(y)\ dy \ = \ \frac{1}{t} \Big(\mathcal{J}(x) - \mathcal{J}(x-t) \Big),$$

where $\mathcal{J}$ is an antiderivative of $\mathcal{I}$. Thus, as implied by Proposition 16.3,

$$\lim_{t \to 0} \mathcal{I} * \gamma_t(x) \ = \ \lim_{t \to 0} \frac{\mathcal{J}(x) - \mathcal{J}(x-t)}{t} \ \underset{(*)}{=} \ \mathcal{J}'(x) \ \underset{(\dagger)}{=} \ \mathcal{I}(x).$$

(Here $(*)$ is just the definition of differentiation, and $(\dagger)$ is because $\mathcal{J}$ is an antiderivative of $\mathcal{I}$.) ♢

16.2(b) ...in many dimensions

Prerequisites: §16.2(a) **Recommended:** §16.3(a)

A nonnegative function $\gamma : \mathbb{R}^D \times (0,\infty) \longrightarrow [0,\infty)$ is called an **approximation of the identity**[1] if it has the following two properties:

(AI1) $\displaystyle\int_{\mathbb{R}^D} \gamma_t(\mathbf{x})\, d\mathbf{x} \;=\; 1$ for all $t \in [0,\infty]$.

(AI2) For any $\epsilon > 0$, $\displaystyle\lim_{t\to 0} \int_{\mathbb{B}(0;\epsilon)} \gamma_t(\mathbf{x})\, d\mathbf{x} \;=\; 1.$

Property **(AI1)** says that γ_t is a probability density. **(AI2)** says that γ_t concentrates all of its "mass" at zero as $t \to 0$.

Example 16.5: Define $\gamma : \mathbb{R}^2 \times (0,\infty) \longrightarrow \mathbb{R}$ by $\gamma_t(x,y) = \begin{cases} \frac{1}{4t^2} & \text{if } |x| \le t \text{ and } |y| \le t; \\ 0 & \text{otherwise.} \end{cases}$.

Then γ is an approximation of the identity on $\mathbb{R}^2$. (**Exercise 16.3**) ♢

Proposition 16.6: *Let $\gamma : \mathbb{R}^D \times (0,\infty) \longrightarrow \mathbb{R}$ be an approximation of the identity.*

(a) *Let $\mathcal{I} : \mathbb{R}^D \longrightarrow \mathbb{R}$ be a bounded continuous function. Then for every $\mathbf{x} \in \mathbb{R}^D$, $\displaystyle\lim_{t\to 0} \mathcal{I} * \gamma_t(\mathbf{x}) \;= \mathcal{I}(\mathbf{x})$.*

(b) *Let $\mathcal{I} : \mathbb{R}^D \longrightarrow \mathbb{R}$ be any bounded integrable function. If $\mathbf{x} \in \mathbb{R}^D$ is any continuity-point of $\mathcal{I}$, then $\displaystyle\lim_{t\to 0} \mathcal{I} * \gamma_t(\mathbf{x}) \;= \mathcal{I}(\mathbf{x})$.*

Proof: **Exercise 16.4** Hint: the argument is basically identical to that of Proposition 16.3; just replace the interval $(-\epsilon,\epsilon)$ with a ball of radius ϵ. □

In other words, as $t \to 0$, the convolution $\mathcal{I} * \gamma_t$ resembles $\mathcal{I}$ with arbitrarily high accuracy. Similar convergence results can be proved in other norms (eg. $\mathbf{L}^2$ convergence, uniform convergence).

When solving partial differential equations, approximations of identity are invariably used in conjunction with the following result:

Proposition 16.7: *Let L be a linear differential operator on $\mathcal{C}^\infty(\mathbb{R}^D;\mathbb{R})$.*

(a) *If $\gamma : \mathbb{R}^D \longrightarrow \mathbb{R}$ is a solution to the homogeneous equation "$\mathsf{L}\,\gamma = 0$", then for any function $\mathcal{I} : \mathbb{R}^D \longrightarrow \mathbb{R}$, the function $u = \mathcal{I} * \gamma$ satisfies: $\mathsf{L}\mathcal{I} = 0$.*

(b) *If $\gamma : \mathbb{R}^D \times (0,\infty) \longrightarrow \mathbb{R}$ satisfies the evolution equation "$\partial_t^n \gamma \;=\; \mathsf{L}\,\gamma$", and we define $\gamma_t(\mathbf{x}) = \gamma(\mathbf{x};t)$, for any function $\mathcal{I} : \mathbb{R}^D \longrightarrow \mathbb{R}$, then the function $u_t = \mathcal{I} * \gamma_t$ satisfies: $\partial_t^n\, u \;=\; \mathsf{L}\,u$.*

[1] Sometimes, this is called a **Dirac sequence** [Lan85], because, as $t \to 0$, it "converges" to the infamous "Dirac δ-function". In harmonic analysis, this object is sometimes called a **summability kernel** [Kat76], because it is used to make certain Fourier series summable to help prove convergence results.

Johann Carl Friedrich Gauss (1777-1855)

Karl Theodor Wilhelm Weierstrass (1815-1897)

Proof: **Exercise 16.5** Hint: Generalize the proof of Proposition 16.9 on the following page, by replacing the one-dimensional convolution integral with a D-dimensional convolution integral, and by replacing the Laplacian with an arbitrary linear operator L. □

Corollary 16.8: *Suppose γ is an approximation of the identity and satisfies the evolution equation "$\partial_t^n \gamma = \mathsf{L}\,\gamma$". For any $\mathcal{I} : \mathbb{R}^D \longrightarrow \mathbb{R}$, define $u : \mathbb{R}^D \times [0, \infty) \longrightarrow \mathbb{R}$ by:*

- $u(\mathbf{x}; 0) = \mathcal{I}(\mathbf{x})$.
- $u_t = \mathcal{I} * \gamma_t$, *for all* $t > 0$.

Then u is a solution to the equation "$\partial_t^n u = \mathsf{L}\, u$", and u satisfies the initial conditions $u(\mathbf{x}, 0) = \mathcal{I}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^D$.

Proof: Combine Propositions 16.6 and 16.7. □

We say that γ is the **fundamental solution** (or **solution kernel**, or **Green's function** or **impulse function**) for the PDE. For example, the D-dimensional Gauss-Weierstrass kernel is a fundamental solution for the D-dimensional Heat Equation.

16.3 The Gaussian Convolution Solution (Heat Equation)

16.3(a) ...in one dimension

Prerequisites: §2.2(a), §16.2(a), §1.8 **Recommended:** §16.1, §18.1(b)

Given two functions $\mathcal{I}, \mathcal{G} : \mathbb{R} \longrightarrow \mathbb{R}$, recall (from §16.1) that their **convolution** is the function $\mathcal{I} * \mathcal{G} : \mathbb{R} \longrightarrow \mathbb{R}$ defined:

$$\mathcal{I} * \mathcal{G}(x) \quad = \quad \int_{-\infty}^{\infty} \mathcal{I}(y) \cdot \mathcal{G}(x-y) \; dy$$

Recall the **Gauss-Weierstrass kernel** from Example 2.1 on page 23:

$$\mathcal{G}(x;t) \quad = \quad \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-x^2}{4t}\right) \qquad \text{(for all } x \in \mathbb{R} \text{ and } t > 0)$$

Define $\mathcal{G}_t(x) = \mathcal{G}(x;t)$. We will treat $\mathcal{G}_t(x)$ as an *impulse-response function* to solve the one-dimensional Heat equation.

Proposition 16.9: *Let $\mathcal{I} : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded integrable function. Define $u : \mathbb{R} \times (0, \infty) \longrightarrow \mathbb{R}$ by $u(x;t) := \mathcal{I} * \mathcal{G}_t(x)$ for all $x \in \mathbb{R}$ and $t > 0$. Then u is a solution to the one-dimensional Heat Equation.*

Proof: For any fixed $y \in \mathbb{R}$, define $u_y(x;t) \; = \; \mathcal{I}(y) \cdot \mathcal{G}_t(x-y)$.

Claim 1: *$u_y(x;t)$ is a solution of the one-dimensional Heat Equation.*

Proof: First note that $\partial_t \, \mathcal{G}_t(x-y) \; = \; \partial_x^2 \, \mathcal{G}_t(x-y)$ (**Exercise 16.6**).
Now, y is a constant, so we treat $\mathcal{I}(y)$ as a constant when differentiating by x or by t. Thus,

$$\partial_t \, u_y(x,t) \quad = \quad \mathcal{I}(y) \cdot \partial_t \, \mathcal{G}_t(x-y) \quad = \quad \mathcal{I}(y) \cdot \partial_x^2 \, \mathcal{G}_t(x-y) \quad = \quad \partial_x^2 \, u_y(x,t) \quad = \quad \triangle u_y(x,t),$$

as desired. $\diamond_{\text{Claim 1}}$

Now, $u(x,t) \; = \; \mathcal{I} * \mathcal{G}_t \; = \; \displaystyle\int_{-\infty}^{\infty} \mathcal{I}(y) \cdot \mathcal{G}_t(x-y) \; dy \; = \; \int_{-\infty}^{\infty} u_y(x;t) \; dy$. Thus,

$$\partial_t \; u(x,t) \quad \overline{\overline{\text{(P1.9)}}} \quad \int_{-\infty}^{\infty} \partial_t \; u_y(x;t) \; dy \quad \overline{\overline{\text{(C1)}}} \quad \int_{-\infty}^{\infty} \triangle u_y(x;t) \; dy \quad \overline{\overline{\text{(P1.9)}}} \quad \triangle \, u(x,t).$$

Here, **(C1)** is by Claim 1, and **(P1.9)** is by Proposition 1.9 on page 18.

(**Exercise 16.7** Verify that the conditions of Proposition 1.9 are satisfied.) □

Remark: One way to visualize the 'Gaussian convolution' $u(x;t) \; = \; \mathcal{I} * \mathcal{G}_t(x)$ is as follows. Consider a finely spaced "ϵ-mesh" of points on the real line,

$$\epsilon \cdot \mathbb{Z} \quad = \quad \{n\epsilon \; ; \; n \in \mathbb{Z}\},$$

For every $n \in \mathbb{Z}$, define the function $\mathcal{G}_t^{(n)}(x) \; = \; \mathcal{G}_t(x - n\epsilon)$. For example, $\mathcal{G}_t^{(5)}(x) \; = \; \mathcal{G}_t(x - 5\epsilon)$ looks like a copy of the Gauss-Weierstrass kernel, but centered at 5ϵ (see Figure 16.8A).

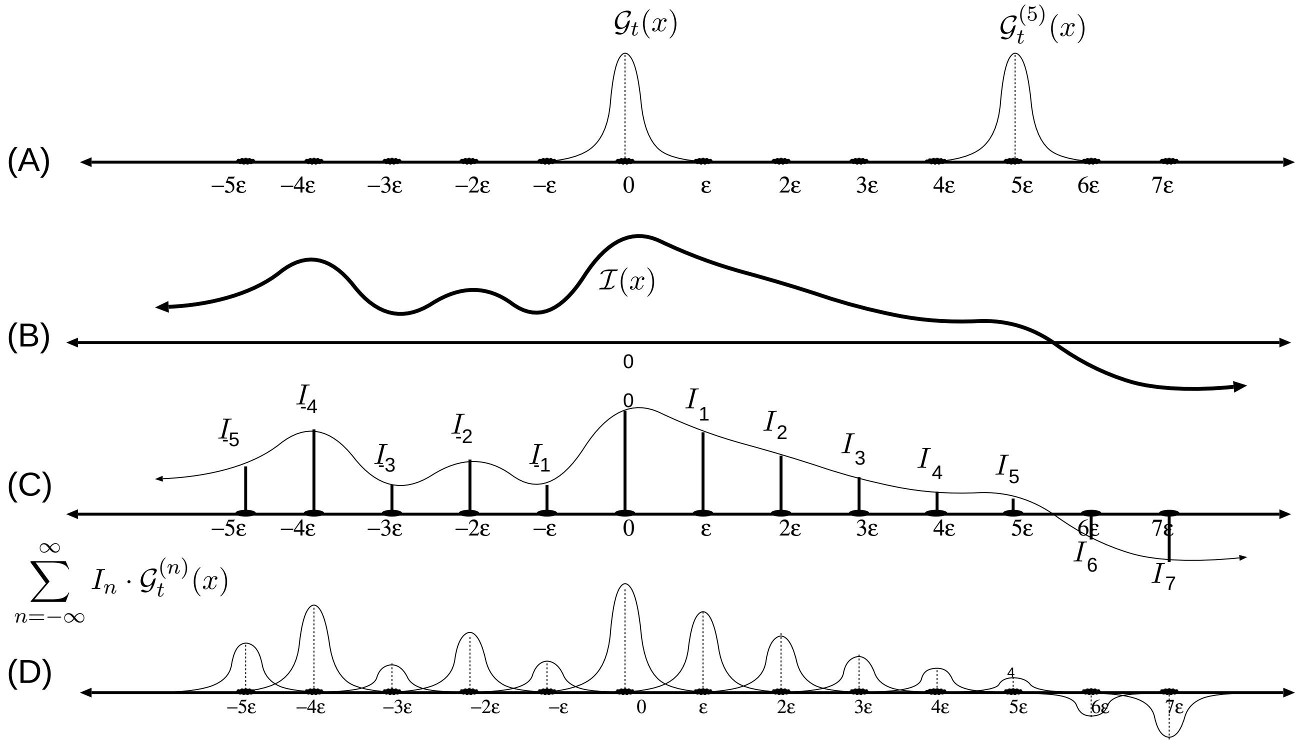


Figure 16.8: Discrete convolution: a superposition of Gaussians

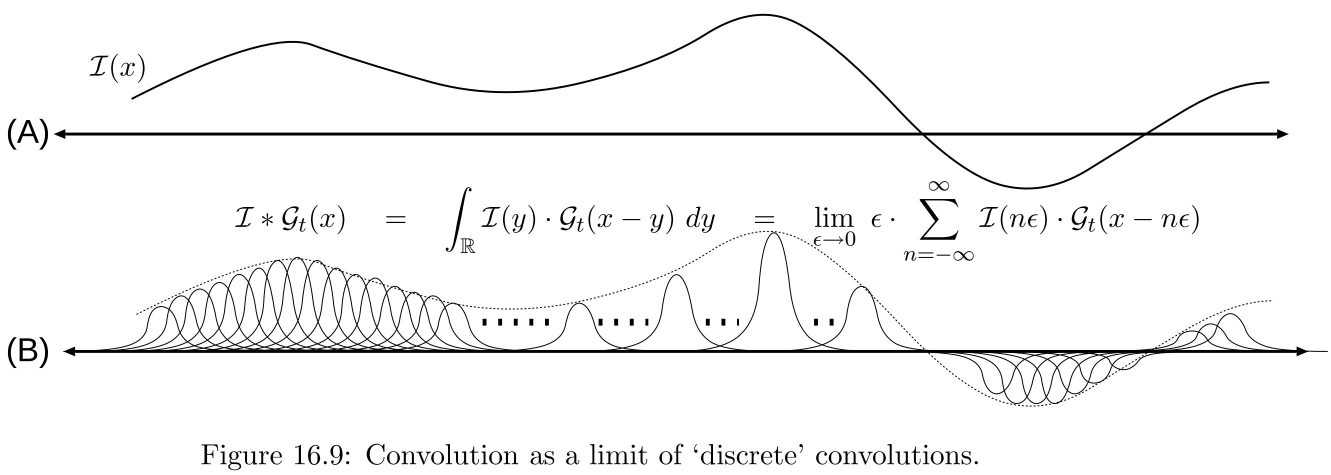


Figure 16.9: Convolution as a limit of 'discrete' convolutions.

For each $n \in \mathbb{Z}$, let $I_n = \mathcal{I}(n \cdot \epsilon)$ (see Figure 16.8C). Now consider the infinite linear combination of Gauss-Weierstrass kernels (see Figure 16.8D):

$$u_\epsilon(x;t) \quad = \quad \epsilon \cdot \sum_{n=-\infty}^{\infty} I_n \cdot \mathcal{G}_t^{(n)}(x)$$

Now imagine that the ϵ-mesh become 'infinitely dense', by letting $\epsilon \to 0$. Define $u(x;t) = \lim_{\epsilon \to 0} u_\epsilon(x;t)$. I claim that $u(x;t) = \mathcal{I} * \mathcal{G}_t(x)$. To see this, note that

$$\begin{aligned} u(x;t) &= \lim_{\epsilon \to 0} \epsilon \cdot \sum_{n=-\infty}^{\infty} I_n \cdot \mathcal{G}_t^{(n)}(x) \quad = \quad \lim_{\epsilon \to 0} \epsilon \cdot \sum_{n=-\infty}^{\infty} \mathcal{I}(n\epsilon) \cdot \mathcal{G}_t(x - n\epsilon) \\ &= \int_{-\infty}^{\infty} \mathcal{I}(y) \cdot \mathcal{G}_t(x-y)\; dy \quad = \quad \mathcal{I} * \mathcal{G}_t(y), \end{aligned}$$

as shown in Figure 16.9.

Proposition 16.10: *The Gauss-Weirstrass kernel is an approximation of identity* (see §16.2(a)), *meaning that it satisfies the following two properties:*

(AI1) *$\mathcal{G}_t(x) \geq 0$ everywhere, and $\int_{-\infty}^{\infty} \mathcal{G}_t(x)\; dx \; = \; 1$ for any fixed $t > 0$.*

(AI2) *For any $\epsilon > 0$, $\lim_{t \to 0} \int_{-\epsilon}^{\epsilon} \mathcal{G}_t(x)\; dx \; = \; 1$.*

Proof: **Exercise 16.8** □

Corollary 16.11: *Let $\mathcal{I} : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded integrable function. Define the function $u : \mathbb{R} \times [0, \infty) \longrightarrow \mathbb{R}$ by*

- *$u(x;0) = \mathcal{I}(x)$.*
- *$u_t = \mathcal{I} * \mathcal{G}_t$, for all $t > 0$.*

Then u is a solution to the one-dimensional Heat Equation. Furthermore:

(a) *If $\mathcal{I}$ is continuous, then u continuous on $\mathbb{R} \times [0, \infty)$, and satisfies the initial conditions $u(x,0) = \mathcal{I}(x)$ for all $x \in \mathbb{R}$.*

(b) *If $\mathcal{I}$ is not continuous, then u is still continuous on $\mathbb{R} \times (0, \infty)$, and satisfies the initial conditions $u(x,0) = \mathcal{I}(x)$ for any $x \in \mathbb{R}$ where f is continuous.*

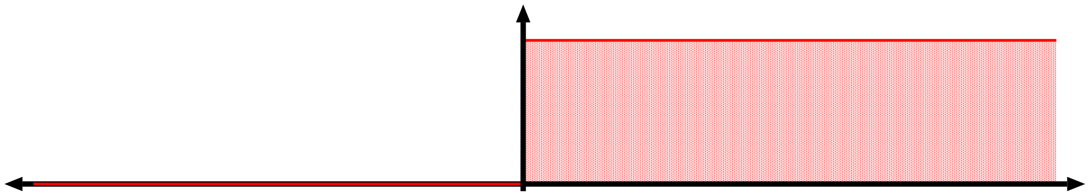


Figure 16.10: The Heaviside step function $\mathcal{H}(x)$.

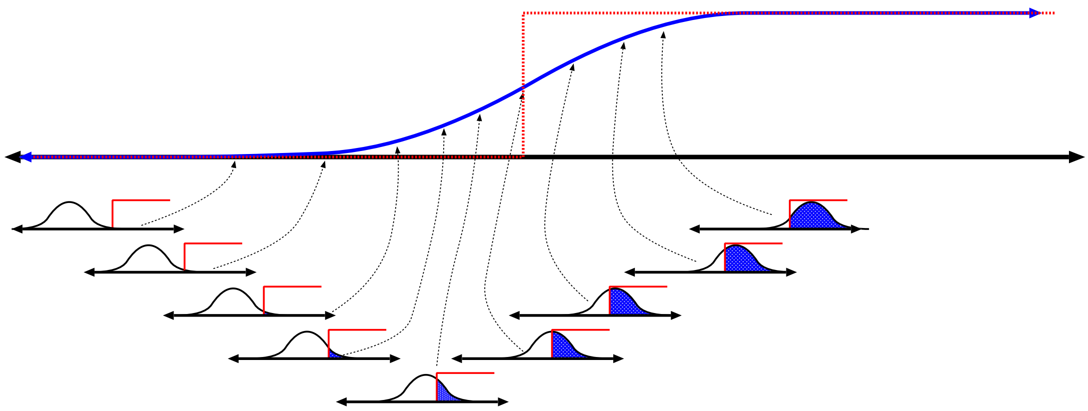


Figure 16.11: $u_t(x) = (\mathcal{H} * \mathcal{G}_t)(x)$ evaluated at several $x \in \mathbb{R}$.

Proof: Propositions 16.9 says that u is a solution to the Heat Equation. Combine Proposition 16.10 with Proposition 16.3 on page 303 to conclude that u is continuous with initial conditions $u(x;0) = \mathcal{I}(x)$. □

Because of Corollary 16.11, we say that $\mathcal{G}$ is the **fundamental solution** (or **solution kernel**, or **Green's function** or **impulse function**) for the Heat equation.

Example 16.12: The Heaviside Step function

Consider the Heaviside **step function** $\mathcal{H}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ (see Figure 16.10). The solution to the one-dimensional Heat equation with initial conditions $u(x,0) = \mathcal{H}(x)$ is given:

$$u(x,t) \underset{\text{(P16.9)}}{=\!=} \mathcal{H} * \mathcal{G}_t(x) \underset{\text{(P16.1)}}{=\!=} \mathcal{G}_t * \mathcal{H}(x) = \int_{-\infty}^{\infty} \mathcal{G}_t(y) \cdot \mathcal{H}(x-y)\, dy$$

$$= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp\left(\frac{-y^2}{4t}\right) \mathcal{H}(x-y)\, dy \underset{(1)}{=\!=} \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{x} \exp\left(\frac{-y^2}{4t}\right) dy$$

$$\underset{(2)}{=\!=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x/\sqrt{2t}} \exp\left(\frac{-z^2}{2}\right) dz = \Phi\left(\frac{x}{\sqrt{2t}}\right).$$

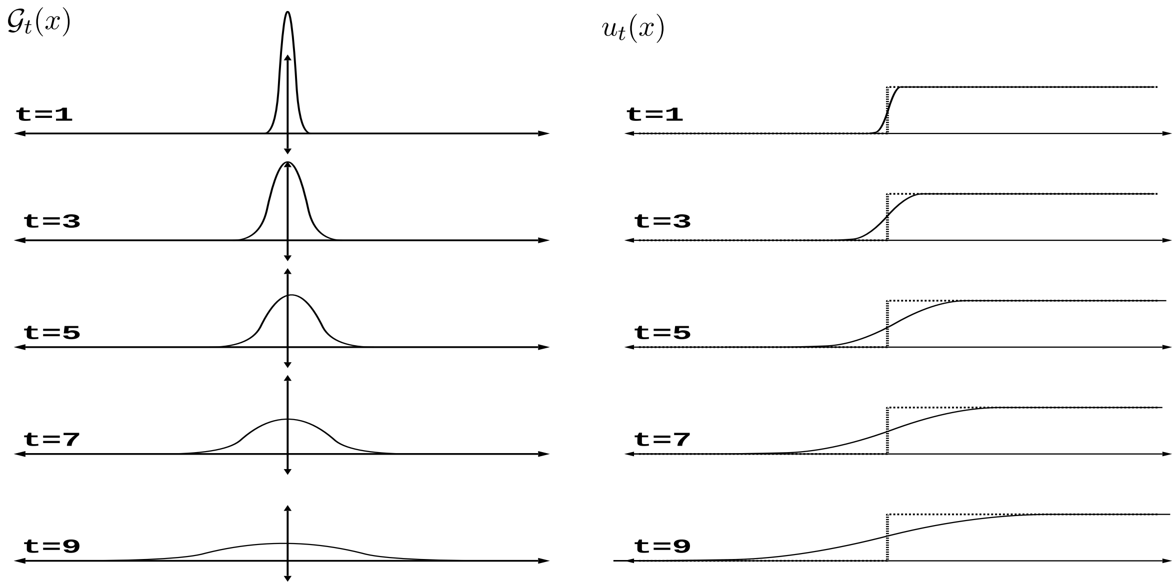


Figure 16.12: $u_t(x) \;=\; (\mathcal{H} * \mathcal{G}_t)(x)$ for several $t > 0$.

Here, **(P16.9)** is by Prop. 16.9 on page 308; **(P16.1)** is by Prop. 16.1 on page 301; **(1)** is because $\mathcal{H}(x-y) = \begin{cases} 1 & \text{if } y \leq x \\ 0 & \text{if } y > x \end{cases}$, and **(2)** is where we make the substitution $z = \frac{y}{\sqrt{2t}}$; thus, $dy = \sqrt{2t}\; dz$.

Here, $\Phi(x)$ is the **cumulative distribution function** of the standard normal probability measure[2], defined:

$$\boxed{\Phi(x) \quad = \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(\frac{-z^2}{2}\right)\; dz}$$

(see Figure 16.11). At time zero, $u(x,0) = \mathcal{H}(x)$ is a step function. For $t > 0$, $u(x,t)$ looks like a compressed version of $\Phi(x)$: a steep sigmoid function. As t increases, this sigmoid becomes broader and flatter. (see Figure 16.12). ♢

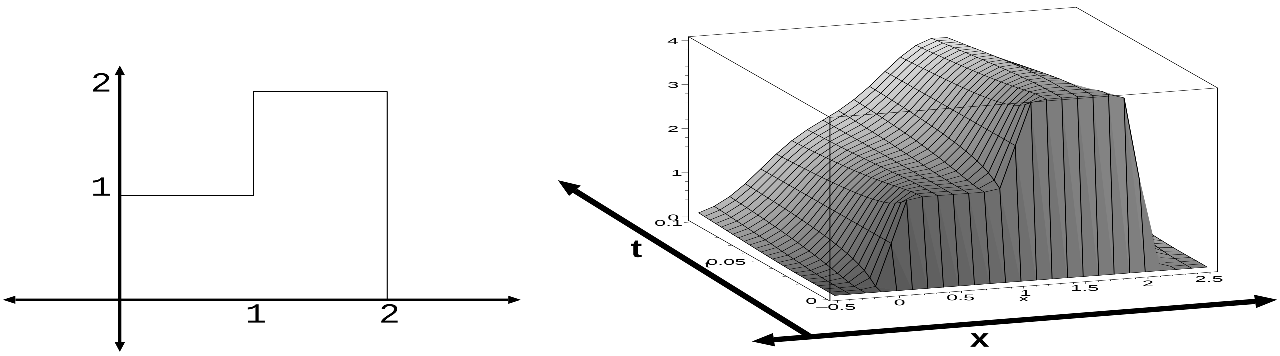
When computing convolutions, you can often avoid a lot of messy integrals by exploiting the following properties:

Proposition 16.13: *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be integrable functions. Then:*

(a) *If $h : \mathbb{R} \longrightarrow \mathbb{R}$ is another integrable function, then $f * (g + h) \;=\; (f * g) + (f * h)$.*

(b) *If $r \in \mathbb{R}$ is a constant, then $f * (r \cdot g) \;=\; r \cdot (f * g)$.*

[2]This is sometimes called the **error function** or **sigmoid** function. Unfortunately, no simple formula exists for $\Phi(x)$. It can be computed with arbitrary accuracy using a Taylor series, and tables of values for $\Phi(x)$ can be found in most statistics texts.



(A) (B)

Figure 16.13: **(A)** A staircase function. **(B)** The resulting solution to the Heat equation.

(c) *Suppose* $d \in \mathbb{R}$ *is some 'displacement', and we define* $f_{\triangleright d}(x) = f(x-d)$. *Then* $(f_{\triangleright d} * g)(x) = (f * g)(x-d)$. *(ie.* $(f_{\triangleright d}) * g = (f * g)_{\triangleright d}$.*)*

Proof: See Practice Problems #2 and # 3 on page 333 of §16.8. □

Example 16.14: A staircase function

Suppose $\mathcal{I}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x < 1 \\ 2 & \text{if } 1 \le x < 2 \\ 0 & \text{if } 2 \le x \end{cases}$ (see Figure 16.13A). Let $\Phi(x)$ be the sigmoid function from Example 16.12. Then

$$u(x,t) = \Phi\left(\frac{x}{\sqrt{2t}}\right) + \Phi\left(\frac{x-1}{\sqrt{2t}}\right) - 2\cdot\Phi\left(\frac{x-2}{\sqrt{2t}}\right) \qquad \text{(see Figure 16.13B)}$$

To see this, observe that we can write:

$$\mathcal{I}(x) = \mathcal{H}(x) + \mathcal{H}(x-1) - 2\cdot\mathcal{H}(x-2) \qquad (16.13)$$
$$= \mathcal{H} + \mathcal{H}_{\triangleright 1}(x) - 2\mathcal{H}_{\triangleright 2}(x), \qquad (16.14)$$

where eqn. (16.14) uses the notation of Proposition 16.13(c). Thus,

$$\begin{aligned} u(x;t) &\underset{\text{(P16.9)}}{=\!=} \mathcal{I} * \mathcal{G}_t(x) \underset{\text{(e16.14)}}{=\!=} \Big(\mathcal{H} + \mathcal{H}_{\triangleright 1} - 2\mathcal{H}_{\triangleright 2}\Big) * \mathcal{G}_t(x) \\ &\underset{\text{(16.13a,b)}}{=\!=} \mathcal{H} * \mathcal{G}_t(x) + \mathcal{H}_{\triangleright 1} * \mathcal{G}_t(x) - 2\mathcal{H}_{\triangleright 2} * \mathcal{G}_t(x) \\ &\underset{\text{(16.13c)}}{=\!=} \mathcal{H} * \mathcal{G}_t(x) + \mathcal{H} * \mathcal{G}_t(x-1) - 2\mathcal{H} * \mathcal{G}_t(x-2) \\ &\underset{\text{(x16.12)}}{=\!=} \boxed{\Phi\left(\frac{x}{\sqrt{2t}}\right) + \Phi\left(\frac{x-1}{\sqrt{2t}}\right) - 2\Phi\left(\frac{x-2}{\sqrt{2t}}\right).} \end{aligned} \qquad (16.15)$$

Here, **(P16.9)** is by Prop. 16.9 on page 308; **(e16.14)** is by eqn. (16.14); **(16.13a,b)** is by Proposition 16.13(a) and (b); **(16.13c)** is by Proposition 16.13(c); and **(X16.12)** is by Example 16.12.

Another approach: Begin with eqn. (16.13), and, rather than using Proposition 16.13, use instead the linearity of the Heat Equation, along with Theorem 5.10 on page 83, to deduce that the solution must have the form:

$$u(x,t) = u_0(x,t) \quad + \quad u_1(x,t) \quad - \quad 2 \cdot u_2(x,t) \tag{16.16}$$

where

- $u_0(x,t)$ is the solution with initial conditions $u_0(x,0) = \mathcal{H}(x)$,
- $u_1(x,t)$ is the solution with initial conditions $u_1(x,0) = \mathcal{H}(x-1)$,
- $u_2(x,t)$ is the solution with initial conditions $u_2(x,0) = \mathcal{H}(x-2)$,

But then we know, from Example 16.12 that

$$u_0(x,t) \;=\; \Phi\left(\frac{x}{\sqrt{2t}}\right); \quad u_1(x,t) \;=\; \Phi\left(\frac{x-1}{\sqrt{2t}}\right); \quad \text{and} \quad u_2(x,t) \;=\; \Phi\left(\frac{x-2}{\sqrt{2t}}\right); \tag{16.17}$$

Now combine (16.16) with (16.17) to again obtain the solution (16.15). ♢

Remark: The Gaussian convolution solution to the Heat Equation is revisited in § 18.1(b) on page 358, using the methods of Fourier transforms.

16.3(b) ...in many dimensions

Prerequisites: §2.2(b), §16.2(b) **Recommended:** §16.1, §16.3(a)

Given two functions $\mathcal{I}, \mathcal{G} : \mathbb{R}^D \longrightarrow \mathbb{R}$, their **convolution** is the function $\mathcal{I} * \mathcal{G} : \mathbb{R}^D \longrightarrow \mathbb{R}$ defined:

$$\mathcal{I} * \mathcal{G}(\mathbf{x}) \;=\; \int_{\mathbb{R}^D} \mathcal{I}(\mathbf{y}) \cdot \mathcal{G}(\mathbf{x} - \mathbf{y}) \; d\mathbf{y}$$

Note that $\mathcal{I} * \mathcal{G}$ is a function of $\mathbf{x}$. The variable $\mathbf{y}$ appears on the right hand side, but as an *integration* variable.

Consider the the D-dimensional Gauss-Weierstrass kernel:

$$\mathcal{G}(\mathbf{x};t) \quad = \quad \frac{1}{(4\pi t)^{D/2}} \exp\left(\frac{-\|\mathbf{x}\|^2}{4t}\right)$$

Let $\mathcal{G}_t(\mathbf{x}) = \mathcal{G}(\mathbf{x};t)$. We will treat $\mathcal{G}_t(x)$ as an *impulse-response function* to solve the D-dimensional Heat equation.

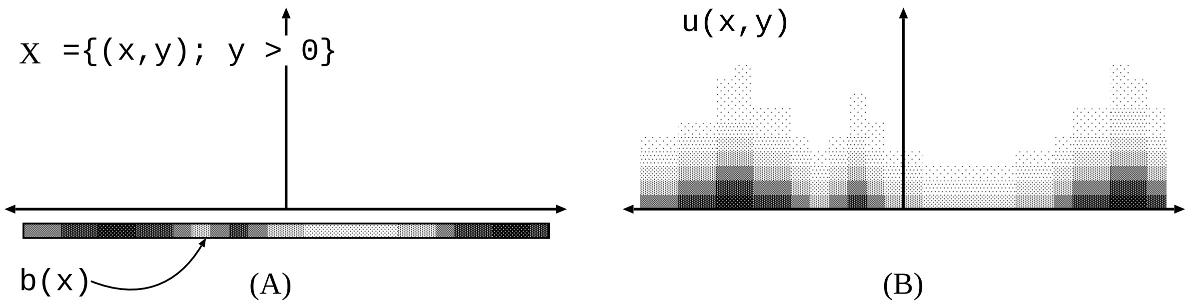


Figure 16.14: The Dirichlet problem on a half-plane.

Theorem 16.15: *Suppose $\mathcal{I} : \mathbb{R}^D \longrightarrow \mathbb{R}$ is a bounded continuous function. Define the function $u : \mathbb{R}^D \times [0, \infty)$ by:*

- $u(\mathbf{x}; 0) = \mathcal{I}(\mathbf{x})$.
- $u_t = \mathcal{I} * \mathcal{G}_t$, *for all* $t > 0$.

Then u is the continuous solution to the Heat equation on $\mathbb{R}^D$ with initial conditions $\mathcal{I}$.

Proof:

Claim 1: *$u(\mathbf{x}; t)$ is a solution to the D-dimensional Heat Equation.*

Proof: **Exercise 16.9** Hint: Combine Example (2c) on page 25 with Proposition 16.7(b) on page 306. $\diamond_{\text{Claim 1}}$

Claim 2: *$\mathcal{G}$ is an approximation of the identity on $\mathbb{R}^D$.*

Proof: **Exercise 16.10** $\diamond_{\text{Claim 2}}$

Now apply Corollary 16.8 on page 307 □

Because of Corollary 16.15, we say that $\mathcal{G}$ is the **fundamental solution** for the Heat equation.

16.4 Poisson's Solution (Dirichlet Problem on the Half-plane)

Prerequisites: §2.3, §6.5, §1.8, §16.2(a) **Recommended:** §16.1

Consider the **half-plane** domain $\mathbb{H} = \left\{(x, y) \in \mathbb{R}^2 \ ; \ y \geq 0\right\}$. The boundary of this domain is just the x axis: $\partial\mathbb{H} = \{(x, 0) \ ; \ x \in \mathbb{R}\}$. Thus, we impose boundary conditions by choosing some function $b(x)$ for $x \in \mathbb{R}$. Figure 16.14 illustrates the corresponding **Dirichlet problem**: find a function $u(x, y)$ for $(x, y) \in \mathbb{H}$ so that

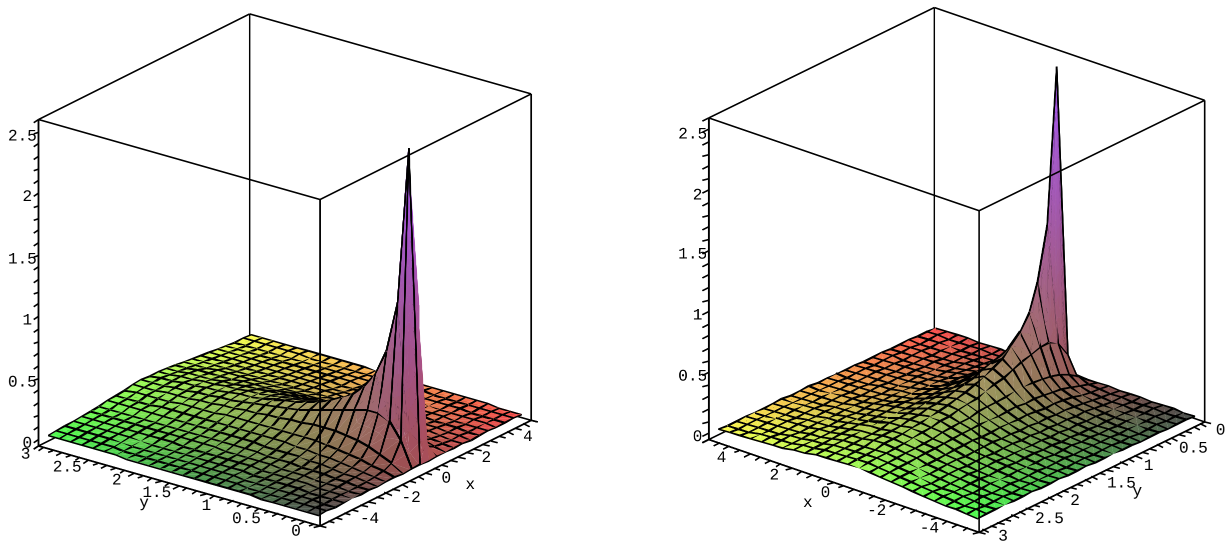


Figure 16.15: Two views of the Poisson kernel $\mathcal{K}_y(x)$.

1. u is *harmonic* —ie. u satisfies the Laplace equation: $\triangle u(x, y) = 0$ for all $x \in \mathbb{R}$ and $y > 0$.

2. u satisfies the *nonhomogeneous Dirichlet boundary condition*: $u(x, 0) = b(x)$, for all $x \in \mathbb{R}$.

Physical Interpretation: Imagine that $\mathbb{H}$ is an infinite 'ocean', so that $\partial\mathbb{H}$ is the beach. Imagine that $b(x)$ is the concentration of some chemical which has soaked into the sand of the beach. The harmonic function $u(x, y)$ on $\mathbb{H}$ describes the equilibrium concentration of this chemical, as it seeps from the sandy beach and diffuses into the water[3]. The boundary condition '$u(x, 0) = b(x)$' represents the chemical content of the sand. Note that $b(x)$ is constant in time; this represents the assumption that the chemical content of the sand is large compared to the amount seeping into the water; hence, we can assume the sand's chemical content remains effectively constant over time, as small amounts diffuse into the water.

We will solve the half-plane Dirichlet problem using the impulse-response method. For any $y > 0$, define the **Poisson kernel** $\mathcal{K}_y : \mathbb{R} \longrightarrow \mathbb{R}$ by:

$$\mathcal{K}_y(x) \quad = \quad \frac{y}{\pi(x^2 + y^2)}. \qquad \text{(Figure 16.15)} \qquad (16.18)$$

Observe that:

[3]Of course this an unrealistic model: in a *real* ocean, currents, wave action, and weather transport chemicals far more quickly than mere diffusion alone.

- $\mathcal{K}_y(x)$ is smooth for all $y > 0$ and $x \in \mathbb{R}$.
- $\mathcal{K}_y(x)$ has a singularity at $(0,0)$. That is: $\lim_{(x,y)\to(0,0)} \mathcal{K}_y(x) = \infty$,
- $\mathcal{K}_y(x)$ decays near infinity. That is, for any fixed $y > 0$, $\lim_{x\to\pm\infty} \mathcal{K}_y(x) = 0$, and also, for any fixed $x \in \mathbb{R}$, $\lim_{y\to\infty} \mathcal{K}_y(x) = 0$.

Thus, $\mathcal{K}_y(x)$ has the profile of an *impulse-response function* as described in § 16.1 on page 298. Heuristically speaking, you can think of $\mathcal{K}_y(x)$ as the solution to the Dirichlet problem on $\mathbb{H}$, with boundary condition $b(x) = \delta_0(x)$, where δ_0 is the infamous 'Dirac delta function'. In other words, $\mathcal{K}_y(x)$ is the equilibrium concentration of a chemical diffusing into the water from an 'infinite' concentration of chemical localized at a single point on the beach (say, a leaking barrel of toxic waste).

Proposition 16.16: Poisson Kernel Solution to Half-Plane Dirichlet problem

Let $b : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded, continuous, integrable function. Then the unique bounded, continuous solution $u : \mathbb{H} \longrightarrow \mathbb{R}$ to the corresponding Dirichlet problem is obtained as follows.

For all $x \in \mathbb{R}$ and $y > 0$, we define

$$u(x,y) \quad := \quad b * \mathcal{K}_y(x) \quad = \quad \int_{-\infty}^{\infty} b(z) \cdot \mathcal{K}_y(x-z)\, dz \quad = \quad \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{b(z)}{(x-z)^2 + y^2}\, dz.$$

while for all $x \in \mathbb{R}$ (with $y = 0$), we define $u(x,0) := b(x)$.

Proof: (sketch)

Claim 1: *Define $\mathcal{K}(x,y) = \mathcal{K}_y(x)$ for all $(x,y) \in \mathbb{H}$, except $(0,0)$. Then the function $\mathcal{K} : \mathbb{H} \longrightarrow \mathbb{R}$ is harmonic on the interior of $\mathbb{H}$.*

Proof: See Practice Problem # 14 on page 335 of §16.8. $\diamond_{\text{Claim 1}}$

Claim 2: *Thus, the function $u : \mathbb{H} \longrightarrow \mathbb{R}$ is harmonic on the interior of $\mathbb{H}$.*

Proof: <u>**Exercise 16.11**</u> Hint: Combine Claim 1 with Proposition 1.9 on page 18 $\diamond_{\text{Claim 2}}$

Recall that we defined u on the boundary of $\mathbb{H}$ by $u(x,0) = b(x)$. It remains to show that u is *continuous* when defined in this way.

Claim 3: *For any $x \in \mathbb{R}$, $\lim_{y\to 0} u(x,y) = b(x)$.*

Proof: <u>**Exercise 16.12**</u> Show that the kernel $\mathcal{K}_y$ is an *approximation of the identity* as $y \to 0$. Then apply Proposition 16.3 on page 303 to conclude that $\lim_{y\to 0} (b * \mathcal{K}_y)(x) = b(x)$ for all $x \in \mathbb{R}$. $\diamond_{\text{Claim 3}}$

Finally, this solution is unique by Theorem 6.14(a) on page 106. □

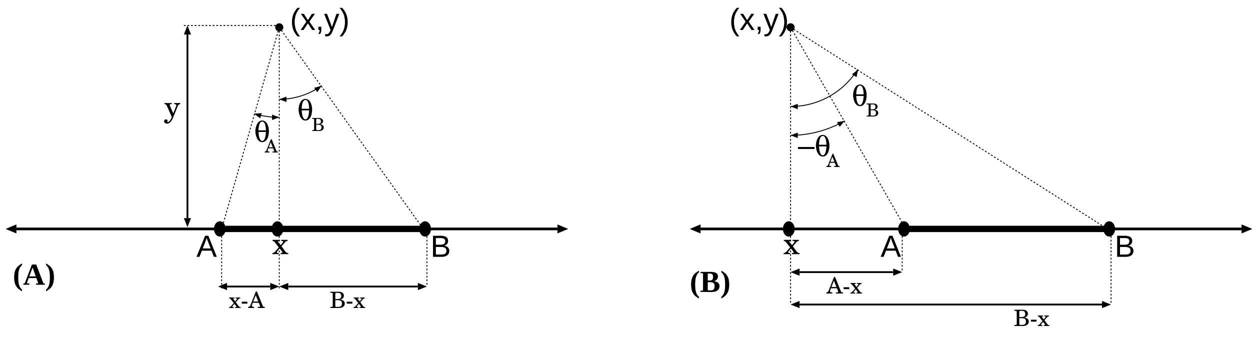


Figure 16.16: Example 16.17.

Example 16.17: Let $A < B$ be real numbers, and suppose $b(x) = \begin{cases} 1 & \text{if } A < x < B; \\ 0 & \text{otherwise.} \end{cases}$.

Then Proposition 18.19 yields solution

$$U(x,y) \underset{\text{(P18.19)}}{=\!=\!=} b * \mathcal{K}_y(x) \underset{\text{(e18.4)}}{=\!=\!=} \frac{y}{\pi}\int_A^B \frac{1}{(x-z)^2+y^2}\,dz \underset{\text{(S)}}{=\!=} \frac{y^2}{\pi}\int_{\frac{A-x}{y}}^{\frac{B-x}{y}} \frac{1}{y^2w^2+y^2}\,dw$$
$$= \frac{1}{\pi}\int_{\frac{A-x}{y}}^{\frac{B-x}{y}} \frac{1}{w^2+1}\,dw = \frac{1}{\pi}\arctan(w)\Big|_{w=\frac{A-x}{y}}^{w=\frac{B-x}{y}}$$
$$= \frac{1}{\pi}\arctan\left(\frac{B-x}{y}\right) - \arctan\left(\frac{A-x}{y}\right) \underset{\text{(T)}}{=\!=} \frac{1}{\pi}\Big(\theta_B - \theta_A\Big),$$

where θ_B and θ_A are as in Figure 16.16. Here, **(P18.19)** is Prop. 18.19; **(e18.4)** is eqn.(18.4); **(S)** is the substitution $w = \frac{z-x}{y}$, so that $dw = \frac{1}{y}\,dz$ and $dz = y\,dw$; and **(T)** follows from elementary trigonometry.

Note that, if $A < x$ (as in Fig. 16.16A), then $A - x < 0$, so θ_A is negative, so that $U(x,y) = \frac{1}{\pi}\Big(\theta_B + |\theta_A|\Big)$. If $A > x$, then we have the situation in Fig. 16.16B. In either case, the interpretation is the same:

$$U(x,y) = \frac{1}{\pi}\Big(\theta_B - \theta_A\Big) = \frac{1}{\pi}\left(\begin{array}{l}\textit{the angle subtended by the interval } [A,B], \textit{ as} \\ \textit{seen by an observer standing at the point } (x,y)\end{array}\right).$$

This is reasonable, because if this observer moves far away from the interval $[A,B]$, or views it at an acute angle, then the subtended angle $\Big(\theta_B - \theta_A\Big)$ will become small —hence, the value of $U(x,y)$ will also become small. ♢

Remark: We will revisit the Poisson kernel solution to the half-plane Dirichlet problem in § 18.3(b) on page 365, where we will prove Proposition 16.16 using Fourier transform methods.

Siméon Denis Poisson
Born: June 21, 1781 in Pithiviers, France
Died: April 25, 1840 in Sceaux (near Paris)

16.5 (∗) Properties of Convolution

Prerequisites: §16.1 **Recommended:** §16.3

We have introduced the convolution operator to solve the Heat Equation, but it is actually ubiquitous, not only in the theory of PDEs, but in other areas of mathematics, especially probability theory and group representation theory. We can define an *algebra* of functions using the operations of convolution and addition; this algebra is as natural as the one you would form using 'normal' multiplication and addition[4]

Proposition 16.18: Algebraic Properties of Convolution

Let $f, g, h : \mathbb{R}^D \longrightarrow \mathbb{R}$ be integrable functions. Then the convolutions of f, g, and h have the following relations:

Commutativity: $f * g \;=\; g * f$.

Associativity: $f * (g * h) \;=\; (f * g) * h$.

Distribution: $f * (g + h) \;=\; (f * g) + (f * h)$.

Linearity: $f * (r \cdot g) \;=\; r \cdot (f * g)$ *for any constant* $r \in \mathbb{R}$.

Proof: **Commutativity** is just Proposition 16.1. In the case $D = 1$, the proofs of the other three properties are Practice Problems #1 and #2 in §16.8. The proofs for $D \geq 2$ are **Exercise 16.13** . □

[4]Indeed, in a sense, it is *the same* algebra, seen through the prism of the Fourier transform; see § 17 on page 337.

Remark: Let $\mathbf{L}^1(\mathbb{R}^D)$ be the set of all integrable functions on $\mathbb{R}^D$. The properties of **Commutativity**, **Distribution**, and **Distribution** mean that the set $\mathbf{L}^1(\mathbb{R}^D)$, together with the operations '+' (pointwise addition) and '$*$' (convolution), is a *ring* (in the language of abstract algebra). This, together with **Linearity**, makes $\mathbf{L}^1(\mathbb{R}^D)$ an *algebra* over $\mathbb{R}$.

Example 16.12 on page 311 exemplifies the extremely convenient "smoothing" properties of convolution. Basically, if we convolve a "rough" function with a "smooth" function, then this action "smooths out" the rough function.

Proposition 16.19: Regularity Properties of Convolution

Let $f, g : \mathbb{R}^D \longrightarrow \mathbb{R}$ be integrable functions.

(a) *If f is continuous, then so is $f * g$ (regardless of whether g is.)*

(b) *If f is differentiable, then so is $f * g$. Furthermore, $\partial_d(f * g) \;=\; (\partial_d f) * g$.*

(c) *If f is N times differentiable, then so is $f * g$, and*

$$\partial_1^{n_1}\partial_2^{n_2}\ldots\partial_D^{n_D}(f * g) \quad = \quad \left(\partial_1^{n_1}\partial_2^{n_2}\ldots\partial_D^{n_D} f\right) * g,$$

for any $n_1, n_2, \ldots, n_D$ so that $n_1 + \ldots + n_D \leq N$.

(d) *More generally, if $\mathcal{L}$ is any linear differential operator of degree N or less, with constant coefficients, then $\mathcal{L}(f * g) \;=\; (\mathcal{L} f) * g$.*

(e) *Thus, if f is a solution to the homogeneous linear equation "$\mathcal{L} f = 0$", then so is $f * g$.*

(f) *If f is infinitely differentiable, then so is $f * g$.*

Proof: **Exercise 16.14** □

This has a convenient consequence: any function, no matter how "rough", can be approximated arbitrarily closely by smooth functions.

Proposition 16.20: *Suppose $f : \mathbb{R}^D \longrightarrow \mathbb{R}$ is integrable. Then there is a sequence $f_1, f_2, f_3, \ldots$ of infinitely differentiable functions which converges pointwise to f. In other words, for every $\mathbf{x} \in \mathbb{R}^D$, $\lim_{n\to\infty} f_n(\mathbf{x}) = f(\mathbf{x})$.*

Proof: **Exercise 16.15** Hint: Use the fact that the Gauss-Weierstrass kernel is infinitely differentiable, and is also an approximation of identity. Then use **Part 6** of the previous theorem. □

Remark: We have formulated this result in terms of *pointwise* convergence, but similar results hold for $\mathbf{L}^2$ convergence, $\mathbf{L}^1$ convergence, uniform convergence, etc. We're neglecting these to avoid technicalities.

16.6 d'Alembert's Solution (One-dimensional Wave Equation)

d'Alembert's method provides a solution to the one-dimensional wave equation with any initial conditions, using combinations of *travelling waves* and *ripples*. First we'll discuss this in the infinite domain $\mathbb{X} = \mathbb{R}$ is infinite in length; then we'll consider a finite domain like $\mathbb{X} = [a, b]$.

16.6(a) Unbounded Domain

Prerequisites: §3.2(a) **Recommended:** §16.1

Consider the one-dimensional **wave equation**

$$\partial_t^2 u(x,t) \ = \ \triangle u(x,t) \tag{16.19}$$

where x is a point in a one-dimensional domain $\mathbb{X}$; thus $\triangle u(x,t) = \partial_x^2 u(x,t)$. If $\mathbb{X} = [0, L]$, you can imagine acoustic vibrations in a violin string. If $\mathbb{X} = \mathbb{R}$ you can can imagine electrical waves propagating through a (very long) copper wire.

Lemma 16.21: (Travelling Wave Solution)

Let $f_0 : \mathbb{R} \longrightarrow \mathbb{R}$ be any twice-differentiable function. For any $x \in \mathbb{R}$ and any $t \geq 0$, let $w_L(x,t) = f_0(x+t)$ and $w_R(x,t) = f_0(x-t)$ (see Figure 16.17). Then w_L and w_R are solutions to the Wave Equation, with

$$\begin{array}{rrcccl} \textbf{Initial Position:} & w_L(x,0) & = & f_0(x) & = & w_R(x,0), \\ \textbf{Initial Velocities:} & \partial_t w_L(x,0) & = & f_0'(x); & & \partial_t w_R(x,0) \ = \ -f_0'(x). \end{array}$$

Thus, if we define $w(x,t) = \frac{1}{2}\Big(w_L(x,t) + w_R(x,t)\Big)$, then w is the unique solution to the Wave Equation, with **Initial Position** $w(x,0) = f_0(x)$ *and* **Initial Velocity** $\partial_t w(x,0) = 0$.

Proof: See Practice Problem #5 in §16.8. □

Physically, w_L represents a leftwards-travelling wave; basically you take a copy of the function f_0 and just rigidly translate it to the left. Similarly, w_R represents a rightwards-travelling wave.

Remark: Naïvely, it seems that $w_L(x,t) = f_0(x+t)$ should be a *rightwards* travelling wave, while w_R should be *leftwards* travelling wave. Yet the opposite is true. Think about this until you understand it. It may be helpful to do the following: Let $f_0(x) = x^2$. Plot $f_0(x)$, and then plot $w_L(x,5) = f(x+5) = (x+5)^2$. Observe the 'motion' of the parabola.

Jean Le Rond d'Alembert
Born: November 17, 1717 in Paris
Died: October 29, 1783 in Paris

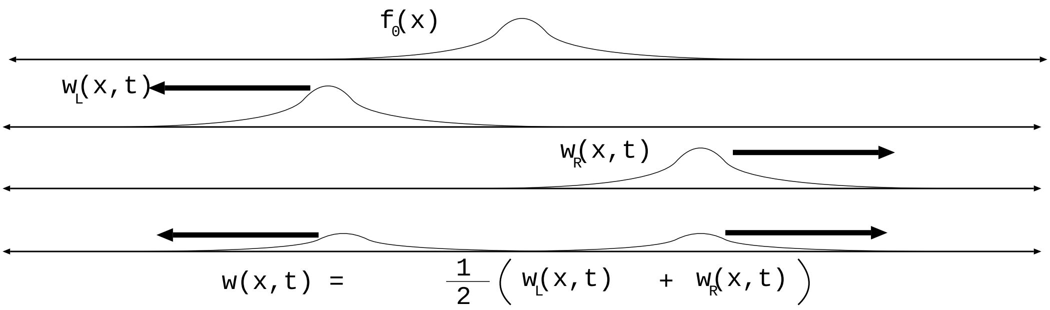


Figure 16.17: The d'Alembert travelling wave solution; $f_0(x) = \frac{1}{x^2+1}$ from Example 2a.

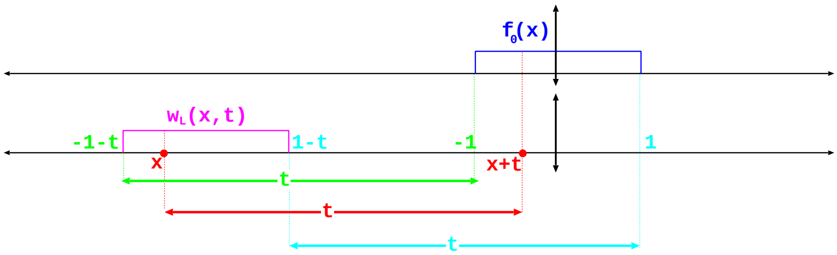


Figure 16.18: The travelling box wave $w_{_L}(x,t) = f_0(x+t)$ from Example 2c.

Example 16.22:

(a) If $f_0(x) = \frac{1}{x^2+1}$, then $w(x) = \frac{1}{2}\left(\frac{1}{(x+t)^2+1} + \frac{1}{(x-t)^2+1}\right)$ (Figure 16.17)

(b) If $f_0(x) = \sin(x)$, then

$$\begin{aligned} w(x;t) &= \frac{1}{2}\Big(\sin(x+t) \ + \ \sin(x-t)\Big) \\ &= \frac{1}{2}\Big(\sin(x)\cos(t)+\cos(x)\sin(t) \ + \ \sin(x)\cos(t)-\cos(x)\sin(t)\Big) \\ &= \frac{1}{2}\Big(2\sin(x)\cos(t)\Big) \quad = \quad \cos(t)\sin(x), \end{aligned}$$

In other words, two sinusoidal waves, traveling in opposite directions, when superposed, result in a sinusoidal *standing* wave.

(c) (see Figure 16.18) Suppose $f_0(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$. Then:

$$w_{_L}(x,t) = f_0(x+t) = \begin{cases} 1 & \text{if } -1 < x+t < 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } -1-t < x < 1-t; \\ 0 & \text{otherwise.} \end{cases}$$

(Notice that the solutions $w_{_L}$ and $w_{_R}$ are continuous (or differentiable) only when f_0 is continuous (or differentiable). But the formulae of Lemma 16.21 make sense even when the original Wave Equation itself ceases to make sense, as in Example (c). This is an example of a *generalized solution* of the Wave equation.) ◇

Lemma 16.23: (Ripple Solution)

Let $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. For any $x \in \mathbb{R}$ and any $t \geq 0$, define $v(x,t) = \frac{1}{2}\int_{x-t}^{x+t} f_1(y)\, dy$. Then v is the unique continuous solution to the Wave Equation, with

Initial Position: $v(x,0) = 0$; **Initial Velocity:** $\partial_t v(x,0) = f_1(x)$.

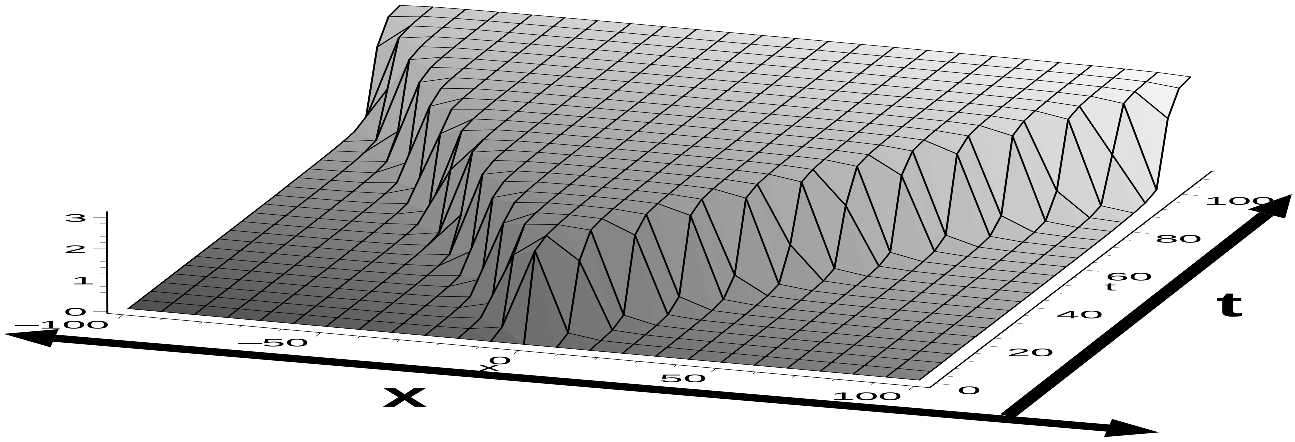


Figure 16.19: The ripple solution with initial velocity $f_1(x) = \frac{1}{1+x^2}$. (see Example 2a)

Proof: See Practice Problem #6 in §16.8. □

Physically, v represents a "ripple". You can imagine that f_1 describes the energy profile of an "impulse" which is imparted into the vibrating medium at time zero; this energy propagates outwards, leaving a disturbance in its wake (see Figure 16.21)

Example 16.24:

(a) If $f_1(x) = \dfrac{1}{1+x^2}$, then the d'Alembert solution to the initial velocity problem is

$$\begin{aligned} v(x,t) &= \frac{1}{2}\int_{x-t}^{x+t} \overline{f}_1(y)\ dy \quad = \quad \frac{1}{2}\int_{x-t}^{x+t} \frac{1}{1+y^2}\ dy \\ &= \frac{1}{2}\arctan(y)\Big|_{y=x-t}^{y=x+t} \quad = \quad \frac{1}{2}\Big(\arctan(x+t) - \arctan(x-t)\Big). \end{aligned}$$

(see Figure 16.19).

(b) If $f_1(x) = \cos(x)$, then

$$\begin{aligned} v(x,t) &= \frac{1}{2}\int_{x-t}^{x+t} \cos(y)\ dy \quad = \quad \frac{1}{2}\Big(\sin(x+t) - \sin(x-t)\Big) \\ &= \frac{1}{2}\Big(\sin(x)\cos(t) + \cos(x)\sin(t) \ - \ \sin(x)\cos(t) + \cos(x)\sin(t)\Big) \\ &= \frac{1}{2}\Big(2\cos(x)\sin(t)\Big) \quad = \quad \sin(t)\cos(x). \end{aligned}$$

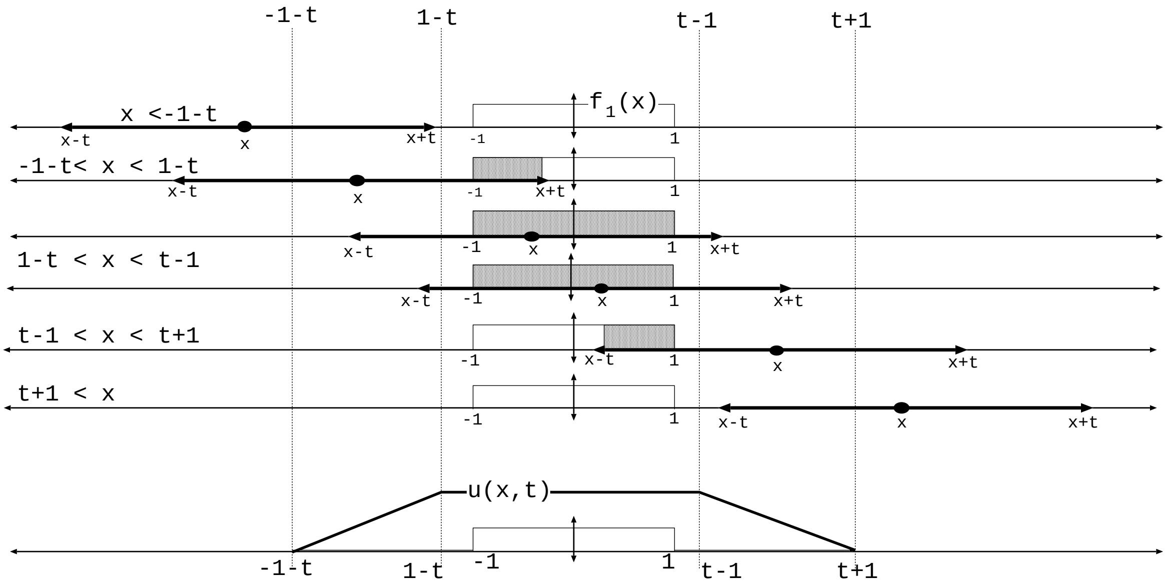


Figure 16.20: The d'Alembert ripple solution from Example 2c, evaluated for various $x \in \mathbb{R}$, assuming $t > 2$.

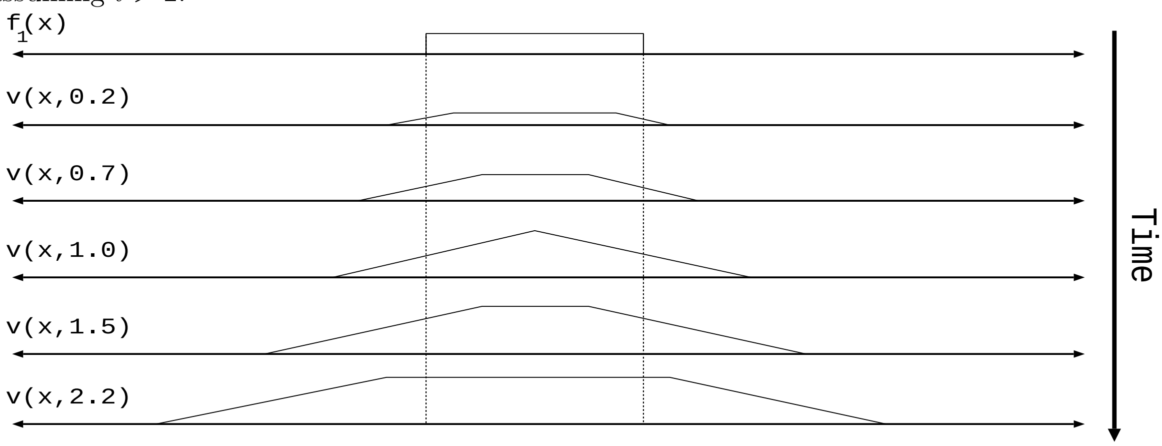


Figure 16.21: The d'Alembert ripple solution from Example 2c, evolving in time.

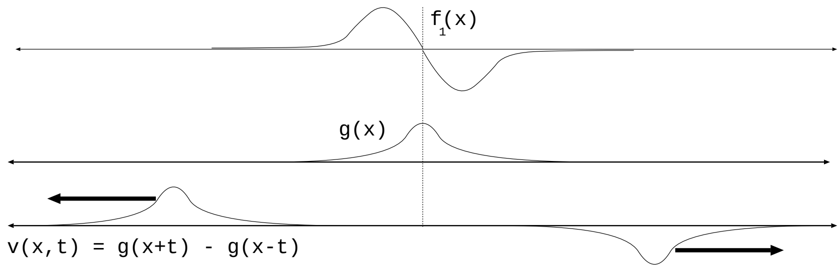


Figure 16.22: The ripple solution with initial velocity: $f_1(x) = \frac{-2x}{(x^2+1)^2}$ (Example 2d).

(c) Let $f_1(x) = \begin{cases} 2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ (Figures 16.20 and 16.21). If $t > 2$, then $v(x,t) =$

$$\begin{cases} 0 & \text{if } x+t < -1; \\ x+t+1 & \text{if } -1 \leq x+t < 1; \\ 2 & \text{if } x-t \leq -1 < 1 \leq x+t; \\ t+1-x & \text{if } -1 \leq x-t < 1; \\ 0 & \text{if } 1 \leq x-t. \end{cases} = \begin{cases} 0 & \text{if } x < -1-t; \\ x+t+1 & \text{if } -1-t \leq x < 1-t; \\ 2 & \text{if } 1-t \leq x < t-1; \\ t+1-x & \text{if } t-1 \leq x < t+1; \\ 0 & \text{if } t+1 \leq x. \end{cases}$$

Exercise 16.16 Verify this formula. Find a similar formula for when $t < 2$.

Notice that, in this example, the wave of displacement propagates outwards through the medium, and the medium *remains displaced.* The model contains no "restoring force" which would cause the displacement to return to zero.

(d) If $f_1(x) = \dfrac{-2x}{(x^2+1)^2}$, then $g(x) = \dfrac{1}{x^2+1}$, and $v(x) = \dfrac{1}{2}\left(\dfrac{1}{(x+t)^2+1} - \dfrac{1}{(x-t)^2+1}\right)$ (see Figure 16.22) ♢

Remark: If $g : \mathbb{R} \longrightarrow \mathbb{R}$ is an antiderivative of f_1 (ie. $g'(x) = f_1(x)$, then $v(x,t) = g(x+t) - g(x-t)$. Thus, the d'Alembert "ripple" solution looks like the d'Alembert "travelling wave" solution, but with the rightward travelling wave being vertically *inverted.*

Exercise 16.17 Express the d'Alembert "ripple" solution as a *convolution*, as described in § 16.1 on page 298. **Hint:** Find an impulse-response function $\Gamma_t(x)$, such that $f_1 * \Gamma_t(x) = \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} f_1(y)\, dy$.

Proposition 16.25: (d'Alembert Solution on an infinite wire)

Let $f_0 : \mathbb{R} \longrightarrow \mathbb{R}$ be twice-differentiable, and $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable. For any $x \in \mathbb{R}$ and $t \geq 0$, define $u(x,t)$ by:

$$u(x,t) = \frac{1}{2}\Big(w_L(x,t) \quad + \quad w_R(x,t)\Big) \quad + \quad v(x,t)$$

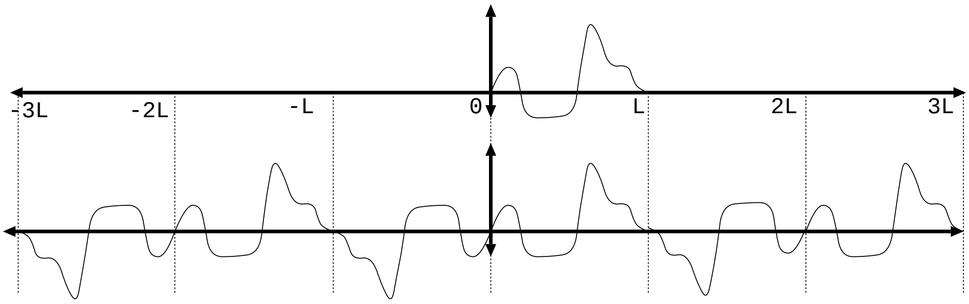


Figure 16.23: The odd periodic extension.

where w_L, w_R, and v are as in Lemmas 16.21 and 16.23. Then $u(x,t)$ satisfies the Wave Equation, with

$$\textbf{Initial Position:}\quad v(x,0) = f_0(x); \qquad \textbf{Initial Velocity:}\quad \partial_t v(x,0) = f_1(x).$$

Furthermore, all *solutions to the Wave Equation with these initial conditions are of this type.*

Proof: This follows from Lemmas 16.21 and 16.23. □

Remark: There is no nice extension of the d'Alembert solution in higher dimensions. The closest analogy is Poisson's **spherical mean** solution to the three-dimensional wave equation in free space, which is discussed in § 18.2(b) on page 361.

16.6(b) Bounded Domain

Prerequisites: §16.6(a), §1.5, §6.5(a)

The d'Alembert solution in §16.6(a) works fine if $\mathbb{X} = \mathbb{R}$, but what if $\mathbb{X} = [0, L)$? We must "extend" the initial conditions in some way. If $f : [0, L) \longrightarrow \mathbb{R}$ is any function, then an **extension** of f is any function $\overline{f} : \mathbb{R} \longrightarrow \mathbb{R}$ so that $\overline{f}(x) = f(x)$ whenever $0 \leq x \leq L$. If f is continuous and differentiable, then we normally require its extension to also be continuous and differentiable.

The extension we want is the **odd, L-periodic extension**.

We want $\overline{f}$ to satisfy the following (see Figure 16.23):

1. $\overline{f}(x) = f(x)$ whenever $0 \leq x \leq L$.

2. $\overline{f}$ is an *odd* function, meaning: $\overline{f}(-x) = -\overline{f}(x)$.

3. $\overline{f}$ is *L-periodic*, meaning $\overline{f}(x + L) = \overline{f}(x)$

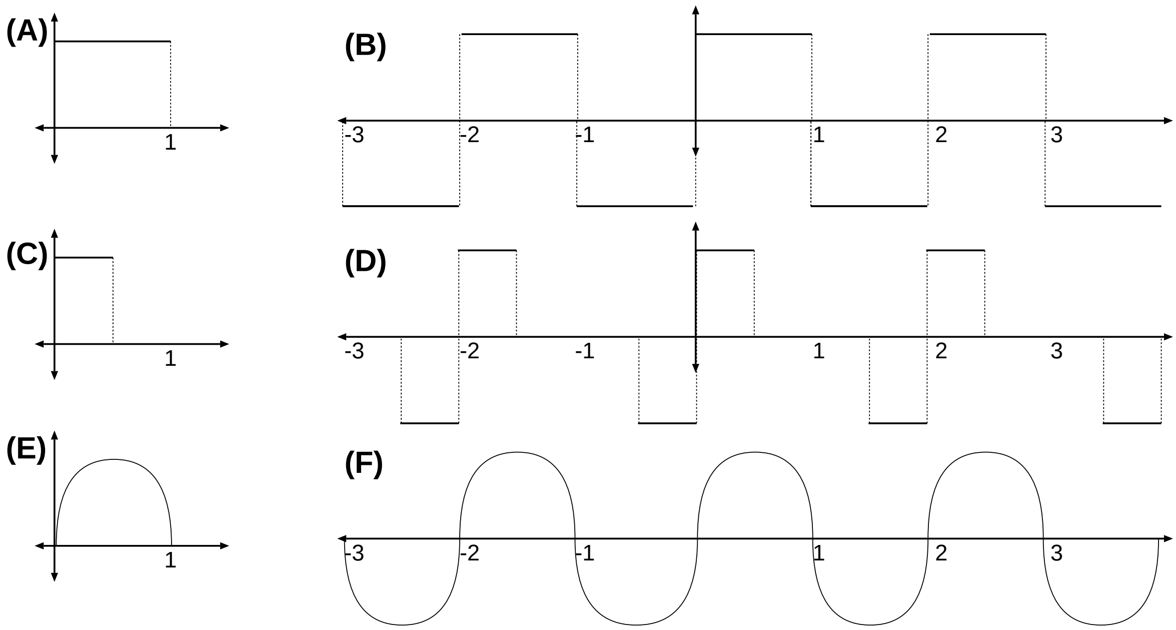


Figure 16.24: The odd, $2L$-periodic extension.

Example 16.26:

(a) Suppose $L = 1$, and $f(x) = 1$ for all $x \in [0, 1)$ (Figure 16.24A). Then the odd, 2-periodic extension is defined:

$$\overline{f}(x) = \begin{cases} 1 & \text{if } x \in \ldots \cup [-2, -1) \cup [0, 1) \cup [2, 3) \cup \ldots \\ -1 & \text{if } x \in \ldots \cup [-1, 0) \cup [1, 2) \cup [3, 4) \cup \ldots \end{cases} \qquad \text{(Figure 16.24B)}$$

(b) Suppose $L = 1$, and $f(x) = \begin{cases} 1 & \text{if } x \in \left[0, \frac{1}{2}\right) \\ 0 & \text{if } x \in \left[\frac{1}{2}, 1\right) \end{cases}$ (Figure 16.24C). Then the odd, 2-periodic extension is defined:

$$\overline{f}(x) = \begin{cases} 1 & \text{if } x \in \ldots \cup \left[-2, -1\frac{1}{2}\right) \cup \left[0, \frac{1}{2}\right) \cup \left[2, 2\frac{1}{2}\right) \cup \ldots \\ -1 & \text{if } x \in \ldots \cup \left[-\frac{1}{2}, 0\right) \cup \left[1\frac{1}{2}, 2\right) \cup \left[3\frac{1}{2}, 4\right) \cup \ldots \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Figure 16.24D)}$$

(c) Suppose $L = \pi$, and $f(x) = \sin(x)$ for all $x \in [0, \pi)$ (Figure 16.24E) Then the odd, 2π-periodic extension is given by $\overline{f}(x) = \sin(x)$ for all $x \in \mathbb{R}$ (Figure 16.24E).

Exercise 16.18 Verify this. $\diamondsuit$

There's a general formula for the odd periodic extension (although it often isn't very useful):

Proposition 16.27: *Let $f : [0, L) \longrightarrow \mathbb{R}$ be any function*

(a) *The odd, $2L$-periodic extension of f is defined:*

$$\overline{f}(x) = \begin{cases} f(x) & \text{if} \quad 0 \leq x < L \\ -f(-x) & \text{if} \quad -L \leq x < 0 \\ f(x-2nL) & \text{if} \quad 2nL \leq x \leq (2n+1)L, \quad \text{for some } n \\ f(2nL-x) & \text{if} \quad (2n-1)L \leq x \leq 2nL, \quad \text{for some } n \end{cases}$$

(b) $\overline{f}$ is continuous *at 0, L, $2L$ etc. if and only if $f(0) = f(L) = 0$.*

(c) $\overline{f}$ is differentiable *at 0, L, $2L$, etc. if and only if $\overline{f}$ is continuous (as in part* **(b)***) and if, in addition, $f'(0) = f'(L)$.*

Proof: <u>**Exercise 16.19**</u> □

Proposition 16.28: (d'Alembert Solution on a violin string)

Let $f_0 : [0, L) \longrightarrow \mathbb{R}$ and $f_1 : [0, L) \longrightarrow \mathbb{R}$ be functions, and let their odd periodic extensions be $\overline{f}_0 : \mathbb{R} \longrightarrow \mathbb{R}$ and $\overline{f}_1 : \mathbb{R} \longrightarrow \mathbb{R}$.

(a) *Define $w(x, t)$ by:*

$$w(x,t) = \frac{1}{2}\left(\overline{f}_0(x-t) + \overline{f}_0(x+t)\right)$$

Then $w(x, t)$ is the unique solution to the Wave Equation with **initial conditions:**

$$w(x, 0) = f_0(x) \qquad \text{and} \qquad \partial_t\, w(x, 0) = 0, \qquad \text{for all } x \in [0, L],$$

and **homogeneous Dirichlet boundary conditions:**

$$w(0, t) = 0 = w(L, t), \qquad \text{for all } t \geq 0.$$

Also, w is continuous *if and only if f_0 itself satisfies homogeneous Dirichlet boundary conditions, and* differentiable *if and only if $f_0'(0) = f_0'(L)$.*

(b) *Define $v(x, t)$ by:*

$$v(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \overline{f}_1(y)\, dy$$

Then $v(x, t)$ is the unique solution to the Wave Equation with **initial conditions:**

$$v(x, 0) = 0 \qquad \text{and} \qquad \partial_t\, v(x, 0) = f_1(x), \qquad \text{for all } x \in [0, L],$$

and **homogeneous Dirichlet boundary conditions:**

$$v(0, t) = 0 = v(L, t), \qquad \text{for all } t \geq 0.$$

v is always continuous*, but v is* differentiable *if and only if f_1 satisfies homogeneous Dirichlet boundary conditions.*

(c) *Let* $u(x,t) = w(x,t) + v(x,t)$*. Then* $u(x,t)$ *is the unique solution to the Wave Equation with* **initial conditions:**

$$u(x,0) = f_0(x) \quad \text{and} \quad \partial_t u(x,0) = f_1(x), \qquad \text{for all } x \in [0,L],$$

and **homogeneous Dirichlet boundary conditions:**

$$u(0,t) = 0 = u(L,t), \qquad \text{for all } t \geq 0.$$

Furthermore, u *is continuous iff* f_0 *satisfies homogeneous Dirichlet conditions, and differentiable iff* f_1 *also satisfies homogeneous Dirichlet conditions, and* $f_0'(0) = f_0'(L)$*.*

Proof: The fact that u, w, and v are solutions to their respective initial value problems follows from the two lemmas. The conditions for continuity/differentiability follow from the fact that we are using the odd periodic extension.

Exercise 16.20 Prove that these solutions satisfy homogeneous Dirichlet conditions.

Exercise 16.21 Proof that these solutions are *unique*. □

16.7 Poisson's Solution (Dirichlet Problem on the Disk)

Prerequisites: §2.3, §1.6(b), §6.5, §1.8 **Recommended:** §16.1, §14.2(e)

Let $\mathbb{D} = \left\{(x,y) \in \mathbb{R}^2 \; ; \; \sqrt{x^2+y^2} \leq R\right\}$ be the **disk** of radius R in $\mathbb{R}^2$. Thus, $\mathbb{D}$ has boundary $\partial\mathbb{D} = \mathbb{S} = \left\{(x,y) \in \mathbb{R}^2 \; ; \; \sqrt{x^2+y^2} = R\right\}$ (the circle of radius R). Suppose $b : \partial\mathbb{D} \longrightarrow \mathbb{R}$ is some function on the boundary. The **Dirichlet problem** on $\mathbb{D}$ asks for a function $u : \mathbb{D} \longrightarrow \mathbb{R}$ such that:

- u is *harmonic*—ie. u satisfies the Laplace equation $\triangle u \equiv 0$.
- u satisfies the *nonhomogeneous Dirichlet Boundary Condition* $u(x,y) = b(x,y)$ for all $(x,y) \in \partial\mathbb{D}$.

If $u(x,y)$ represents the concentration of some chemical diffusing in from the boundary, then the value of $u(x,y)$ at any point (x,y) in the interior of the disk should represent some sort of 'average' of the chemical reaching (x,y) from all points on the boundary. This is the inspiration of *Poisson's Solution*. We define the **Poisson kernel** $\mathcal{P} : \mathbb{D} \times \mathbb{S} \longrightarrow \mathbb{R}$ as follows:

$$\mathcal{P}(\mathbf{x},\mathbf{s}) \quad = \quad \frac{R^2 - \|\mathbf{x}\|^2}{\|\mathbf{x}-\mathbf{s}\|^2}, \qquad \text{for all } \mathbf{x} \in \mathbb{D} \text{ and } \mathbf{s} \in \mathbb{S}.$$

As shown in Figure 16.25(A), the denominator, $\|\mathbf{x}-\mathbf{s}\|^2$, is just the squared-distance from $\mathbf{x}$ to $\mathbf{s}$. The numerator, $R^2 - \|\mathbf{x}\|^2$, roughly measures the distance from $\mathbf{x}$ to the boundary $\mathbb{S}$; if $\mathbf{x}$ is close to $\mathbb{S}$, then $R^2 - \|\mathbf{x}\|^2$ becomes very small. Intuitively speaking, $\mathcal{P}(\mathbf{x},\mathbf{s})$ measures the 'influence' of the boundary condition at the point $\mathbf{s}$ on the value of u at $\mathbf{x}$; see Figure 16.26.

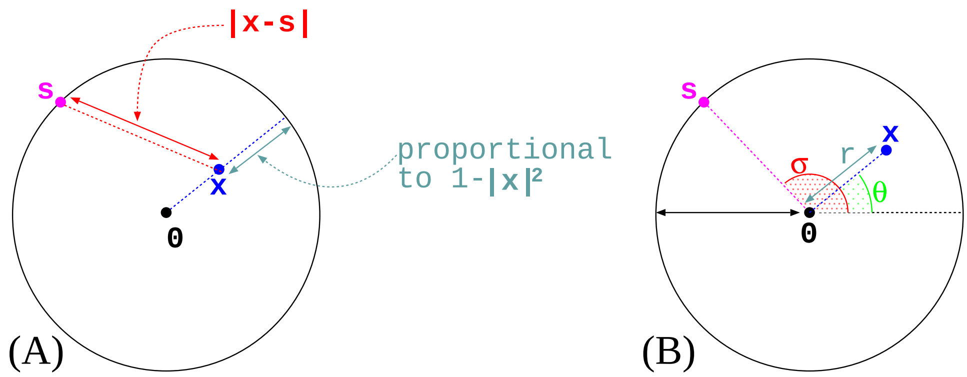


Figure 16.25: The Poisson kernel

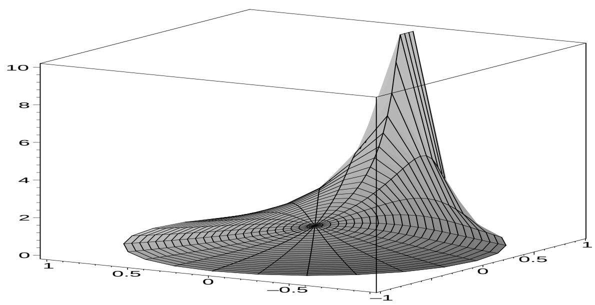


Figure 16.26: The Poisson kernel $\mathcal{P}(\mathbf{x}, \mathbf{s})$ as a function of $\mathbf{x}$. (for some fixed value of $\mathbf{s}$). This surface illustrates the 'influence' of the boundary condition at the point $\mathbf{s}$ on the point $\mathbf{x}$. (The point $\mathbf{s}$ is located at the 'peak' of the surface.)

In polar coordinates (Figure 16.25B), we can parameterize $\mathbf{s} \in \mathbb{S}$ with a single angular coordinate $\sigma \in [-\pi, \pi)$, so that $\mathbf{s} = \Big(R\cos(\sigma),\ R\sin(\sigma)\Big)$. If $\mathbf{x}$ has coordinates (x, y), then Poisson's kernel takes the form:

$$\mathcal{P}(\mathbf{x}, \mathbf{s}) \;=\; \mathcal{P}_\sigma(x, y) \;=\; \frac{R^2 - x^2 - y^2}{\big(x - R\cos(\sigma)\big)^2 \;+\; \big(y - R\sin(\sigma)\big)^2}$$

Proposition 16.29: Poisson's Integral Formula

Let $\mathbb{D} = \left\{(x, y)\ ;\ x^2 + y^2 \leq R^2\right\}$ be the disk of radius R, and let $b : \partial\mathbb{D} \longrightarrow \mathbb{R}$ be continuous. The unique continuous, **bounded** *solution to the corresponding Dirichlet problem is given:*

For any (x, y) on the interior *of $\mathbb{D}$*
$$u(x, y) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} b(\sigma)\cdot\mathcal{P}_\sigma(x, y)\; d\sigma, \qquad (16.20)$$

while, for $(x, y) \in \partial\mathbb{D}$, we define $u(x, y) = b(x, y)$.

More abstractly, for any $\mathbf{x} \in \mathbb{D}$, $u(\mathbf{x}) \quad = \quad \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{S}} b(\mathbf{s}) \cdot \mathcal{P}(\mathbf{x}, \mathbf{s}) \; d\mathbf{s} & \textit{if} \quad \|x\| < R; \\ b(\mathbf{x}) & \textit{if} \quad \|x\| = R. \end{cases}$

Proof: (sketch) For simplicity, assume $R = 1$ (the proof for $R \neq 1$ is similar). Thus,

$$\mathcal{P}_\sigma(x, y) \quad = \quad \frac{1 - x^2 - y^2}{(x - \cos(\sigma))^2 + (y - \sin(\sigma))^2}$$

Claim 1: *Fix* $\sigma \in [-\pi, \pi)$. *The function* $\mathcal{P}_\sigma : \mathbb{D} \longrightarrow \mathbb{R}$ *is harmonic on the interior of* $\mathbb{D}$.

Proof: Exercise 16.22 $\diamond_{\text{Claim 1}}$

Claim 2: *Thus, the function* $u(x, y)$ *is harmonic on the interior of* $\mathbb{D}$.

Proof: Exercise 16.23 Hint: Combine Claim 1 with Proposition 1.9 on page 18 $\diamond_{\text{Claim 2}}$

Recall that we defined u on the boundary $\mathbb{S}$ of $\mathbb{D}$ by $u(\mathbf{s}) = b(\mathbf{s})$. It remains to show that u is *continuous* when defined in this way.

Claim 3: *For any* $\mathbf{s} \in \mathbb{S}$, $\displaystyle\lim_{(x,y)\to\mathbf{s}} u(x, y) \quad = \quad b(\mathbf{s})$.

Proof: Exercise 16.24 (Hard)

Hint: Write (x, y) in polar coordinates as (r, θ). Thus, our claim becomes $\displaystyle\lim_{\theta\to\sigma} \lim_{r\to 1} u(r, \theta) \quad = \quad b(\sigma)$.

(a) Show that $\mathcal{P}_\sigma(x, y) \quad = \quad \mathcal{P}_r(\theta - \sigma)$, where, for any $r \in [0, 1)$, we define

$$\mathcal{P}_r(\phi) \quad = \quad \frac{1 - r^2}{1 - 2r\cos(\phi) + r^2}, \quad \text{for all } \phi \in [-\pi, \pi).$$

(b) Thus, $u(r, \theta) \quad = \quad \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} b(\sigma) \cdot \mathcal{P}_r(\theta - \sigma) \; d\sigma$ is a sort of 'convolution on a circle'. We can write this: $u(r, \theta) \quad = \quad (b \star \mathcal{P}_r)(\theta)$.

(c) Show that the function $\mathcal{P}_r$ is an 'approximation of the identity' as $r \to 1$, meaning that, for any continuous function $b : \mathbb{S} \longrightarrow \mathbb{R}$, $\displaystyle\lim_{r\to 1} (b \star \mathcal{P}_r)(\theta) \quad = \quad b(\theta)$. For your proof, borrow from the proof of Proposition 16.3 on page 303 $\diamond_{\text{Claim 3}}$

Finally, this solution is unique by Theorem 6.14(a) on page 106. □

Remark: The Poisson solution to the Dirichlet problem on a disk is revisited in § 14.2(e) on page 250 using the methods of polar-separated harmonic functions.

16.8 Practice Problems

1. Let $f, g, h : \mathbb{R} \longrightarrow \mathbb{R}$ be integrable functions. Show that $f * (g * h) \quad = \quad (f * g) * h$.

2. Let $f, g, h : \mathbb{R} \longrightarrow \mathbb{R}$ be integrable functions, and let $r \in \mathbb{R}$ be a constant. Prove that $f * (r \cdot g + h) \quad = \quad r \cdot (f * g) + (f * h)$.

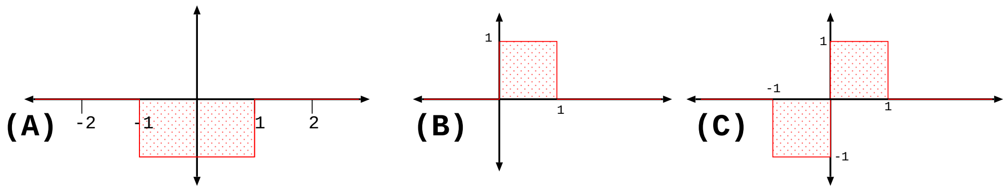


Figure 16.27: Problems #1(a), #1(b), #1(c) and #2(a).

3. Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be integrable functions. Let $d \in \mathbb{R}$ be some 'displacement' and define $f_{\triangleright d}(x) = f(x-d)$. Prove that $(f_{\triangleright d}) * g = (f * g)_{\triangleright d}$.

4. In each of the following, use the method of Gaussian convolutions to find the solution to the one-dimensional **Heat Equation** $\partial_t u(x;t) = \partial_x^2 u(x;t)$ with **initial conditions** $u(x,0) = \mathcal{I}(x)$.

   (a) $\mathcal{I}(x) = \begin{cases} -1 & \text{if} \quad -1 \leq x \leq 1 \\ 0 & \text{if} \quad x < -1 \text{ or } 1 < x \end{cases}$. (see Figure 16.27A).

   (In this case, sketch your solution evolving in time.)

   (b) $\mathcal{I}(x) = \begin{cases} 1 & \text{if} \quad 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$ (see Figure 16.27B).

   (c) $\mathcal{I}(x) = \begin{cases} -1 & \text{if} \quad -1 \leq x \leq 0 \\ 1 & \text{if} \quad 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$ (see Figure 16.27C).

5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be some differentiable function. Define $v(x;t) = \frac{1}{2}\Big(f(x+t) + f(x-t)\Big)$.

   (a) Show that $v(x;t)$ satisfies the one-dimensional Wave Equation $\partial_t^2 v(x;t) = \partial_x^2 v(x;t)$

   (b) Compute the **initial position** $v(x;0)$.

   (c) Compute the **initial velocity** $\partial_t v(x;0)$.

6. Let $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. For any $x \in \mathbb{R}$ and any $t \geq 0$, define $v(x,t) = \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} f_1(y)\, dy$.

   (a) Show that $v(x;t)$ satisfies the one-dimensional Wave Equation $\partial_t^2 v(x;t) = \partial_x^2 v(x;t)$

   (b) Compute the **initial position** $v(x;0)$.

   (c) Compute the **initial velocity** $\partial_t v(x;0)$.

7. In each of the following, use the d'Alembert method to find the solution to the one-dimensional **Wave Equation** $\partial_t^2 u(x;t) = \partial_x^2 u(x;t)$ with **initial position** $u(x,0) = f_0(x)$ and **initial velocity** $\partial_t u(x,0) = f_1(x)$.

   In each case, identify whether the solution satisfies homogeneous Dirichlet boundary conditions when treated as a function on the interval $[0, \pi]$. Justify your answer.

(a) $f_0(x) = \begin{cases} 1 & \text{if} \quad 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$; and $f_1(x) = 0$ (see Figure 16.27B).

(b) $f_0(x) = \sin(3x)$ and $f_1(x) = 0$.

(c) $f_0(x) = 0$ and $f_1(x) = \sin(5x)$.

(d) $f_0(x) = \cos(2x)$ and $f_1(x) = 0$.

(e) $f_0(x) = 0$ and $f_1(x) = \cos(4x)$.

(f) $f_0(x) = x^{1/3}$ and $f_1(x) = 0$.

(g) $f_0(x) = 0$ and $f_1(x) = x^{1/3}$.

(h) $f_0(x) = 0$ and $f_1(x) = \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$.

8. Let $\mathcal{G}_t(x) = \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-x^2}{4t}\right)$ be the Gauss-Weierstrass Kernel. Fix $s, t > 0$; we claim that $\mathcal{G}_s * \mathcal{G}_t = \mathcal{G}_{s+t}$. (For example, if $s = 3$ and $t = 5$, this means that $\mathcal{G}_3 * \mathcal{G}_5 = \mathcal{G}_8$).

   (a) Prove that $\mathcal{G}_s * \mathcal{G}_t = \mathcal{G}_{s+t}$ by directly computing the convolution integral.

   (b) Use Corollary 16.11 on page 310 to find a short and elegant proof that $\mathcal{G}_s * \mathcal{G}_t = \mathcal{G}_{s+t}$ *without* computing any convolution integrals.

   Remark: Because of this result, probabilists say that the set $\{\mathcal{G}_t\}_{t\in(0,\infty)}$ forms a *stable family of probability distributions* on $\mathbb{R}$. Analysts say that $\{\mathcal{G}_t\}_{t\in(0,\infty)}$ is a *one-parameter semigroup* under convolution.

9. Let $\mathcal{G}_t(x, y) = \dfrac{1}{4\pi t} \exp\left(\dfrac{-(x^2 + y^2)}{4t}\right)$ be the 2-dimensional Gauss-Weierstrass Kernel. Suppose $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a harmonic function. Show that $h * \mathcal{G}_t = h$ for all $t > 0$.

10. Let $\mathbb{D}$ be the unit disk. Let $b : \partial\mathbb{D} \longrightarrow \mathbb{R}$ be some function, and let $u : \mathbb{D} \longrightarrow \mathbb{R}$ be the solution to the corresponding Dirichlet problem with boundary conditions $b(\sigma)$. Prove that

$$u(0,0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} b(\sigma)\, d\sigma.$$

   Remark: This is a special case of the Mean Value Theorem for Harmonic Functions (Theorem 2.13 on page 33), but do *not* simply 'quote' Theorem 2.13 to solve this problem. Instead, apply Proposition 16.29 on page 331.

11. Let $\gamma_t(x) = \begin{cases} \frac{1}{t} & \text{if} \quad 0 \leq x \leq t; \\ 0 & \text{if} \quad x < 0 \text{ or } t < x. \end{cases}$ (Figure 16.7). Show that γ is an approximation of identity.

12. Let $\gamma_t(x) = \begin{cases} \frac{1}{t} & \text{if} \quad |x| \leq t \\ 0 & \text{if} \quad t < |x| \end{cases}$. Show that γ is an approximation of identity.

13. Let $\mathbb{D} = \left\{\mathbf{x} \in \mathbb{R}^2 \; ; \; |\mathbf{x}| \leq 1\right\}$ be the unit disk.

(a) Let $u : \mathbb{D} \longrightarrow \mathbb{R}$ be the unique solution to the **Laplace equation** ($\triangle u = 0$) satisfying the **nonhomogeneous Dirichlet** boundary conditions $u(\mathbf{s}) = 1$, for all $\mathbf{s} \in \mathbb{S}$. Show that u must be constant: $u(\mathbf{x}) = 1$ for all $\mathbf{x} \in \mathbb{D}$.

(b) Recall that the Poisson Kernel $\mathcal{P} : \mathbb{D} \times \mathbb{S} \longrightarrow \mathbb{R}$ is defined by $\mathcal{P}(\mathbf{x}, \mathbf{s}) = \frac{1-\|\mathbf{x}\|^2}{\|\mathbf{x}-\mathbf{s}\|^2}$, for any $\mathbf{x} \in \mathbb{D}$ and $\mathbf{s} \in \mathbb{S}$. Show that, for any fixed $\mathbf{x} \in \mathbb{D}$, $\frac{1}{2\pi}\int_{\mathbb{S}} \mathcal{P}(\mathbf{x}, \mathbf{s})\, d\mathbf{s} = 1$.

(c) Let $b : \mathbb{S} \longrightarrow \mathbb{R}$ be any function, and $\triangle u = 0$) satisfying the nonhomogeneous Dirichlet boundary conditions $u(\mathbf{s}) = b(\mathbf{s})$, for all $\mathbf{s} \in \mathbb{S}$.
Let $m := \min_{\mathbf{s}\in\mathbb{S}} b(\mathbf{s})$, and $M := \max_{\mathbf{s}\in\mathbb{S}} b(\mathbf{s})$. Show that:

$$\text{For all } \mathbf{x} \in \mathbb{D}, \qquad m \quad \leq \quad u(\mathbf{x}) \quad \leq \quad M.$$

[In other words, the harmonic function u must take its *maximal* and *minimal* values on the *boundary* of the domain $\mathbb{D}$. This is a special case of the *Maximum Modulus Principle* for harmonic functions.]

14. Let $\mathbb{H} := \left\{(x, y) \in \mathbb{R}^2 \ ; \ y \geq 0\right\}$ be the *half-plane*. Recall that the *half-plane Poisson kernel* is the function $\mathcal{K} : \mathbb{H} \longrightarrow \mathbb{R}$ defined $\mathcal{K}(x, y) := \dfrac{y}{\pi(x^2 + y^2)}$ for all $(x, y) \in \mathbb{H}$ except $(0, 0)$ (where it is not defined). Show that $\mathcal{K}$ is harmonic on the interior of $\mathbb{H}$.

Notes: ....

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VII Fourier Transforms on Unbounded Domains

In Part III, we saw that trigonometric functions like *sin* and *cos* formed orthogonal bases of $\mathbf{L}^2(\mathbb{X})$, where $\mathbb{X}$ was one of several bounded subsets of $\mathbb{R}^D$. Thus, any function in $\mathbf{L}^2(\mathbb{X})$ could be expressed using a *Fourier series*. In Parts IV and V, we used these Fourier series to solve initial/boundary value problems on $\mathbb{X}$.

A *Fourier transform* is similar to a Fourier series, except that now $\mathbb{X}$ is an *unbounded* set (e.g. $\mathbb{X} = \mathbb{R}$ or $\mathbb{R}^D$). This introduces considerable technical complications. Nevertheless, the underlying philosophy is the same; we will construct something analogous to an orthogonal basis for $\mathbf{L}^2(\mathbb{X})$, and use this to solve partial differential equations on $\mathbb{X}$.

It is technically convenient (although not strictly necessary) to replace *sin* and *cos* with the complex exponential functions like $\exp(x\mathbf{i}) = \cos(x) + \mathbf{i}\sin(x)$. The material on Fourier series in Part III could have also been developed using these complex exponentials, but in that context, this would have been a needless complication. In the context of Fourier transforms, however, it is actually a simplification.

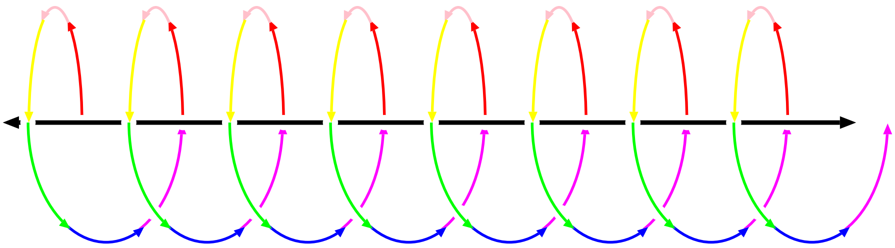

Figure 17.1: $\mathcal{E}_\mu(x) \;:=\; \exp(-\mu \cdot x \cdot \mathbf{i})$ as a function of x.

17 Fourier Transforms

17.1 One-dimensional Fourier Transforms

Prerequisites: §1.3 **Recommended:** §8.1, §9.4

Fourier *series* help us to represent functions on a *bounded* domain, like $\mathbb{X} = [0,1]$ or $\mathbb{X} = [0,1]\times[0,1]$. But what if the domain is *unbounded*, like $\mathbb{X} = \mathbb{R}$? Now, instead of using a *discrete* collection of Fourier coefficients like $\{A_0, A_1, B_1, A_2, B_2, \ldots\}$ or $\{\widehat{f}_{-1}, \widehat{f}_0, \widehat{f}_1, \widehat{f}_2, \ldots\}$, we must use a continuously parameterized family.

For every $\mu \in \mathbb{R}$, we define the function $\mathcal{E}_\mu : \mathbb{R} \longrightarrow \mathbb{C}$ by $\mathcal{E}_\mu(x) \;:=\; \exp(\mu \mathbf{i} x)$. You can visualize this function as a 'ribbon' which spirals with frequency μ around the unit circle in the complex plane (see Figure 17.1). Indeed, using de Moivre's formulae, it is not hard to check that $\mathcal{E}_\mu(x) \;=\; \cos(\mu x) + \mathbf{i}\sin(\mu x)$ (**Exercise 17.1**). In other words, the real and imaginary parts of $\mathcal{E}_\mu(x)$ like like a cosine wave and a sine wave, respectively, both of frequency μ.

Heuristically speaking, the (continuously parameterized) family of functions $\{\mathcal{E}_\mu\}_{\mu\in\mathbb{R}}$ acts as a kind of 'orthogonal basis' for a certain space of functions from $\mathbb{R}$ into $\mathbb{C}$ (although making this rigorous is very complicated). This is the motivating idea behind the next definition:

Definition 17.1: Fourier Transform

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be some function. The **Fourier transform** *of f is the function $\widehat{f} : \mathbb{R} \longrightarrow \mathbb{C}$ defined:*

$$\text{For any } \mu \in \mathbb{R}, \quad \widehat{f}(\mu) \;:=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\overline{\mathcal{E}_\mu(x)}\,dx \;=\; \boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\cdot\exp(-\mu\cdot x\cdot\mathbf{i})\,dx}$$

(In other words, $\widehat{f}(\mu) := \dfrac{1}{2\pi}\langle f, \mathcal{E}_\mu\rangle$, in the notation of § 7.2 on page 113)

Notice that this integral may not converge, in general. We need $f(x)$ to "decay fast enough" as x goes to $\pm\infty$. To be precise, we need f to be an **absolutely integrable** function, meaning that

$$\int_{-\infty}^{\infty} |f(x)|\;dx \;<\; \infty.$$

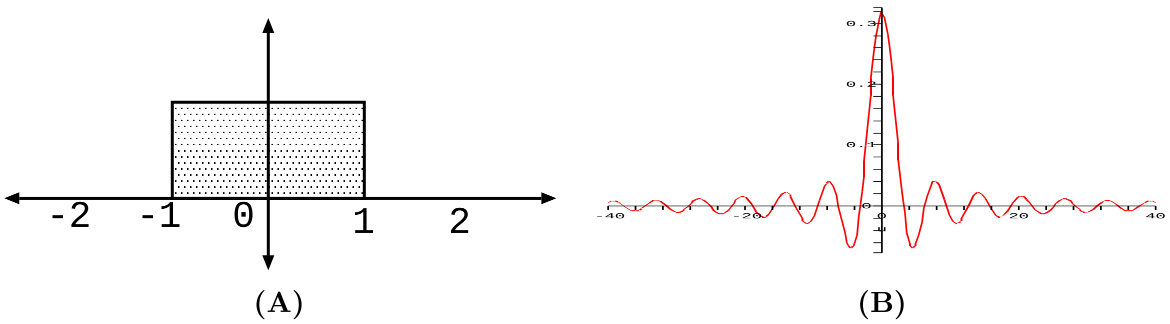

Figure 17.2: **(A)** Example 17.4. **(B)** The Fourier transform $\widehat{f}(x) = \frac{\sin(\mu)}{\pi\mu}$ from Example 17.4.

We indicate this by writing: "$f \in \mathbf{L}^1(\mathbb{R})$".

The Fourier transform $\widehat{f}(\mu)$ plays the same role that the Fourier coefficients $\{\widehat{f}_{-1}, \widehat{f}_0, \widehat{f}_1, \widehat{f}_2, \ldots\}$ played for a function on an interval. In particular, we can express $f(x)$ as a a sort of generalized "Fourier series". We would like to write something like:

$$\text{“ } f(x) \quad = \quad \sum_{\mu \in \mathbb{R}} \widehat{f}(\mu) \mathcal{E}_\mu(x) \text{ ”}$$

However, this expression makes no mathematical sense, because you can't *sum* over all real numbers (there are too many). Instead of *summing* over all Fourier coefficients, we must *integrate*....

Theorem 17.2: Fourier Inversion Formula

Suppose that $f \in \mathbf{L}^1(\mathbb{R})$. Then for any fixed $x \in \mathbb{R}$ so that f is continuous at x,

$$f(x) \quad = \quad \lim_{M \to \infty} \int_{-M}^{M} \widehat{f}(\mu) \cdot \mathcal{E}_\mu(x) \, d\mu \quad = \quad \lim_{M \to \infty} \int_{-M}^{M} \widehat{f}(\mu) \cdot \exp(\mu \cdot x \cdot \mathbf{i}) \, d\mu. \tag{17.1}$$

□

It follows that, under mild conditions, a function can be uniquely identified from its Fourier transform:

Corollary 17.3: *Suppose $f, g \in \mathcal{C}(\mathbb{R}) \cap \mathbf{L}^1(\mathbb{R})$* [i.e. f and g are continuous and absolutely integrable functions]. *Then:* $\left(\widehat{f} = \widehat{g} \right) \iff \left(f = g \right)$.

Proof: <u>**Exercise 17.2**</u> □

Example 17.4: Suppose $f(x) = \begin{cases} 1 & \text{if } -1 < x < 1; \\ 0 & \text{otherwise} \end{cases}$ [see Figure 17.2(A)]. Then

$$\begin{aligned} \text{For all } \mu \in \mathbb{R}, \quad \widehat{f}(\mu) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\exp(-\mu\cdot x\cdot \mathbf{i})\ dx \ = \ \frac{1}{2\pi}\int_{-1}^{1}\exp(-\mu\cdot x\cdot\mathbf{i})\ dx \\ &= \frac{1}{-2\pi\mu\mathbf{i}}\exp\Big(-\mu\cdot x\cdot\mathbf{i}\Big)_{x=-1}^{x=1} \ = \ \frac{1}{-2\pi\mu\mathbf{i}}\left(e^{-\mu\mathbf{i}} - e^{\mu\mathbf{i}}\right) \\ &= \frac{1}{\pi\mu}\left(\frac{e^{\mu\mathbf{i}} - e^{-\mu\mathbf{i}}}{2\mathbf{i}}\right) \ \underset{\text{(dM)}}{=\!=} \ \frac{1}{\pi\mu}\sin(\mu) \quad \text{[see Fig.17.2(B)]} \end{aligned}$$

where **(dM)** is de Moivre's formula[1].

Thus, the Fourier Inversion Formula says, that, if $-1 < x < 1$, then

$$\lim_{M\to\infty}\int_{-M}^{M}\frac{\sin(\mu)}{\pi\mu}\exp(\mu\cdot x\cdot\mathbf{i})\ d\mu \ = \ 1,$$

while, if $x < -1$ or $x > 1$, then $\lim_{M\to\infty}\int_{-M}^{M}\frac{\sin(\mu)}{\pi\mu}\exp(\mu\cdot x\cdot\mathbf{i})\ d\mu = 0$. If $x = \pm 1$, then the Fourier inversion integral will converge to neither of these values. ♢

Example 17.5: Suppose $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1; \\ 0 & \text{otherwise} \end{cases}$ [see Figure 17.3(A)]. Then $\widehat{f}(\mu) = \frac{1-e^{-\mu\mathbf{i}}}{2\pi\mu\mathbf{i}}$ [see Figure 17.3(B)]; the verification of this is practice problem # 1 on page 352 of §17.6. Thus, the Fourier inversion formula says, that, if $0 < x < 1$, then

$$\lim_{M\to\infty}\int_{-M}^{M}\frac{1-e^{-\mu\mathbf{i}}}{2\pi\mu\mathbf{i}}\exp(\mu\cdot x\cdot\mathbf{i})\ d\mu \ = \ 1,$$

while, if $x < 0$ or $x > 1$, then $\lim_{M\to\infty}\int_{-M}^{M}\frac{1-e^{-\mu\mathbf{i}}}{2\pi\mu\mathbf{i}}\exp(\mu\cdot x\cdot\mathbf{i})\ d\mu = 0$. If $x = 0$ or $x = 1$, then the Fourier inversion integral will converge to neither of these values. ♢

In the Fourier Inversion Formula, it is important that the positive and negative bounds of the integral go to infinity at the same rate in the limit (17.1); such a limit is called a *Cauchy principal value*. In particular, it is *not* the case that $f(x) = \lim_{N,M\to\infty}\int_{-N}^{M}\widehat{f}(\mu)\exp(\mu\cdot x\cdot\mathbf{i})\ d\mu$; in general, *this* integral may not converge. The reason is this: even if f is absolutely integrable, its Fourier transform $\widehat{f}$ may *not* be. This is what introduces the complications in the Fourier inversion formula. If we assume that $\widehat{f}$ is *also* absolutely integrable, then things become easier.

Theorem 17.6: Strong Fourier Inversion Formula

Suppose that $f \in \mathbf{L}^1(\mathbb{R})$, and that $\widehat{f}$ is also in $\mathbf{L}^1(\mathbb{R})$. Then f must be continuous everywhere. For any fixed $x \in \mathbb{R}$, $f(x) = \int_{-\infty}^{\infty}\widehat{f}(\mu)\cdot\exp(\mu\cdot x\cdot\mathbf{i})\ d\mu.$ □

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[1] See formula sheet.

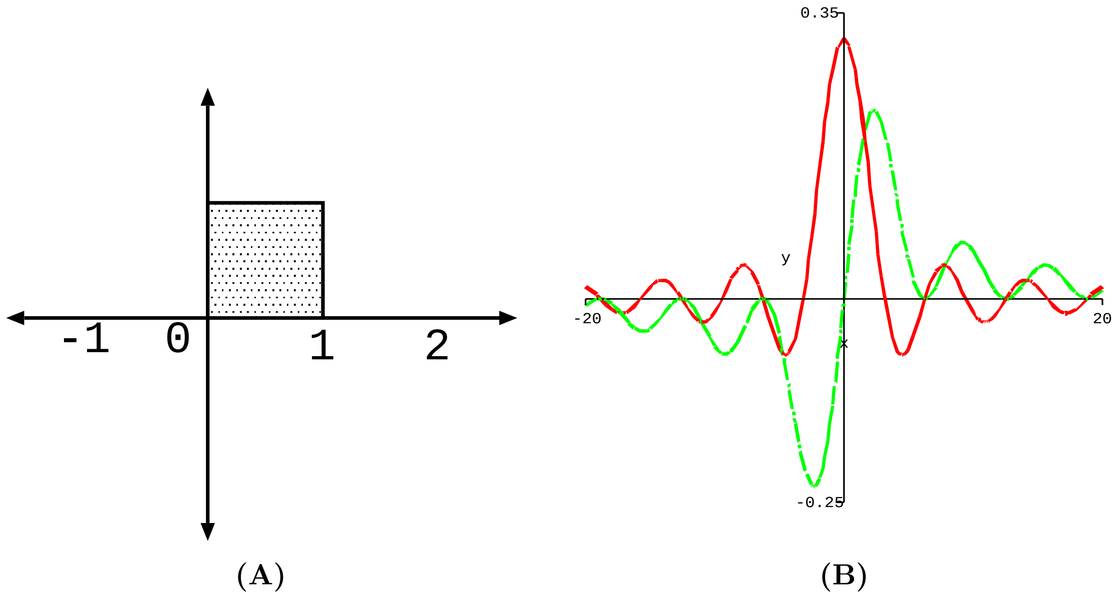


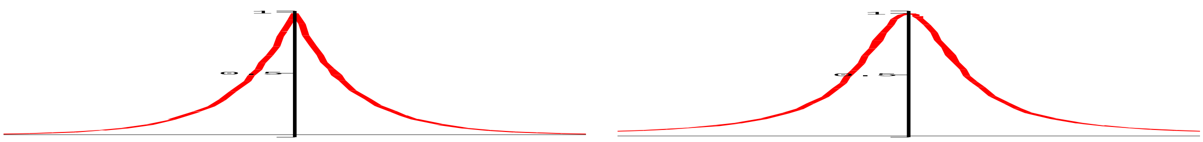
Figure 17.3: **(A)** Example 17.5. **(B)** The real and imaginary parts of the Fourier transform $\widehat{f}(x) = \frac{1-e^{-\mu \mathbf{i}}}{2\pi\mu\mathbf{i}}$ from Example 17.5.

Corollary 17.7: *Suppose $f \in \mathbf{L}^1(\mathbb{R})$, and there exists some $g \in \mathbf{L}^1(\mathbb{R})$ such that $f = \widehat{g}$. Then $\widehat{f}(\mu) = \frac{1}{2\pi} g(-\mu)$ for all $\mu \in \mathbb{R}$.*

Proof: **Exercise 17.3** □

Example 17.8: Let $\alpha > 0$ be a constant, and suppose $f(x) = e^{-\alpha\cdot|x|}$. [see Figure 17.4(A)]. Then

$$\begin{aligned}
2\pi\widehat{f}(\mu) &= \int_{-\infty}^{\infty} e^{-\alpha\cdot|x|}\exp(-\mu x\mathbf{i})\, dx \\
&= \int_{0}^{\infty} e^{-\alpha\cdot x}\exp(-\mu x\mathbf{i})\, dx \;+\; \int_{-\infty}^{0} e^{\alpha\cdot x}\exp(-\mu x\mathbf{i})\, dx \\
&= \int_{0}^{\infty} \exp(-\alpha x - \mu x\mathbf{i})\, dx \;+\; \int_{-\infty}^{0} \exp(\alpha x - \mu x\mathbf{i})\, dx \\
&= \frac{1}{-(\alpha+\mu\mathbf{i})}\exp\Big(-(\alpha+\mu\mathbf{i})\cdot x\Big)_{x=0}^{x=\infty} \;+\; \frac{1}{\alpha-\mu\mathbf{i}}\exp\Big((\alpha-\mu\mathbf{i})\cdot x\Big)_{x=-\infty}^{x=0} \\
&\underset{(*)}{=} \frac{-1}{\alpha+\mu\mathbf{i}}(0-1) + \frac{1}{\alpha-\mu\mathbf{i}}(1-0) \;=\; \frac{1}{\alpha+\mu\mathbf{i}} + \frac{1}{\alpha-\mu\mathbf{i}} \;=\; \frac{\alpha-\mu\mathbf{i}+\alpha+\mu\mathbf{i}}{(\alpha+\mu\mathbf{i})(\alpha-\mu\mathbf{i})} \\
&= \frac{2\alpha}{\alpha^2+\mu^2}
\end{aligned}$$



(A) **(B)**

Figure 17.4: **(A)** The symmetric exponential tail function $f(x) = e^{-\alpha\cdot|x|}$ from Example 17.8. **(B)** The Fourier transform $\widehat{f}(x) = \frac{a}{\pi(x^2+a^2)}$ of the symmetric exponential tail function from Example 17.8.

Thus, we conclude: $\widehat{f}(\mu) = \dfrac{\alpha}{\pi(\alpha^2+\mu^2)}$. [see Figure 17.4(B)].

To see equality $(*)$, recall that $\left|\exp\left(-(\alpha+\mu\mathbf{i})\cdot x\right)\right| = e^{-\alpha\cdot x}$. Thus, $\lim_{\mu\to\infty}\left|\exp\left(-(\alpha+\mu\mathbf{i})\cdot x\right)\right| = \lim_{\mu\to\infty} e^{-\alpha\cdot x} = 0$. Likewise, $\lim_{\mu\to-\infty}\left|\exp\left((\alpha-\mu\mathbf{i})\cdot x\right)\right| = \lim_{\mu\to-\infty} e^{\alpha\cdot x} = 0$. ♢

Example 17.9: Conversely, suppose $\alpha > 0$, and $g(x) = \dfrac{1}{(\alpha^2+x^2)}$. Then $\widehat{g}(\mu) = \dfrac{1}{2\alpha}e^{-\alpha\cdot|\mu|}$; the verification of this is practice problem # 6 on page 353 of §17.6. ♢

17.2 Properties of the (one-dimensional) Fourier Transform

Prerequisites: §17.1, §1.8

Theorem 17.10: Riemann-Lebesgue Lemma

If $f \in \mathbf{L}^1(\mathbb{R})$, then $\widehat{f}$ is **continuous** *and* **bounded**. *To be precise: If $B = \int_{-\infty}^{\infty} |f(x)|\, dx$, then, for all $\mu \in \mathbb{R}$, $\left|\widehat{f}(\mu)\right| < B$.*

Also, $\widehat{f}$ **asymptotically decays** *near infinity:* $\lim_{\mu\to\pm\infty}\left|\widehat{f}(\mu)\right| = 0$. □

Recall that, if $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are two functions, then their **convolution** is the function $(f * g) : \mathbb{R} \longrightarrow \mathbb{R}$ defined:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)\cdot g(x-y)\, dy.$$

Similarly, if f has Fourier transform $\widehat{f}$ and g has Fourier transform $\widehat{g}$, we can convolve $\widehat{f}$ and $\widehat{g}$ to get a function $(\widehat{f} * \widehat{g}) : \mathbb{R} \longrightarrow \mathbb{R}$ defined:

$$(\widehat{f} * \widehat{g})(\mu) = \int_{-\infty}^{\infty} \widehat{f}(\nu)\cdot \widehat{g}(\mu-\nu)\, d\nu.$$

(see § 16.3(a) on page 307 for more discussion of convolutions).

Theorem 17.11: Algebraic Properties of the Fourier Transform

Suppose $f, g \in \mathbf{L}^1(\mathbb{R})$ are two functions.

(a) *If $h(x) = f(x) + g(x)$, then for all $\mu \in \mathbb{R}$, $\widehat{h}(\mu) = \widehat{f}(\mu) + \widehat{g}(\mu)$.*

(b) *Suppose that $h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(y) \cdot g(x-y)\, dy$. Then for all $\mu \in \mathbb{R}$, $\widehat{h}(\mu) = 2\pi \cdot \widehat{f}(\mu) \cdot \widehat{g}(\mu)$.*

(c) *Conversely, suppose $h(x) = f(x) \cdot g(x)$. If $\widehat{f}, \widehat{g}$ and $\widehat{h}$ are in $\mathbf{L}^1(\mathbb{R})$, then for all $\mu \in \mathbb{R}$, $\widehat{h}(\mu) = (\widehat{f} * \widehat{g})(\mu)$.*

Proof: See practice problems #11 to # 13 on page 353 of §17.6 □

This theorem allows us to compute the Fourier transform of a complicated function by breaking it into a sum/product of simpler pieces.

Theorem 17.12: Translation and Phase Shift

Suppose $f \in \mathbf{L}^1(\mathbb{R})$.

(a) *If $\tau \in \mathbb{R}$ is fixed, and g is defined by: $g(x) = f(x+\tau)$, then for all $\mu \in \mathbb{R}$, $\widehat{g}(\mu) = e^{\tau\mu\mathbf{i}} \cdot \widehat{f}(\mu)$.*

(b) *Conversely, if $\nu \in \mathbb{R}$ is fixed, and g is defined: $g(x) = e^{\nu x \mathbf{i}} f(x)$, then for all $\mu \in \mathbb{R}$, $\widehat{g}(\mu) = \widehat{f}(\mu - \nu)$.*

Proof: See practice problems #14 and # 15 on page 354 of §17.6. □

Thus, *translating* a function by τ in physical space corresponds to *phase-shifting* its Fourier transform by $e^{\tau\mu\mathbf{i}}$, and vice versa. This means that, via a suitable translation, we can put the "center" of our coordinate system wherever it is most convenient to do so.

Example 17.13: Suppose $g(x) = \begin{cases} 1 & \text{if } -1-\tau < x < 1-\tau; \\ 0 & \text{otherwise} \end{cases}$. Thus, $g(x) = f(x+\tau)$, where $f(x)$ is as in Example 17.4 on page 339. We know that $\widehat{f}(\mu) = \dfrac{\sin(\mu)}{\pi\mu}$; thus, it follows from Theorem 17.12 that $\widehat{g}(\mu) = e^{\tau\mu\mathbf{i}} \cdot \dfrac{\sin(\mu)}{\pi\mu}$. ♢

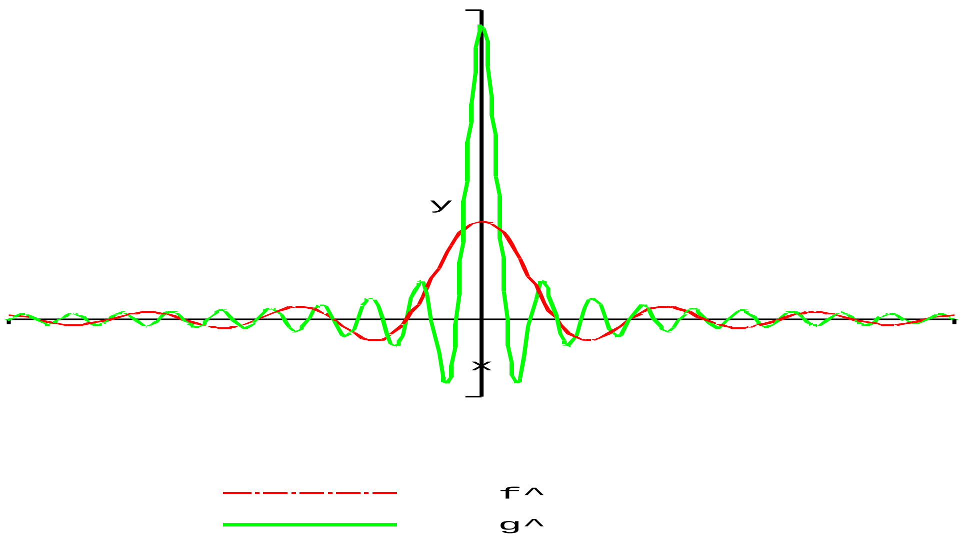


Figure 17.5: Plot of $\widehat{f}$ and $\widehat{g}$ in Example 17.15, where $g(x) = f(x/3)$.

Theorem 17.14: Rescaling Relation

Suppose $f \in \mathbf{L}^1(\mathbb{R})$. *If* $\sigma > 0$ *is fixed, and* g *is defined by:* $g(x) = f\left(\frac{x}{\sigma}\right)$, *then for all* $\mu \in \mathbb{R}$, $\widehat{g}(\mu) = \sigma \cdot \widehat{f}(\sigma \cdot \mu)$.

Proof: See practice problem # 16 on page 354 of §17.6. □

In Theorem 17.14, the function g is the same as function f, but expressed in a coordinate system "rescaled" by a factor of σ.

Example 17.15: Suppose $g(x) = \begin{cases} 1 & \text{if } -\sigma < x < \sigma; \\ 0 & \text{otherwise} \end{cases}$. Thus, $g(x) = f(x/\sigma)$, where $f(x)$ is as in Example 17.4 on page 339. We know that $\widehat{f}(\mu) = \frac{\sin(\mu)}{\mu\pi}$; thus, it follows from Theorem 17.14 that $\widehat{g}(\mu) = \sigma \cdot \frac{\sin(\sigma\mu)}{\sigma\mu\pi} = \frac{\sin(\sigma\mu)}{\mu\pi}$. See Figure 17.5. ♢

Theorem 17.16: Differentiation and Multiplication

Suppose $f \in \mathbf{L}^1(\mathbb{R})$.

(a) *If* $f \in \mathcal{C}^1(\mathbb{R})$ [i.e. f is differentiable], *and* $g(x) = f'(x)$, *then for all* $\mu \in \mathbb{R}$, $\widehat{g}(\mu) = \mathbf{i}\mu \cdot \widehat{f}(\mu)$.

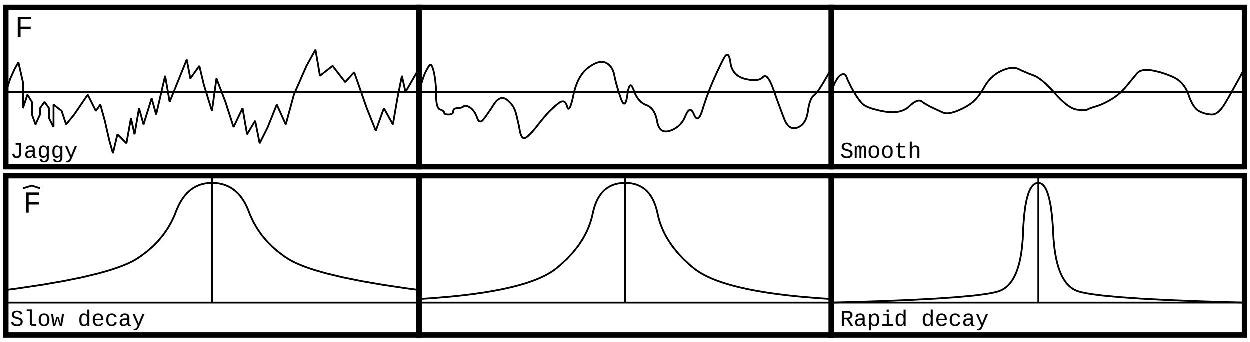


Figure 17.6: Smoothness vs. asymptotic decay in the Fourier Transform.

(b) *More generally, if* $f \in \mathcal{C}^n(\mathbb{R})$ [i.e. f is n times differentiable], *and* $g(x) = \frac{d^n}{dx^n} f(x)$, *then for all* $\mu \in \mathbb{R}$,

$$\widehat{g}(\mu) \;=\; (\mathbf{i}\mu)^n \cdot \widehat{f}(\mu).$$

Thus, $\widehat{f}(\mu)$ *asymptotically decays faster than* $\frac{1}{\mu^n}$ *as* $\mu \to \pm\infty$. *That is,* $\lim_{\mu \to \pm\infty} \mu^n \widehat{f}(\mu) = 0$.

(c) *Conversely, let* $g(x) = x^n \cdot f(x)$, *and suppose that* f *decays "quickly enough" that* g *is also in* $\mathbf{L}^1(\mathbb{R})$ [for example, this is the case if $\lim_{x \to \pm\infty} x^{n+1} f(x) = 0$]. *Then the function* $\widehat{f}$ *is* n *times differentiable, and, for all* $\mu \in \mathbb{R}$,

$$\widehat{g}(\mu) \;=\; \mathbf{i}^n \cdot \frac{d^n}{d\mu^n} \widehat{f}(\mu).$$

Proof: **(a)**, Assume for simplicity that $\lim_{x \to \pm\infty} |f(x)| = 0$. (This isn't always true, but the hypothesis that $f \in \mathbf{L}^1(\mathbb{R})$ means it is 'virtually' true, and the general proof has a very similar flavour.) Then the proof is practice problem # 17 on page 354 of §17.6.

(b) is just the result of iterating **(a)** n times.

(c) is **Exercise 17.4** Hint: *either* 'reverse' the result of **(a)** using the Fourier Inversion Formula (Theorem 17.2 on page 338), *or* use Proposition 1.9 on page 18 to directly differentiate the integral defining $\widehat{f}(\mu)$. □

This theorem says that the Fourier transform converts *differentiation-by-x* into *multiplication-by-μ*. This implies that the *smoothness* of a function f is closely related to the *asymptotic decay rate* of its Fourier transform. The "smoother" f is (ie. the more times we can differentiate it), the more *rapidly* $\widehat{f}(\mu)$ decays as $\mu \to \infty$ (see Figure 17.6).

Physically, we can interpret this as follows. If we think of f as a "signal", then $\widehat{f}(\mu)$ is the amount of "energy" at the "frequency" μ in the spectral decomposition of this signal. Thus, the magnitude of $\widehat{f}(\mu)$ for extremely large μ is the amount of "very high frequency" energy in f, which corresponds to very finely featured, "jaggy" structure in the shape of f. If f is "smooth", then we expect there will be very little of this "jagginess"; hence the high frequency part of the energy spectrum will be very small.

Conversely, the asymptotic decay rate of f determines the smoothness of its Fourier transform. This makes sense, because the Fourier inversion formula can be (loosely) intepreted as saying that f is itself a sort of "backwards" Fourier transform of $\widehat{f}$.

One very important Fourier transform is the following:

Theorem 17.17: Fourier Transform of a Gaussian

(a) *If* $f(x) = \exp(-x^2)$, *then* $\widehat{f}(\mu) = \frac{1}{2\sqrt{\pi}} \cdot f\left(\frac{\mu}{2}\right) = \frac{1}{2\sqrt{\pi}} \cdot \exp\left(\frac{-\mu^2}{4}\right)$.

(b) *Fix* $\sigma > 0$. *If* $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right)$ *is a normal probability distribution with mean* 0 *and variance* σ^2*, then*

$$\widehat{f}(\mu) = \frac{1}{2\pi} \exp\left(\frac{-\sigma^2\mu^2}{2}\right).$$

(c) *Fix* $\sigma > 0$ *and* $\tau \in \mathbb{R}$. *If* $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-|x-\tau|^2}{2\sigma^2}\right)$ *is a normal probability distribution with mean* τ *and variance* σ^2*, then*

$$\widehat{f}(\mu) = \frac{e^{-\mathbf{i}\tau\mu}}{2\pi} \exp\left(\frac{-\sigma^2\mu^2}{2}\right).$$

Proof: We'll start with **Part (a)**. Let $g(x) = f'(x)$. Then by Theorem 17.16**(a)**,

$$\widehat{g}(\mu) = \mathbf{i}\mu \cdot \widehat{f}(\mu). \tag{17.2}$$

However direct computation says $g(x) = -2x \cdot f(x)$, so $\frac{-1}{2} g(x) = x \cdot f(x)$, so Theorem 17.16**(c)** implies

$$\frac{\mathbf{i}}{2}\widehat{g}(\mu) = (\widehat{f})'(\mu). \tag{17.3}$$

Combining (17.3) with (17.2), we conclude:

$$(\widehat{f})'(\mu) \underset{(17.3)}{=\!=} \frac{\mathbf{i}}{2}\widehat{g}(\mu) \underset{(17.2)}{=\!=} \frac{\mathbf{i}}{2} \cdot \mathbf{i}\mu \cdot \widehat{f}(\mu) = \frac{-\mu}{2}\widehat{f}(\mu). \tag{17.4}$$

Define $h(\mu) = \widehat{f}(\mu) \cdot \exp\left(\frac{\mu^2}{4}\right)$. If we differentiate $h(\mu)$, we get:

$$h'(\mu) \underset{(\text{dL})}{=\!=} \widehat{f}(\mu) \cdot \frac{\mu}{2} \exp\left(\frac{\mu^2}{4}\right) \underbrace{- \frac{\mu}{2}\widehat{f}(\mu)}_{(*)} \cdot \exp\left(\frac{\mu^2}{4}\right) = 0.$$

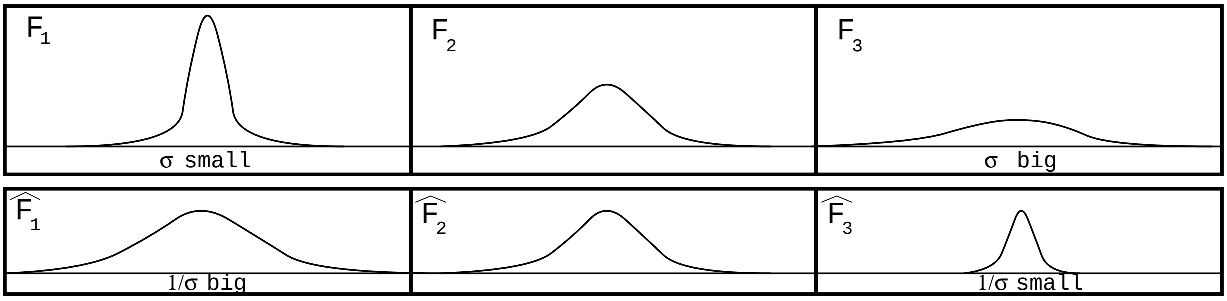


Figure 17.7: The Uncertainty Principle.

Here, **(dL)** is differentiating using the Liebniz rule, and $(*)$ is by eqn.(17.4).

In other words, $h(\mu) = H$ is a constant. Thus,

$$\widehat{f}(\mu) \quad = \quad \frac{h(\mu)}{\exp\left(\mu^2/4\right)} \quad = \quad H \cdot \exp\left(\frac{-\mu^2}{4}\right) \quad = \quad H \cdot f\left(\frac{\mu}{2}\right).$$

To evaluate H, set $\mu = 0$, to get

$$H \quad = \quad H \cdot \exp\left(\frac{-0^2}{4}\right) \quad = \quad \widehat{f}(0) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, dx \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-x^2\right)$$
$$= \quad \frac{1}{2\sqrt{\pi}}.$$

(where the last step is **Exercise 17.5**). Thus, we conclude: $\widehat{f}(\mu) = \frac{1}{2\sqrt{\pi}} \cdot f\left(\frac{\mu}{2}\right)$.

Part (b) follows by applying Theorem 17.14 on page 343.

Part (c) then follows by applying Theorem 17.12 on page 342. (**Exercise 17.6**) □

Loosely speaking, this theorem says, "The Fourier transform of a Gaussian is another Gaussian"[2]. However, notice that, in **Part (b)** of the theorem, as the **variance** of the Gaussian (that is, σ^2) gets bigger, the "variance" of it's Fourier transform (which is effectively $\frac{1}{\sigma^2}$) gets *smaller* (see Figure 17.7). If we think of the Gaussian as the probability distribution of some unknown piece of information, then the variance measures the degree of "uncertainty". Hence, we conclude: the greater the uncertainty embodied in the Gaussian f, the *less* the uncertainty embodied in $\widehat{f}$, and vice versa. This is a manifestation of the so-called *Uncertainty Principle* (see page 73).

[2]This is only *loosely* speaking, however, because a proper Gaussian contains the multiplier "$\frac{1}{\sigma\sqrt{2\pi}}$" to make it a probability distribution, whereas the Fourier transform does not.

Proposition 17.18: Inversion and Conjugation

Suppose $f \in \mathbf{L}^1(\mathbb{R})$, and define $g(x) = f(-x)$. Then for all $\mu \in \mathbb{R}$, $\widehat{g}(\mu) = \overline{\widehat{f}(\mu)}$, (where $\overline{z}$ is the complex conjugate of z).

If f is **even** *(ie. $f(-x) = f(x)$), then $\widehat{f}$ is purely real-valued.*

If f is **odd** *(ie. $f(-x) = -f(x)$), then $\widehat{f}$ is purely imaginary-valued.*

Proof: **Exercise 17.7** □

Example 17.19: Autocorrelation and Power Spectrum

If $f : \mathbb{R} \longrightarrow \mathbb{R}$, then the **autocorrelation function** of f is defined by

$$\mathbf{A}f(x) = \int_{-\infty}^{\infty} f(y) \cdot f(x+y)\ dy$$

Heuristically, if we think of $f(x)$ as a "random signal", then $\mathbf{A}f(x)$ measures the degree of correlation in the signal across time intervals of length x —ie. it provides a crude measure of how well you can predict the value of $f(y+x)$ given information about $f(x)$. In particular, if f has some sort of "T-periodic" component, then we expect $\mathbf{A}f(x)$ to be large when $x = nT$ for any $n \in \mathbb{Z}$.

If we define $g(x) = f(-x)$, then we can see that

$$\mathbf{A}f(x) \;=\; f * g(-x).$$

(**Exercise 17.8**) Thus, applying Proposition 17.18 (to $f * g$) and then Theorem 17.11**(b)**, and then Proposition 17.18 again (to f), we conclude that, for any $\mu \in \mathbb{R}$,

$$\widehat{\mathbf{A}f}(\mu) \;=\; \overline{\widehat{f * g}\ (\mu)} \;=\; \overline{\widehat{f}(\mu) \cdot \widehat{g}(\mu)} \;=\; \overline{\widehat{f}(\mu) \cdot \overline{\widehat{f}(\mu)}} \;=\; \overline{\widehat{f}(\mu)} \cdot \widehat{f}(\mu) \;=\; \left|\widehat{f}(\mu)\right|^2$$

The function $\left|\widehat{f}(\mu)\right|^2$ measures the absolute magnitude of the Fourier transform of $\widehat{f}$, and is sometimes called the **power spectrum** of $\widehat{f}$. ♢

17.3 Two-dimensional Fourier Transforms

Prerequisites: §17.1 **Recommended:** §10.1

Definition 17.20: 2-dimensional Fourier Transform

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ be some function. The **Fourier transform** *of f is the function $\widehat{f} : \mathbb{R}^2 \longrightarrow \mathbb{C}$ defined:*

For all $(\mu, \nu) \in \mathbb{R}^2$,

$$\widehat{f}(\mu,\nu) \;=\; \boxed{\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \cdot \exp\Big(-(\mu x + \nu y)\cdot \mathbf{i}\Big)\ dx\ dy}$$

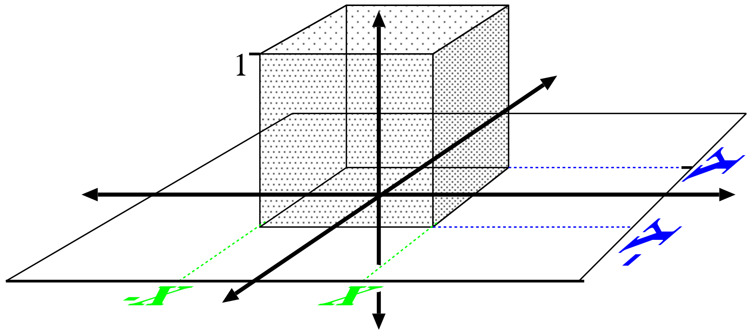


Figure 17.8: Example 17.23

Again, we need f to be an **absolutely integrable** function:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)| \; dx \; dy \;<\; \infty.$$

We indicate this by writing: "$f \in \mathbf{L}^1(\mathbb{R}^2)$".

Theorem 17.21: 2-dimensional Fourier Inversion Formula

Suppose that $f \in \mathbf{L}^1(\mathbb{R}^2)$. Then for any fixed $(x,y) \in \mathbb{R}$ so that f is continuous at (x,y),

$$f(x,y) \quad = \quad \lim_{R\to\infty} \int_{-R}^{R}\int_{-R}^{R} \widehat{f}(\mu,\nu)\cdot \exp\Big((\mu x + \nu y)\cdot \mathbf{i}\Big) \; d\mu \; d\nu. \tag{17.5}$$

or, alternately:

$$f(x,y) \;=\; \lim_{R\to\infty} \int_{\mathbb{D}(R)} \widehat{f}(\mu,\nu)\cdot \exp\Big((\mu x + \nu y)\cdot \mathbf{i}\Big) \; d\mu \; d\nu. \tag{17.6}$$

*Here $\mathbb{D}(R) = \big\{(\mu,\nu) \; ; \; \mu^2+\nu^2 \leq R\big\}$ is the **disk** of radius R.* □

Corollary 17.22: *If $f,g \in \mathcal{C}(\mathbb{R}^2)$ are continuous, integrable functions, then* $\Big(\; \widehat{f} \;=\; \widehat{g}\;\Big) \iff \Big(\; f \;=\; g\;\Big)$. □

Example 17.23: Let $X, Y > 0$, and let $f(x,y) \;=\; \begin{cases} 1 & \text{if} \quad -X \leq x \leq X \text{ and } -Y \leq y \leq Y; \\ 0 & \text{otherwise.} \end{cases}$.

(Figure 17.8) Then:

$$\begin{aligned} \widehat{f}(\mu,\nu) \quad &= \quad \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\cdot \exp\Big(-(\mu x+\nu y)\cdot \mathbf{i}\Big) \; dx \; dy \\ &= \quad \frac{1}{4\pi^2}\int_{-X}^{X}\int_{-Y}^{Y} \exp(-\mu x\mathbf{i})\cdot\exp(-\nu y\mathbf{i}) \; dx \; dy \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4\pi^2}\left(\int_{-X}^{X} \exp(-\mu x \mathbf{i})\ dx\right)\cdot\left(\int_{-Y}^{Y} \exp(-\nu y \mathbf{i}) dy\right) \\
&= \frac{1}{4\pi^2}\cdot\left(\frac{-1}{\mu\mathbf{i}}\exp\Big(-\mu x\mathbf{i}\Big)_{x=-X}^{x=X}\right)\cdot\left(\frac{1}{\nu\mathbf{i}}\exp\Big(-\nu y\mathbf{i}\Big)_{y=-Y}^{y=Y}\right) \\
&= \frac{1}{4\pi^2}\left(\frac{e^{\mu X\mathbf{i}}-e^{-\mu X\mathbf{i}}}{\mu\mathbf{i}}\right)\left(\frac{e^{\nu Y\mathbf{i}}-e^{-\nu Y\mathbf{i}}}{\nu\mathbf{i}}\right) \quad = \quad \frac{1}{\pi^2\mu\nu}\left(\frac{e^{\mu X\mathbf{i}}-e^{-\mu X\mathbf{i}}}{2\mathbf{i}}\right)\left(\frac{e^{\nu Y\mathbf{i}}-e^{-\nu Y\mathbf{i}}}{2\mathbf{i}}\right) \\
&\overline{\overline{\scriptstyle(\mathrm{dM})}} \frac{1}{\pi^2\mu\nu}\sin(\mu X)\cdot\sin(\nu Y),
\end{aligned}
$$

where **(dM)** is by double application of de Moivre's formula. Thus, the Fourier inversion formula says, that, if $-X < x < X$ and $-Y < y < Y$, then

$$\lim_{R\to\infty}\int_{\mathbb{D}(R)} \frac{\sin(\mu X)\cdot\sin(\nu Y)}{\pi^2\cdot\mu\cdot\nu}\exp\Big((\mu x+\nu y)\cdot\mathbf{i}\Big)\ d\mu\ d\nu \;=\; 1$$

while, if $(x, y) \not\in [-X, X] \times [-X, Y]$, then

$$\lim_{R\to\infty}\int_{\mathbb{D}(R)} \frac{\sin(\mu X)\cdot\sin(\nu Y)}{\pi^2\cdot\mu\cdot\nu}\exp\Big((\mu x+\nu y)\cdot\mathbf{i}\Big)\ d\mu\ d\nu \;=\; 0.$$

At points on the *boundary* of the box $[0, X] \times [0, Y]$, however, the Fourier inversion integral will converge to neither of these values. ♢

Example 17.24: If $f(x, y) = \dfrac{1}{2\sigma^2\pi}\exp\left(\dfrac{-x^2-y^2}{2\sigma^2}\right)$ is a two-dimensional Gaussian distribution, then $\widehat{f}(\mu,\nu) = \dfrac{1}{4\pi^2}\exp\left(\dfrac{-\sigma^2}{2}\left(\mu^2+\nu^2\right)\right)$. (**Exercise 17.9**) ♢

17.4 Three-dimensional Fourier Transforms

Prerequisites: §17.1 **Recommended:** §10.2, §17.3

In three or more dimensions, it is cumbersome to write vectors as an explicit list of coordinates. We will adopt a more compact notation. **Bold-face** letters will indicate vectors, and normal letters, their components. For example:

$$\mathbf{x} = (x_1, x_2, x_3), \quad \mathbf{y} = (y_1, y_2, y_3), \quad \boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3), \quad \text{and} \quad \boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$$

We will also use the notation:

$$\langle \mathbf{x}, \mathbf{y}\rangle = x_1\cdot y_1 + x_2\cdot y_2 + x_3\cdot y_3$$

for inner products, and the notation

$$\int_{\mathbb{R}^3} f(\mathbf{x})\ d\mathbf{x} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3)\ dx_1\ dx_2\ dx_3$$

for integrals.

Definition 17.25: 3-dimensional Fourier Transform

Let $f : \mathbb{R}^3 \longrightarrow \mathbb{C}$ *be some function. The* **Fourier transform** *of* f *is the function* $\widehat{f} : \mathbb{R}^3 \longrightarrow \mathbb{C}$ *defined:*

$$\text{For all } \boldsymbol{\mu} \in \mathbb{R}^3, \qquad \widehat{f}(\boldsymbol{\mu}) \quad = \quad \boxed{\frac{1}{8\pi^3} \int_{\mathbb{R}^3} f(\mathbf{x}) \cdot \exp\Big(- \langle \mathbf{x}, \boldsymbol{\mu} \rangle \cdot \mathbf{i} \Big) \; d\mathbf{x}}$$

Again, we need f to be an **absolutely integrable** function:

$$\int_{\mathbb{R}^3} |f(\mathbf{x})| \; d\mathbf{x} \; < \; \infty$$

We indicate this by writing: "$f \in \mathbf{L}^1(\mathbb{R}^3)$".

Theorem 17.26: 3-dimensional Fourier Inversion Formula

Suppose that $f \in \mathbf{L}^1(\mathbb{R}^3)$. *Then for any fixed* $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}$ *so that* f *is continuous at* $\mathbf{x}$,

$$f(\mathbf{x}) \quad = \quad \lim_{R\to\infty} \int_{-R}^{R} \int_{-R}^{R} \int_{-R}^{R} \widehat{f}(\boldsymbol{\mu}) \; \cdot \exp\Big(\langle \boldsymbol{\mu}, \mathbf{x} \rangle \cdot \mathbf{i} \Big) \; d\boldsymbol{\mu}; \qquad (17.7)$$

or, alternately

$$f(\mathbf{x}) \quad = \quad \lim_{R\to\infty} \int_{\mathbb{B}(R)} \widehat{f}(\mu) \cdot \exp\Big(\langle \boldsymbol{\mu}, \mathbf{x} \rangle \cdot \mathbf{i} \Big) \; d\boldsymbol{\mu}; \qquad (17.8)$$

Here $\mathbb{B}(R) = \big\{(\mu_1, \mu_2, \mu_3) \; ; \; \mu_1^2 + \mu_2^2 + \mu_3^2 \leq R\big\}$ *is the* **ball** *of radius* R. □

Corollary 17.27: *If* $f, g \in \mathcal{C}(\mathbb{R}^3)$ *are continuous, integrable functions, then* $\Big(\widehat{f} \; = \; \widehat{g} \Big) \iff \Big(f \; = \; g \Big)$. □

Example 17.28: A Ball

For any $\mathbf{x} \in \mathbb{R}^3$, let $f(\mathbf{x}) \; = \; \begin{cases} 1 & \text{if } \|\mathbf{x}\| \leq R; \\ 0 & \text{otherwise.} \end{cases}$. Thus, $f(\mathbf{x})$ is nonzero on a ball of radius R around zero. Then

$$\widehat{f}(\boldsymbol{\mu}) \; = \; \frac{1}{2\pi^2} \left(\frac{\sin(\mu R)}{\mu^3} \; - \; \frac{R\cos(\mu R)}{\mu^2} \right),$$

where $\mu = \|\boldsymbol{\mu}\|$. ♢

<u>**Exercise 17.10**</u> Verify Example 17.28. **Hint:** Argue that, by spherical symmetry, we can rotate $\boldsymbol{\mu}$ without changing the integral, so we can assume that $\boldsymbol{\mu} = (\mu, 0, 0)$. Switch to the spherical coordinate system $(x_1, x_2, x_3) \; = \; (r \cdot \cos(\phi), \; r \cdot \sin(\phi)\sin(\theta), \; r \cdot \sin(\phi)\cos(\theta))$, to express the Fourier integral as

$$\frac{1}{8\pi^3} \int_0^R \int_0^\pi \int_{-\pi}^{\pi} \exp\left(\mu \cdot r \cdot \cos(\phi) \cdot \mathbf{i}\right) \cdot r\sin(\phi) \; d\theta \; d\phi \; dr.$$

Use **Claim 1** from Theorem 18.15 on page 362 to simplify this to $\frac{1}{2\pi^2\mu}\int_0^R r \cdot \sin(\mu \cdot r)\ dr$. Now apply integration by parts.

Exercise 17.11 The Fourier transform of Example 17.28 contains the terms $\frac{\sin(\mu R)}{\mu^3}$ and $\frac{\cos(\mu R)}{\mu^2}$, both of which go to infinity as $\mu \to 0$. However, these two infinities "cancel out". Use l'Hôpital's rule to show that $\lim_{\boldsymbol{\mu}\to 0} \widehat{f}(\boldsymbol{\mu}) = \frac{1}{24\pi^3}$.

Example 17.29: A spherically symmetric function

Suppose $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ was a spherically symmetric function; in other words, $f(\mathbf{x}) = \phi(\|\mathbf{x}\|)$ for some function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}$. Then for any $\boldsymbol{\mu} \in \mathbb{R}^3$,

$$\widehat{f}(\boldsymbol{\mu}) = \frac{1}{2\pi^2}\int_0^\infty \phi(r) \cdot r \cdot \sin(\|\boldsymbol{\mu}\| \cdot r)\ dr.$$

(**Exercise 17.12**) ♢

Fourier transforms can be defined in an analogous way in higher dimensions. From now on, we will suppress the explicit "Cauchy principal value" notation when writing the Fourier inversion formula, and simply write it as "$\int_{-\infty}^{\infty}$", or whatever.

17.5 Fourier (co)sine Transforms on the Half-Line

Prerequisites: §17.1

To represent a function on the *symmetric* interval $[-L, L]$, we used a full Fourier series (with both "sine" and "cosine" terms). However, to represent a function on the interval $[0, L]$, we found it only necessary to employ half as many terms, using either the Fourier sine series or the Fourier cosine series. A similar phenomenon occurs when we go from functions on the *whole* real line to functions on the positive *half*-line

Let $\mathbb{R}^+ = \{x \in \mathbb{R}\ ;\ x \geq 0\}$ be the **half-line**: the set of all nonnegative real numbers. Let

$$\mathbf{L}^1(\mathbb{R}^+) = \left\{ f : \mathbb{R}^+ \longrightarrow \mathbb{R}\ ;\ \int_0^\infty |f(x)|\ dx\ < \infty \right\}$$

be the set of **absolutely integrable** functions on the half-line.

The "boundary" of the half-line is just the point 0. Thus, we will say that a function f satisfies homogeneous **Dirichlet** boundary conditions if $f(0) = 0$. Likewise, f satisfies homogeneous **Neumann** boundary conditions if $f'(0) = 0$.

Definition 17.30: Fourier (co)sine Transform

If $f \in \mathbf{L}^1(\mathbb{R}^+)$, then the **Fourier Cosine Transform** *of f is the function $\widehat{f}_{\cos} : \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined:*

$$\widehat{f}_{\cos}(\mu) = \boxed{\frac{2}{\pi}\int_0^\infty f(x)\cdot\cos(\mu x)\ dx}$$

the **Fourier Sine Transform** *of f is the function $\widehat{f}_{\sin} : \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined:*

$$\widehat{f}_{\sin}(\mu) = \boxed{\frac{2}{\pi}\int_0^\infty f(x)\cdot\sin(\mu x)\ dx}$$

Theorem 17.31: Fourier (co)sine Inversion Formula

Suppose that $f \in \mathbf{L}^1(\mathbb{R}^+)$. Then for any fixed $x > 0$ so that f is continuous at x,

$$\begin{aligned} f(x) &= \lim_{M\to\infty}\int_0^M \widehat{f}_{\cos}(\mu)\cdot\cos(\mu\cdot x)\ d\mu, \\ \text{and } f(x) &= \lim_{M\to\infty}\int_0^M \widehat{f}_{\sin}(\mu)\cdot\sin(\mu\cdot x)\ d\mu, \end{aligned}$$

Also, if $f(0) = 0$, then the Fourier sine series also converges at 0. If $f(0) \neq 0$, then the Fourier cosine series converges at 0. □

17.6 Practice Problems

1. Suppose $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1; \\ 0 & \text{otherwise} \end{cases}$, as in Example 17.5 on page 339. Check that $\widehat{f}(\mu) = \dfrac{1 - e^{-\mu\mathbf{i}}}{2\pi\mu\mathbf{i}}$

2. Compute the one-dimensional Fourier transforms of $g(x)$, when:

   (a) $g(x) = \begin{cases} 1 & \text{if } -\tau < x < 1-\tau; \\ 0 & \text{otherwise} \end{cases}$,

   (b) $g(x) = \begin{cases} 1 & \text{if } 0 < x < \sigma; \\ 0 & \text{otherwise} \end{cases}$.

3. Let $X, Y > 0$, and let $f(x, y) = \begin{cases} 1 & \text{if } 0 \le x \le X \text{ and } 0 \le y \le Y; \\ 0 & \text{otherwise.} \end{cases}$. Compute the two-dimensional Fourier transform of $f(x, y)$. What does the Fourier Inversion formula tell us?

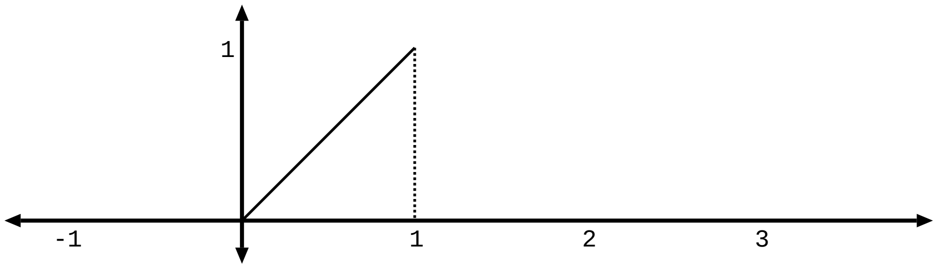


Figure 17.9: Problem #4

4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined: $f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$ (Fig.17.9)
   Compute the **Fourier Transform** of f.

5. Let $f(x) = x \cdot \exp\left(\frac{-x^2}{2}\right)$. Compute the Fourier transform of f.

6. Let $\alpha > 0$, and let $g(x) = \frac{1}{\alpha^2 + x^2}$. Example 17.9 claims that $\widehat{g}(\mu) = \frac{1}{2\alpha} e^{-\alpha|\mu|}$. Verify this. **Hint:** *Use the Fourier Inversion Theorem.*

7. Fix $y > 0$, and let $\mathcal{K}_y(x) = \frac{y}{\pi(x^2 + y^2)}$ (this is the *half-space Poisson Kernel* from §16.4 and §18.3(b)).

   Compute the one-dimensional Fourier transform $\widehat{\mathcal{K}}_y(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{K}_y(x) \exp\Big(-\mu \mathbf{i} x\Big)\, d\mu$.

8. Let $f(x) = \frac{2x}{(1 + x^2)^2}$. Compute the Fourier transform of f.

9. Let $f(x) = \begin{cases} 1 & \text{if } -4 < x < 5; \\ 0 & \text{otherwise.} \end{cases}$ Compute the Fourier transform $\widehat{f}(\mu)$.

10. Let $f(x) = \frac{x\cos(x) - \sin(x)}{x^2}$. Compute the Fourier transform $\widehat{f}(\mu)$.

11. Let $f, g \in \mathbf{L}^1(\mathbb{R})$, and let $h(x) = f(x) + g(x)$. Show that, for all $\mu \in \mathbb{R}$, $\widehat{h}(\mu) = \widehat{f}(\mu) + \widehat{g}(\mu)$.

12. Let $f, g \in \mathbf{L}^1(\mathbb{R})$, and let $h = f * g$. Show that for all $\mu \in \mathbb{R}$, $\widehat{h}(\mu) = 2\pi \cdot \widehat{f}(\mu) \cdot \widehat{g}(\mu)$.
    Hint: $\exp(-\mathbf{i}\mu x) = \exp(-\mathbf{i}\mu y) \cdot \exp\Big(-\mathbf{i}\mu(x - y)\Big)$.

13. Let $f, g \in \mathbf{L}^1(\mathbb{R})$, and let $h(x) = f(x) \cdot g(x)$. Suppose $\widehat{h}$ is also in $\mathbf{L}^1(\mathbb{R})$. Show that, for all $\mu \in \mathbb{R}$, $\widehat{h}(\mu) = (\widehat{f} * \widehat{g})(\mu)$.
    Hint: *Combine problem #12 with the Strong Fourier Inversion Formula (Theorem 17.6 on page 339).*

14. Let $f \in \mathbf{L}^1(\mathbb{R})$. Fix $\tau \in \mathbb{R}$, and define $g : \mathbb{R} \longrightarrow \mathbb{C}$ by: $g(x) = f(x + \tau)$. Show that, for all $\mu \in \mathbb{R}$, $\widehat{g}(\mu) = e^{\tau\mu\mathbf{i}} \cdot \widehat{f}(\mu)$.

15. Let $f \in \mathbf{L}^1(\mathbb{R})$. Fix $\nu \in \mathbb{R}$ and define $g : \mathbb{R} \longrightarrow \mathbb{C}$ by $g(x) = e^{\nu x\mathbf{i}} f(x)$. Show that, for all $\mu \in \mathbb{R}$, $\widehat{g}(\mu) = \widehat{f}(\mu - \nu)$.

16. Suppose $f \in \mathbf{L}^1(\mathbb{R})$. Fix $\sigma > 0$, and define $g : \mathbb{R} \longrightarrow \mathbb{C}$ by: $g(x) = f\left(\frac{x}{\sigma}\right)$. Show that, for all $\mu \in \mathbb{R}$, $\widehat{g}(\mu) = \sigma \cdot \widehat{f}(\sigma \cdot \mu)$.

17. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable, and that $f \in \mathbf{L}^1(\mathbb{R})$ and $g := f' \in \mathbf{L}^1(\mathbb{R})$. Assume that $\lim_{x\to\pm\infty} f(x) = 0$. Show that $\widehat{g}(\mu) = \mathbf{i}\mu \cdot \widehat{f}(\mu)$.

18. Let $\mathcal{G}_t(x) = \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-x^2}{4t}\right)$ be the Gauss-Weierstrass kernel. Recall that $\widehat{\mathcal{G}_t}(\mu) = \frac{1}{2\pi} e^{-\mu^2 t}$. Use this to construct a simple proof that, for any $s, t > 0$, $\mathcal{G}_t * \mathcal{G}_s = \mathcal{G}_{t+s}$.

    (**Hint:** *Use problem #12. Do* **not** *compute any convolution integrals, and do* **not** *use the 'solution to the heat equation' argument from Problem # 8 on page 334.*)

    Remark: Because of this result, probabilists say that the set $\{\mathcal{G}_t\}_{t\in(0,\infty)}$ forms a *stable family of probability distributions* on $\mathbb{R}$. Analysts say that $\{\mathcal{G}_t\}_{t\in(0,\infty)}$ is a *one-parameter semigroup* under convolution.

18 Fourier Transform Solutions to PDEs

The 'Fourier series' solutions to the PDEs on a bounded domain generalize to 'Fourier transform' solutions on the unbounded domain in the obvious way.

18.1 The Heat Equation

18.1(a) Fourier Transform Solution

Prerequisites: §2.2, §17.1, §6.4, §1.8 **Recommended:** §11.1, §12.2, §**??**, §17.3, §17.4

Proposition 18.1: Heat Equation on an Infinite Rod

Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded function (of $\mu \in \mathbb{R}$).

(a) *For all $t > 0$ and all $x \in \mathbb{R}$, define $u : \mathbb{R} \times (0, \infty) \longrightarrow \mathbb{R}$ by*

$$u(x,t) \ := \ \int_{-\infty}^{\infty} F(\mu) \cdot \exp(\mu x \mathbf{i}) \cdot e^{-\mu^2 t} \, d\mu, \qquad \text{for all } x \in \mathbb{R} \text{ and } t > 0.$$

then $u(x,t)$ is a smooth function and satisifies the Heat Equation.

(b) *In particular, suppose $f \in \mathbf{L}^1(\mathbb{R})$, and $\widehat{f}(\mu) = F(\mu)$. If we define $u(x,0) = f(x)$, and $u(x,t)$ by the previous formula when $t > 0$, then $u(x,t)$ is continuous, and is solution to the Heat Equation with initial conditions $u(x,0) = f(x)$.*

Proof: **Exercise 18.1** Hint: Use Proposition 1.9 on page 18 □

Example 18.2: Suppose $f(x) \ = \ \begin{cases} 1 & \text{if } \ -1 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$ We already know from Example 17.4 on page 339 that $\widehat{f}(\mu) \ = \ \dfrac{\sin(\mu)}{\pi \mu}$. Thus,

$$u(x,t) \ = \ \int_{-\infty}^{\infty} \widehat{f}(\mu) \cdot \exp(\mu x \mathbf{i}) \cdot e^{-\mu^2 t} \, d\mu \ = \ \int_{-\infty}^{\infty} \frac{\sin(\mu)}{\pi \mu} \exp(\mu x \mathbf{i}) \cdot e^{-\mu^2 t} \, d\mu,$$

by which, of course, we really mean $\displaystyle \lim_{M \to \infty} \int_{-M}^{M} \frac{\sin(\mu)}{\pi \mu} \exp(\mu x \mathbf{i}) \cdot e^{-\mu^2 t} \, d\mu.$ ◇

Example 18.3: The Gauss-Weierstrass Kernel

For all $x \in \mathbb{R}$ and $t > 0$, define the **Gauss-Weierstrass Kernel**: $\mathcal{G}(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-x^2}{4t}\right)$.
If we fix $t > 0$ and define $\mathcal{G}_t(x) = \mathcal{G}(x,t)$, then, setting $\sigma = \sqrt{2t}$ in Theorem 17.17**(b)**, we get

$$\widehat{\mathcal{G}}_t(\mu) \quad = \quad \frac{1}{2\pi} \exp\left(\frac{-(\sqrt{2t})^2\mu^2}{2}\right) \quad = \quad \frac{1}{2\pi} \exp\left(\frac{-2t\mu^2}{2}\right) \quad = \quad \frac{1}{2\pi} e^{-\mu^2 t}$$

Thus, applying the Fourier Inversion formula (Theorem 17.2 on page 338), we have:

$$\mathcal{G}(x,t) \ = \ \int_{-\infty}^{\infty} \widehat{\mathcal{G}}_t(\mu) \exp(\mu x \mathbf{i}) \, d\mu \ = \ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mu^2 t} \exp(\mu x \mathbf{i}) \, d\mu,$$

which, according to Proposition 18.1, is a smooth solution of the Heat Equation, where we take $F(\mu)$ to be the *constant* function: $F(\mu) = 1/2\pi$. Thus, F is *not* the Fourier transform of any function f. Hence, the Gauss-Weierstrass kernel solves the Heat Equation, but the "initial conditions" $\mathcal{G}_0$ do not correspond to a function, but instead a define more singular object, rather like an infinitely dense concentration of mass at a single point. Sometimes $\mathcal{G}_0$ is called the **Dirac delta function**, but this is a misnomer, since it isn't really a function. Instead, $\mathcal{G}_0$ is an example of a more general class of objects called *distributions*. ♢

Proposition 18.4: Heat Equation on an Infinite Plane

Let $F : \mathbb{R}^2 \longrightarrow \mathbb{C}$ be some bounded function (of $(\mu,\nu) \in \mathbb{R}^2$).

(a) *For all $t > 0$ and all $(x,y) \in \mathbb{R}^2$, define*

$$u(x,y;\ t) \quad = \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu,\nu) \cdot \exp\Big((\mu x + \nu y) \cdot \mathbf{i}\Big) \cdot e^{-(\mu^2+\nu^2)t} \, d\mu \, d\nu.$$

Then u is a smooth function and satisfies the two-dimensional Heat Equation.

(b) *In particular, suppose $f \in \mathbf{L}^1(\mathbb{R}^2)$, and $\widehat{f}(\mu,\nu) = F(\mu,\nu)$. If we define $u(x,y,0) = f(x,y)$, and $u(x,y,t)$ by the previous formula when $t > 0$, then $u(x,y,t)$ is continuous, and is solution to the Heat Equation with initial conditions $u(x,y,0) = f(x,y)$.*

Proof: <u>**Exercise 18.2**</u> Hint: Use Proposition 1.9 on page 18 □

Example 18.5: Let $X, Y > 0$ be constants, and suppose the initial conditions are:

$$f(x,y) \quad = \quad \begin{cases} 1 & \text{if } -X \le x \le X \text{ and } -Y \le y \le Y; \\ 0 & \text{otherwise.} \end{cases}$$

From Example 17.23 on page 348, the Fourier transform of $f(x,y)$ is given:

$$\widehat{f}(\mu,\nu) \quad = \quad \frac{\sin(\mu X) \cdot \sin(\nu Y)}{\pi^2 \cdot \mu \cdot \nu}.$$

Thus, the corresponding solution to the two-dimensional Heat equation is:

$$\begin{aligned} u(x,y,t) &= \int_{\mathbb{R}^2} \widehat{f}(\mu,\nu) \cdot \exp\Big((\mu x + \nu y)\cdot \mathbf{i}\Big) \cdot e^{-(\mu^2+\nu^2)t}\ d\mu\ d\nu \\ &= \int_{\mathbb{R}^2} \frac{\sin(\mu X)\cdot \sin(\nu Y)}{\pi^2 \cdot \mu \cdot \nu} \cdot \exp\Big((\mu x + \nu y)\cdot \mathbf{i}\Big) \cdot e^{-(\mu^2+\nu^2)t}\ d\mu\ d\nu\ . \end{aligned}$$

◇

Proposition 18.6: Heat Equation in Infinite Space

Let $F : \mathbb{R}^3 \longrightarrow \mathbb{C}$ be some bounded function (of $\boldsymbol{\mu} \in \mathbb{R}^3$).

(a) *For all $t > 0$ and all $(x_1, x_2, x_3) \in \mathbb{R}^3$, define*

$$u(x_1,x_2,x_3;\ t) \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\boldsymbol{\mu}) \cdot \exp\Big(\langle \boldsymbol{\mu}, \mathbf{x}\rangle \cdot \mathbf{i}\Big) \cdot e^{-\|\mu\|^2 t}\ d\boldsymbol{\mu}.$$

where $\|\boldsymbol{\mu}\|^2 = \mu_1^2 + \mu_2^2 + \mu_3^2$. Then u is a smooth function and satisfies the three-dimensional Heat Equation.

(b) *In particular, suppose $f \in \mathbf{L}^1(\mathbb{R}^3)$, and $\widehat{f}(\boldsymbol{\mu}) = F(\boldsymbol{\mu})$. If we define $u(x_1,x_2,x_3,0) = f(x_1,x_2,x_3)$, and $u(x_1,x_2,x_3,t)$ by the previous formula when $t > 0$, then $u(x_1,x_2,x_3,t)$ is continuous, and is solution to the Heat Equation with initial conditions $u(\mathbf{x},0) = f(\mathbf{x})$.*

Proof: <u>**Exercise 18.3**</u> Hint: Use Proposition 1.9 on page 18 □

Example 18.7: A ball of heat

Suppose the initial conditions are: $f(\mathbf{x}) = \begin{cases} 1 & \text{if } \|\mathbf{x}\| \le 1; \\ 0 & \text{otherwise.} \end{cases}$

Setting $R = 1$ in Example 17.28 (p.350) yields the three-dimensional Fourier transform of f:

$$\widehat{f}(\boldsymbol{\mu}) \;=\; \frac{1}{2\pi^2}\left(\frac{\sin\|\boldsymbol{\mu}\|}{\|\boldsymbol{\mu}\|^3} - \frac{\cos\|\boldsymbol{\mu}\|}{\|\boldsymbol{\mu}\|^2}\right).$$

The resulting solution to the Heat Equation is:

$$\begin{aligned} u(\mathbf{x};\ t) &= \int_{\mathbb{R}^3} \widehat{f}(\boldsymbol{\mu}) \cdot \exp\Big(\langle \boldsymbol{\mu}, \mathbf{x}\rangle \cdot \mathbf{i}\Big) \cdot e^{-\|\mu\|^2 t}\ d\boldsymbol{\mu} \\ &= \frac{1}{2\pi^2}\int_{\mathbb{R}^3} \left(\frac{\sin\|\boldsymbol{\mu}\|}{\|\boldsymbol{\mu}\|^3} - \frac{\cos\|\boldsymbol{\mu}\|}{\|\boldsymbol{\mu}\|^2}\right) \cdot \exp\Big(\langle \boldsymbol{\mu}, \mathbf{x}\rangle \cdot \mathbf{i}\Big) \cdot e^{-\|\mu\|^2 t}\ d\boldsymbol{\mu}. \end{aligned}$$

◇

18.1(b) The Gaussian Convolution Formula, revisited

Prerequisites: §16.3(a), §17.2, §18.1(a)

Recall from § 16.3(a) on page 307 that the Gaussian Convolution formula solved the initial value problem for the Heat Equation by "locally averaging" the initial conditions. Fourier methods provide another proof that this is a solution to the Heat Equation.

Theorem 18.8: Gaussian Convolutions and the Heat Equation

Let $f \in \mathbf{L}^1(\mathbb{R})$, and let $\mathcal{G}_t(x)$ be the Gauss-Weierstrass kernel from Example 18.3. For all $t > 0$, define $U_t = f * \mathcal{G}_t$; in other words, for all $x \in \mathbb{R}$,

$$U_t(x) \quad = \quad \int_{-\infty}^{\infty} f(y) \cdot \mathcal{G}_t(x-y) \; dy$$

Also, for all $x \in \mathbb{R}$, define $U_0(x) = f(x)$. Then $U_t(x)$ is a smooth function of two variables, and is the unique solution to the Heat Equation with initial conditions $U(x,0) = f(x)$.

Proof: $U(x,0) = f(x)$ by definition. To show that U satisfies the Heat Equation, we will show that it is in fact *equal* to the Fourier solution $u(x,t)$ described in Theorem 18.1 on page 355. Fix $t > 0$, and let $u_t(x) = u(x,t)$; recall that, by definition

$$u_t(x) \; = \; \int_{-\infty}^{\infty} \widehat{f}(\mu) \cdot \exp(\mu x \mathbf{i}) \cdot e^{-\mu^2 t} \; d\mu \; = \; \int_{-\infty}^{\infty} \widehat{f}(\mu) e^{-\mu^2 t} \cdot \exp(\mu x \mathbf{i}) \; d\mu$$

Thus, Corollary 17.3 on page 338 says that

$$\widehat{u}_t(\mu) \quad = \quad \widehat{f}(\mu) \cdot e^{-t\mu^2} \underset{(*)}{=} 2\pi \cdot \widehat{f}(\mu) \cdot \widehat{\mathcal{G}}_t(\mu). \tag{18.1}$$

Here, $(*)$ is because Example 18.3 on page 356 says that $e^{-t\mu^2} \; = \; 2\pi \cdot \widehat{\mathcal{G}}_t(\mu)$

But remember that $U_t = f * \mathcal{G}_t$, so, Theorem 17.11**(b)** says

$$\widehat{U}_t(\mu) \quad = \quad 2\pi \cdot \widehat{f}(\mu) \cdot \widehat{\mathcal{G}}_t(\mu). \tag{18.2}$$

Thus (18.1) and (18.2) mean that $\widehat{U}_t = \widehat{u}_t$. But then Corollary 17.3 on page 338 implies that $u_t(x) = U_t(x)$. □

For more discussion and examples of the Gaussian convolution approach to the Heat Equation, see § 16.3(a) on page 307.

18.2 The Wave Equation

18.2(a) Fourier Transform Solution

Prerequisites: §3.2, §17.1, §6.4, §1.8 **Recommended:** §11.2, §12.4, §17.3, §17.4, §18.1(a)

Proposition 18.9: Wave Equation on an Infinite Wire

Let $f_0, f_1 \in \mathbf{L}^1(\mathbb{R})$ be twice-differentiable, and suppose f_0 and f_1 have Fourier transforms $\widehat{f_0}$ and $\widehat{f_1}$, respectively. Define $u : \mathbb{R} \times [0,\infty) \longrightarrow \mathbb{R}$ by

$$u(x,t) \;=\; \int_{-\infty}^{\infty} \left(\widehat{f_0}(\mu)\cos(\mu t) + \frac{\widehat{f_1}(\mu)}{\mu}\sin(\mu t) \right) \cdot \exp(\mu x \mathbf{i}) \; d\mu$$

Then $u(x,t)$ is the unique solution to the Wave Equation with **Initial Position:** $u(x,0) = f_0(x)$, *and* **Initial Velocity:** $\partial_t u(x,0) = f_1(x)$.

Proof: **Exercise 18.4** Hint: Show that this solution is equivalant to the d'Alembert solution of Proposition 16.25. □

Example 18.10: Suppose $\alpha > 0$ is a constant, and suppose $f_0(x) = \dfrac{1}{(\alpha^2 + x^2)}$, as in Example 17.9 on page 341, while $f_1 \equiv 0$. We know from Example 17.9 that $\widehat{f_0}(\mu) = \dfrac{1}{2\alpha} e^{-\alpha \cdot |\mu|}$. Thus, Proposition 18.9 says:

$$\begin{aligned} u(x,t) \;&=\; \int_{-\infty}^{\infty} \widehat{f_0}(\mu) \cdot \exp(\mu x \mathbf{i}) \cdot \cos(\mu t) \; d\mu \;=\; \int_{-\infty}^{\infty} \frac{1}{2\alpha} e^{-\alpha\cdot|\mu|} \cdot \exp(\mu x \mathbf{i}) \cdot \cos(\mu t) \; d\mu \\ &=\; \frac{1}{2\alpha} \int_{-\infty}^{\infty} \exp\left(\mu x \mathbf{i} - \alpha \cdot |\mu|\right) \cdot \cos(\mu t) \; d\mu, \end{aligned}$$

by which, of course, we really mean $\displaystyle \frac{1}{2\alpha} \lim_{M\to\infty} \int_{-M}^{M} \exp\left(\mu x \mathbf{i} - \alpha \cdot |\mu|\right) \cdot \cos(\mu t) \; d\mu.$ ♢

Proposition 18.11: Wave Equation on an Infinite Plane

Let $f_0, f_1 \in \mathbf{L}^1(\mathbb{R}^2)$ be twice differentiable, and suppose f_0 and f_1 have Fourier transforms $\widehat{f_0}$ and $\widehat{f_1}$, respectively. Define $u : \mathbb{R}^2 \times [0,\infty) \longrightarrow \mathbb{R}$ by

$$u(x,y,t) \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\widehat{f_0}(\mu,\nu)\cos\left(\sqrt{\mu^2+\nu^2}\;\cdot t\right) + \frac{\widehat{f_1}(\mu,\nu)}{\sqrt{\mu^2+\nu^2}}\sin\left(\sqrt{\mu^2+\nu^2}\;\cdot t\right) \right) \cdot \exp\Big((\mu x + \nu y)\cdot \mathbf{i}\Big) \; d\mu \; d\nu.$$

Then $u(x,y,t)$ is the unique solution to the Wave Equation with **Initial Position:** $u(x,y,0) = f_0(x,y)$, *and* **Initial Velocity:** $\partial_t u(x,y,0) = f_1(x,y)$.

Proof: **Exercise 18.5** Hint: Use Proposition 1.9 on page 18 □

Example 18.12: *Thor the thunder god is angry, and smites the Earth with his mighty hammer. Model the resulting shockwave as it propagates across the Earth's surface.*

Solution: As everyone knows, the Earth is a vast, flat sheet, supported on the backs of four giant turtles. We will thus approximate the Earth as an infinite plane. Thor's hammer has a square head; we will assume the square has sidelength 2 (in appropriate units). Thus, if $u(x,y;t)$ is the shockwave, then we have

$$\begin{array}{rrcl} \textbf{Initial Position:} & u(x,y,0) & = & 0, \\ \textbf{Initial Velocity:} & \partial_t u(x,y,0) & = & f_1(x,y) \quad = \quad \begin{cases} 1 & \text{if } -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1; \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

Setting $X = Y = 1$ in Example 17.23 on page 348, we get the Fourier transform of $f_1(x,y)$:

$$\widehat{f_1}(\mu,\nu) \quad = \quad \frac{\sin(\mu)\cdot\sin(\nu)}{\pi^2\cdot\mu\cdot\nu}$$

Thus, Proposition 18.11 says that the corresponding solution to the two-dimensional wave equation is:

$$\begin{aligned} u(x,y,t) &= \int_{\mathbb{R}^2} \frac{\widehat{f_1}(\mu,\nu)}{\sqrt{\mu^2+\nu^2}} \sin\left(\sqrt{\mu^2+\nu^2}\ \cdot t\right)\cdot \exp\Big((\mu x+\nu y)\cdot \mathbf{i}\Big)\ d\mu\ d\nu \\ &= \frac{1}{\pi^2}\int_{\mathbb{R}^2} \frac{\sin(\mu)\cdot\sin(\nu)}{\mu\nu\cdot\sqrt{\mu^2+\nu^2}} \sin\left(\sqrt{\mu^2+\nu^2}\ \cdot t\right)\cdot \exp\Big((\mu x+\nu y)\cdot \mathbf{i}\Big)\ d\mu\ d\nu. \qquad \diamondsuit \end{aligned}$$

Remark: Strictly speaking, Proposition 18.11 is not applicable to Example 18.12, because the initial conditions are not twice-differentiable. However, we can imagine approximating the discontinuous function $f_1(x,y)$ in Example 18.12 very closely by a smooth function (see Proposition 16.20 on page 320). It is 'physically reasonable' to believe that the resulting solution (obtained from Proposition 18.11) will very closely approximate the 'real' solution (obtained from initial conditions $f_1(x,y)$ of Example 18.12).

This is not a rigorous argument, but, it can be made rigorous, using the concept of *generalized solutions* to PDEs. However, this is beyond the scope of these notes.

Proposition 18.13: Wave Equation in Infinite Space

Let $f_0, f_1 \in \mathbf{L}^1(\mathbb{R}^3)$ be twice differentiable, and suppose f_0 and f_1 have Fourier transforms $\widehat{f_0}$ and $\widehat{f_1}$, respectively. Define $u : \mathbb{R}^3 \times [0,\infty) \longrightarrow \mathbb{R}$ by

$$u(x_1,x_2,x_3,t) \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\widehat{f_0}(\boldsymbol{\mu})\cos\left(\|\boldsymbol{\mu}\|\cdot t\right) \;+\; \frac{\widehat{f_1}(\boldsymbol{\mu})}{\|\boldsymbol{\mu}\|}\sin\left(\|\boldsymbol{\mu}\|\cdot t\right)\right)\cdot\exp\left(\langle\boldsymbol{\mu},\mathbf{x}\rangle\,\mathbf{i}\right)\ d\boldsymbol{\mu}$$

Then $u(\mathbf{x},t)$ is the unique solution to the Wave Equation with

$$\begin{array}{rrcl} \textbf{Initial Position:} & u(x_1,x_2,x_3,0) & = & f_0(x_1,x_2,x_3); \\ \textbf{Initial Velocity:} & \partial_t u(x_1,x_2,x_3,0) & = & f_1(x_1,x_2,x_3). \end{array}$$

Proof: <u>**Exercise 18.6**</u> Hint: Use Proposition 1.9 on page 18 □

Example 18.14: *A firecracker explodes in mid air. Model the resulting soundwave.*

Solution: When the firecracker explodes, it creates a small region of extremely high pressure; this region rapidly expands and becomes a pressure wave which we hear as sound.

Assuming the explosion is spherically symmetrical, we can assume that the pressurized region at the moment of detonation is a small ball. Thus, we have:

$$\textbf{Initial Position:} \quad u(\mathbf{x};0) \;=\; f_0(\mathbf{x}) \;=\; \begin{cases} 1 & \text{if } \|\mathbf{x}\| \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

$$\textbf{Initial Velocity:} \quad \partial_t\, u(\mathbf{x};\ 0) \;=\; 0.$$

Setting $R = 1$ in Example 17.28 on page 350, we find the three-dimensional Fourier transform of f:

$$\widehat{f_0}(\boldsymbol{\mu}) \;=\; \frac{1}{2\pi^2}\left(\frac{\sin\|\boldsymbol{\mu}\|}{\|\boldsymbol{\mu}\|^3} - \frac{\cos\|\boldsymbol{\mu}\|}{\|\boldsymbol{\mu}\|^2}\right).$$

The resulting solution to the wave equation is:

$$u(\mathbf{x};\ t) \;=\; \int_{\mathbb{R}^3} \widehat{f_0}(\boldsymbol{\mu}) \cos\left(\|\boldsymbol{\mu}\|\cdot t\right)\cdot \exp\left(\langle \boldsymbol{\mu}, \mathbf{x}\rangle\, \mathbf{i}\right)\ d\boldsymbol{\mu}$$

$$=\; \frac{1}{2\pi^2}\int_{\mathbb{R}^3}\left(\frac{\sin\|\boldsymbol{\mu}\|}{\|\boldsymbol{\mu}\|^3} - \frac{\cos\|\boldsymbol{\mu}\|}{\|\boldsymbol{\mu}\|^2}\right)\cos\left(\|\boldsymbol{\mu}\|\cdot t\right)\cdot \exp\left(\langle \boldsymbol{\mu}, \mathbf{x}\rangle\, \mathbf{i}\right)\ d\boldsymbol{\mu}\ . \qquad \diamondsuit$$

18.2(b) Poisson's Spherical Mean Solution; Huygen's Principle

Prerequisites: §16.1, §17.4, §18.2(a) **Recommended:** §16.6, §18.1(b)

The Gaussian Convolution formula of §18.1(b) solves the initial value problem for the Heat Equation in terms of a kind of "local averaging" of the initial conditions. Similarly, d'Alembert's formula (§16.6) solves the initial value problem for the one-dimensional Wave Equation in terms of a local average.

There an analogous result for higher-dimensional wave equations. To explain it, we must introduce the concept of *spherical averages*. Suppose $f(x_1, x_2, x_3)$ is a function of three variables. If $\mathbf{x} \in \mathbb{R}^3$ is a point in space, and $R > 0$, then the **spherical average** of f at $\mathbf{x}$, of radius R, is defined:

$$\mathbf{M}_R\, f(\mathbf{x}) \;=\; \frac{1}{4\pi R^2}\int_{\mathbb{S}(R)} f(\mathbf{x}+\mathbf{s})\ d\mathbf{s}$$

Here, $\mathbb{S}(R) = \left\{\mathbf{s} \in \mathbb{R}^3\ ;\ \|\mathbf{s}\| = R\right\}$ is the sphere around 0 of radius R. "$\mathbf{s}$" is a point on the the sphere, and "$d\mathbf{s}$" is the natural measure of surface area relative to which we compute integrals over spheres. The total surface area of the sphere is $4\pi R^2$; notice that we divide out by this quantity to obtain an average.

Theorem 18.15: Poisson's Spherical Mean Solution to Wave Equation

(a) *Suppose $f_1 \in \mathbf{L}^1(\mathbb{R}^3)$. For all $\mathbf{x} \in \mathbb{R}^3$ and $t > 0$, define*

$$v(\mathbf{x};t) \;=\; t \cdot \mathbf{M}_t\, f_1(\mathbf{x})$$

Then $v(\mathbf{x};t)$ is the unique solution to the Wave Equation with

Initial Position: $v(\mathbf{x},0) \;=\; 0$; **Initial Velocity:** $\partial_t\, v(\mathbf{x},0) \;=\; f_1(\mathbf{x})$.

(b) *Suppose $f_0 \in \mathbf{L}^1(\mathbb{R}^3)$. For all $\mathbf{x} \in \mathbb{R}^3$ and $t > 0$, define $W(\mathbf{x};t) \;=\; t \cdot \mathbf{M}_t\, f_0(\mathbf{x})$, and then define*

$$w(\mathbf{x};t) \;=\; \partial_t\, W(\mathbf{x};t)$$

Then $w(\mathbf{x};t)$ is the unique solution to the Wave Equation with

Initial Position: $w(\mathbf{x},0) \;=\; f_0(\mathbf{x})$; **Initial Velocity:** $\partial_t\, w(\mathbf{x},0) \;=\; 0$.

(c) *Let $f_0, f_1 \in \mathbf{L}^1(\mathbb{R}^3)$, and for all $\mathbf{x} \in \mathbb{R}^3$ and $t > 0$, define*

$$u(\mathbf{x};t) \;=\; w(\mathbf{x};t) \;+\; v(\mathbf{x};t)$$

where $w(\mathbf{x};t)$ is as in **Part (b)** *and $v(\mathbf{x};t)$ is as in* **Part (a)**. *Then $u(\mathbf{x};t)$ is the unique solution to the Wave Equation with*

Initial Position: $u(\mathbf{x},0) \;=\; f_0(\mathbf{x})$; **Initial Velocity:** $\partial_t\, u(\mathbf{x},0) \;=\; f_1(\mathbf{x})$.

Proof: We will prove **Part (a)**. First we will need a certain calculation....

Claim 1: *For any $R > 0$, and any $\boldsymbol{\mu} \in \mathbb{R}^3$,* $\displaystyle\int_{\mathbb{S}(R)} \exp\Big(\langle \boldsymbol{\mu}, \mathbf{s}\rangle\, \mathbf{i}\Big)\; d\mathbf{s} \;=\; \frac{4\pi R \cdot \sin\left(\|\boldsymbol{\mu}\| \cdot R\right)}{\|\boldsymbol{\mu}\|}$.

Proof: By spherical symmetry, we can rotate the vector $\boldsymbol{\mu}$ without affecting the value of the integral, so rotate $\boldsymbol{\mu}$ until it becomes $\boldsymbol{\mu} = (\mu, 0, 0)$, with $\mu > 0$. Thus, $\|\boldsymbol{\mu}\| = \mu$, and, if a point $\mathbf{s} \in \mathbb{S}(R)$ has coordinates (s_1, s_2, s_3) in $\mathbb{R}^3$, then $\langle \boldsymbol{\mu}, \mathbf{s}\rangle = \mu \cdot s_1$. Thus, the integral simplifies to:

$$\int_{\mathbb{S}(R)} \exp\left(\langle \boldsymbol{\mu}, \mathbf{s}\rangle\, \mathbf{i}\right)\; d\mathbf{s} \;=\; \int_{\mathbb{S}(R)} \exp\left(\mu \cdot s_1 \cdot \mathbf{i}\right)\; d\mathbf{s}$$

We will integrate using a spherical coordinate system (ϕ, θ) on the sphere, where $0 < \phi < \pi$ and $-\pi < \theta < \pi$, and where

$$(s_1, s_2, s_3) \;=\; R \cdot \left(\cos(\phi),\;\; \sin(\phi)\sin(\theta),\;\; \sin(\phi)\cos(\theta)\right).$$

The surface area element is given

$$d\mathbf{s} \;=\; R^2 \sin(\phi)\; d\theta\; d\phi$$

$$\text{Thus,} \quad \int_{\mathbb{S}(R)} \exp(\mu \cdot s_1 \cdot \mathbf{i}) \; d\mathbf{s} \;=\; \int_0^{\pi}\int_{-\pi}^{\pi} \exp(\mu \cdot R \cdot \cos(\phi) \cdot \mathbf{i}) \cdot R^2 \sin(\phi) \; d\theta \; d\phi$$

$$\underset{(1)}{=} \; 2\pi \int_0^{\pi} \exp(\mu \cdot R \cdot \cos(\phi) \cdot \mathbf{i}) \cdot R^2 \sin(\phi) \; d\phi$$

$$\underset{(2)}{=} \; 2\pi \int_{-R}^{R} \exp(\mu \cdot s_1 \cdot \mathbf{i}) \cdot R \; ds_1$$

$$= \; \frac{2\pi R}{\mu \mathbf{i}} \exp\Big(\mu \cdot s_1 \cdot \mathbf{i}\Big)_{s_1=-R}^{s_1=R}$$

$$= \; 2\frac{2\pi R}{\mu} \cdot \left(\frac{e^{\mu R \mathbf{i}} - e^{-\mu R \mathbf{i}}}{2\mathbf{i}}\right) \; \underset{(3)}{=} \; \frac{4\pi R}{\mu} \sin(\mu R)$$

(1) The integrand is constant in the θ coordinate. **(2)** Making substitution $s_1 = R\cos(\phi)$, so $ds_1 = -R\sin(\phi)\; d\phi$. **(3)** By de Moivre's formulae. $\diamond_{\text{Claim 1}}$

Now, by Proposition 18.13 on page 360, the unique solution to the Wave Equation with zero initial position and initial velocity f_1 is given by:

$$u(\mathbf{x}, t) \;=\; \int_{\mathbb{R}^3} \widehat{f_1}(\boldsymbol{\mu}) \frac{\sin(\|\boldsymbol{\mu}\| \cdot t)}{\|\boldsymbol{\mu}\|} \exp\Big(\langle \boldsymbol{\mu}, \mathbf{x}\rangle \,\mathbf{i}\Big) \; d\boldsymbol{\mu} \tag{18.3}$$

However, if we set $R = t$ in **Claim 1**, we have:

$$\frac{\sin(\|\boldsymbol{\mu}\| \cdot t)}{\|\boldsymbol{\mu}\|} \;=\; \frac{1}{4\pi t}\int_{\mathbb{S}(t)} \exp\Big(\langle \boldsymbol{\mu}, \mathbf{s}\rangle \,\mathbf{i}\Big) \; d\mathbf{s}$$

$$\text{Thus,} \quad \frac{\sin(\|\boldsymbol{\mu}\| \cdot t)}{\|\boldsymbol{\mu}\|} \cdot \exp\Big(\langle \boldsymbol{\mu}, \mathbf{x}\rangle \,\mathbf{i}\Big) \;=\; \exp\Big(\langle \boldsymbol{\mu}, \mathbf{x}\rangle \,\mathbf{i}\Big) \cdot \frac{1}{4\pi t}\int_{\mathbb{S}(t)} \exp\Big(\langle \boldsymbol{\mu}, \mathbf{s}\rangle \,\mathbf{i}\Big) \; d\mathbf{s}$$

$$= \; \frac{1}{4\pi t}\int_{\mathbb{S}(t)} \exp\Big(\langle \boldsymbol{\mu}, \mathbf{x}\rangle \,\mathbf{i} + \langle \boldsymbol{\mu}, \mathbf{s}\rangle \,\mathbf{i}\Big) \; d\mathbf{s}$$

$$= \; \frac{1}{4\pi t}\int_{\mathbb{S}(t)} \exp\Big(\langle \boldsymbol{\mu}, \; \mathbf{x} + \mathbf{s}\rangle \,\mathbf{i}\Big) \; d\mathbf{s}$$

Substituting this into (18.3), we get:

$$u(\mathbf{x}, t) \;=\; \int_{\mathbb{R}^3} \frac{\widehat{f_1}(\boldsymbol{\mu})}{4\pi t} \cdot \left(\int_{\mathbb{S}(t)} \exp\Big(\langle \boldsymbol{\mu}, \; \mathbf{x} + \mathbf{s}\rangle \,\mathbf{i}\Big) \; d\mathbf{s}\right) \; d\boldsymbol{\mu}$$

$$\underset{(1)}{=} \; \frac{1}{4\pi t}\int_{\mathbb{S}(t)} \int_{\mathbb{R}^3} \widehat{f_1}(\boldsymbol{\mu}) \cdot \exp\Big(\langle \boldsymbol{\mu}, \; \mathbf{x} + \mathbf{s}\rangle \,\mathbf{i}\Big) \; d\boldsymbol{\mu} \; d\mathbf{s}$$

$$\underset{(2)}{=} \; \frac{1}{4\pi t}\int_{\mathbb{S}(t)} f_1(\mathbf{x} + \mathbf{s}) \; d\mathbf{s} \;=\; t \cdot \frac{1}{4\pi t^2}\int_{\mathbb{S}(t)} f_1(\mathbf{x} + \mathbf{s}) \; d\mathbf{s} \;=\; t \cdot \mathbf{M}_t\, f_1(\mathbf{x}).$$

(1) We simply interchange the two integrals[1]. **(2)** This is just the Fourier Inversion theorem.

Part (b) is **Exercise 18.7** . **Part (c)** follows by combining **Part (a)** and **Part (b)**. □

[1] This actually involves some subtlety, which we will gloss over.

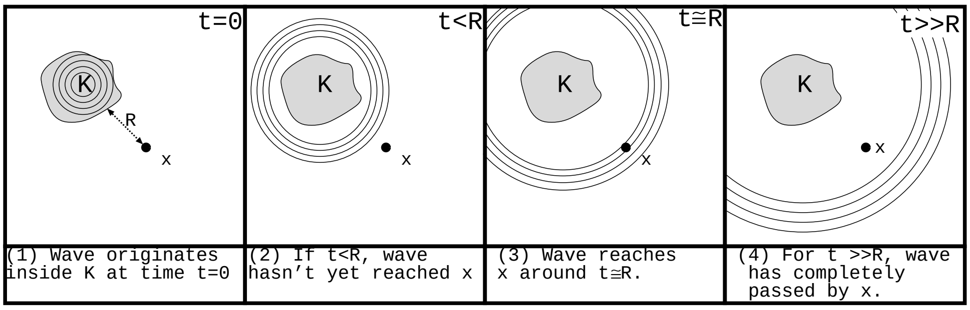


Figure 18.1: Huygen's principle.

Corollary 18.16: Huygen's Principle

Let $f_0, f_1 \in \mathbf{L}^1(\mathbb{R}^3)$, and suppose there is some bounded region $\mathbb{K} \subset \mathbb{R}^3$ so that f_0 and f_1 are zero outside of $\mathbb{K}$ —that is: $f_0(\mathbf{y}) = 0$ and $f_1(\mathbf{y}) = 0$ for all $\mathbf{y} \notin \mathbb{K}$ (see Figure 18.1A). Let $u(\mathbf{x}; t)$ be the solution to the Wave Equation with initial position f_0 and initial velocity f_1, and let $\mathbf{x} \in \mathbb{R}^3$

(a) *Let R be the distance from $\mathbb{K}$ to $\mathbf{x}$. If $t < R$ then $u(\mathbf{x}; t) = 0$ (Figure 18.1B).*

(b) *If t is large enough that $\mathbb{K}$ is* entirely contained *in a ball of radius t around $\mathbf{x}$, then $u(\mathbf{x}; t) = 0$ (Figure 18.1D).*

Proof: Exercise 18.8 □

Part (a) of Huygen's Principle says that, if a sound wave originates in the region $\mathbb{K}$ at time 0, and $\mathbf{x}$ is of distance R then it does not reach the point $\mathbf{x}$ before time R. This is not surprising; it takes time for sound to travel through space. **Part (b)** says that the soundwave propagates through the point $\mathbf{x}$ in a *finite* amount of time, and leaves no wake behind it. This is somewhat more surprising, but corresponds to our experience; sounds travelling through open spaces do not "reverberate" (except due to echo effects). It turns out, however, that **Part (b)** of the theorem is *not* true for waves travelling in *two* dimensions (eg. ripples on the surface of a pond).

18.3 The Dirichlet Problem on a Half-Plane

Prerequisites: §2.3, §17.1, §6.5, §1.8 **Recommended:** §12.1, §13.2, §17.3, §17.4

In §12.1 and §13.2, we saw how to solve Laplace's equation on a bounded domain such as a rectangle or a cube, in the context of Dirichlet boundary conditions. Now consider the **half-plane** domain $\mathbb{H} = \left\{(x, y) \in \mathbb{R}^2 \; ; \; y \geq 0\right\}$. The boundary of this domain is just the x axis: $\partial\mathbb{H} = \{(x, 0) \; ; \; x \in \mathbb{R}\}$; thus, boundary conditions are imposed by chosing some function $b(x)$ for $x \in \mathbb{R}$. Figure 16.14 on page 315 illustrates the corresponding **Dirichlet problem**: find a function $u(x, y)$ for $(x, y) \in \mathbb{H}$ so that

1. u satisfies the Laplace equation: $\triangle u(x,y) = 0$ for all $x \in \mathbb{R}$ and $y > 0$.

2. u satisfies the nonhomogeneous Dirichlet boundary condition: $u(x,0) = b(x)$.

18.3(a) Fourier Solution

Heuristically speaking, we will solve the problem by defining $u(x,y)$ as a continuous sequence of horizontal "fibres", parallel to the x axis, and ranging over all values of $y > 0$. Each fibre is a function only of x, and thus, has a one-dimensional Fourier transform. The problem then becomes determining these Fourier transforms from the Fourier transform of the boundary function b.

Proposition 18.17: Fourier Solution to Half-Plane Dirichlet problem

Let $b \in \mathbf{L}^1(\mathbb{R})$. Suppose that b has Fourier transform $\widehat{b}$, and define $u : \mathbb{H} \longrightarrow \mathbb{R}$ by

$$u(x,y) \quad := \quad \int_{-\infty}^{\infty} \widehat{b}(\mu) \cdot e^{-|\mu|\cdot y} \cdot \exp\Big(\mu \mathbf{i} x\Big) \; d\mu, \qquad \text{for all } x \in \mathbb{R} \text{ and } y \geq 0.$$

Then u the unique solution to the Laplace equation ($\triangle u = 0$) which satisfies the nonhomogeneous Dirichlet boundary condition $u(x,0) = b(x)$, for all $x \in \mathbb{R}$.

Proof: For any fixed $\mu \in \mathbb{R}$, the function $f_\mu(x,y) = \exp\Big(-|\mu|\cdot y\Big)\exp\Big(-\mu \mathbf{i} x\Big)$ is harmonic (see practice problem # 10 on page 369 of §18.6). Thus, Proposition 1.9 on page 18 implies that the function $u(x,y)$ is also harmonic. Finally, notice that, when $y = 0$, the expression for $u(x,0)$ is just the Fourier inversion integral for $b(x)$. □

Example 18.18: Suppose $b(x) \; = \; \begin{cases} 1 & \text{if } -1 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$ We already know from Example 17.4 on page 339 that $\widehat{b}(\mu) \; = \; \dfrac{\sin(\mu)}{\pi\mu}$.

Thus, $u(x,y) \; = \; \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{\sin(\mu)}{\mu} \cdot e^{-|\mu|\cdot y} \cdot \exp\Big(\mu \mathbf{i} x\Big) \; d\mu.$ ♢

18.3(b) Impulse-Response solution

Prerequisites: §18.3(a) **Recommended:** §16.4

For any $y > 0$, define the **Poisson kernel** $\mathcal{K}_y : \mathbb{R} \longrightarrow \mathbb{R}$ by:

$$\mathcal{K}_y(x) \quad = \quad \frac{y}{\pi(x^2+y^2)}. \qquad \text{(see Figure 16.15 on page 316)} \qquad (18.4)$$

In § 16.4 on page 315, we used the Poisson kernel to solve the half-plane Dirichlet problem using *impulse-response* methods (Proposition 16.16 on page 317). We can now use the 'Fourier' solution to provide another proof Proposition 16.16.

Proposition 18.19: Poisson Kernel Solution to Half-Plane Dirichlet problem

Let $b : \mathbb{R} \longrightarrow \mathbb{R}$ be an integrable function. For all $y > 0$ and $x \in \mathbb{R}$, define

$$U(x,y) \quad = \quad b * \mathcal{K}_y(x) \quad = \quad \int_{-\infty}^{\infty} b(z) \cdot \mathcal{K}_y(x-z)\, dz \quad = \quad \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{b(z)}{(x-z)^2 + y^2}\, dz. \quad (18.5)$$

Then $U(x,y)$ is the unique solution to the Laplace equation ($\triangle U = 0$) which satisfies the nonhomogeneous Dirichlet boundary condition $U(x,0) = b(x)$.

Proof: We'll show that the solution $U(x,y)$ in eqn. (18.5) is actually equal to the 'Fourier' solution $u(x,y)$ from Proposition 18.17.

Fix $y > 0$, and define $U_y(x) = U(x,y)$ for all $x \in \mathbb{R}$. Thus,

$$U_y \quad = \quad b * \mathcal{K}_y,$$

so that Theorem 17.11**(b)** (p.342) says:

$$\widehat{U}_y \quad = \quad 2\pi \cdot \widehat{b} \cdot \widehat{\mathcal{K}}_y. \quad (18.6)$$

Now, by practice problem # 7 on page 353 of §17.6, we have:

$$\widehat{\mathcal{K}}_y(\mu) \quad = \quad \frac{e^{-y|\mu|}}{2\pi}, \quad (18.7)$$

Combine (18.6) and (18.7) to get:

$$\widehat{U}_y(\mu) \quad = \quad e^{-y|\mu|} \cdot \widehat{b}(\mu). \quad (18.8)$$

Now apply the Fourier inversion formula (Theorem 17.2 on page 338) to eqn (18.8) to obtain:

$$U_y(x) \quad = \quad \int_{-\infty}^{\infty} \widehat{U}(\mu) \cdot \exp(\mu \cdot x \cdot \mathbf{i})\, d\mu \quad = \quad \int_{-\infty}^{\infty} e^{-y|\mu|} \cdot \widehat{b}(\mu)\, exp(\mu \cdot x \cdot \mathbf{i})\, d\mu \quad = \quad u(x,y),$$

where $u(x,y)$ is the solution from Proposition 18.17. □

18.4 PDEs on the Half-Line

Prerequisites: §2.2(a), §17.5, §6.5, §1.8

Using the Fourier (co)sine series, we can solve PDEs on the half-line.

Theorem 18.20: The Heat Equation; Dirichlet boundary conditions

Let $f \in \mathbf{L}^1(\mathbb{R}^+)$ have Fourier sine transform $\widehat{f}_{\sin}$, and define $u(x,t)$ by:

$$u(x,t) \;=\; \int_0^{\infty} \widehat{f}_{\sin}(\mu) \cdot \sin(\mu \cdot x) \cdot e^{-\mu^2 t}\, d\mu$$

Then $u(x,t)$ is a solution to the Heat Equation, with initial conditions $u(x,0) = f(x)$, and satisfies the homogeneous Dirichlet boundary condition: $u(0,t) = 0$.

Proof: <u>**Exercise 18.9**</u> Hint: Use Proposition 1.9 on page 18 □

Theorem 18.21: The Heat Equation; Neumann boundary conditions

Let $f \in \mathbf{L}^1(\mathbb{R}^+)$ *have Fourier cosine transform* $\widehat{f}_{\cos}$*, and define* $u(x,t)$ *by:*

$$u(x,t) \;=\; \int_0^\infty \widehat{f}_{\cos}(\mu) \cdot \cos(\mu \cdot x) \cdot e^{-\mu^2 t} \, d\mu$$

Then $u(x,t)$ *is a solution to the Heat Equation, with initial conditions* $u(x,0) = f(x)$*, and satisfies the homogeneous Neumann boundary condition:* $\partial_x u(0,t) = 0$*.*

Proof: **Exercise 18.10** Hint: Use Proposition 1.9 on page 18 □

18.5 (∗) The Big Idea

Most of the results of this chapter can be subsumed into a single abstraction, which makes use of the **polynomial formalism** developed in § 15.6 on page 293.

Theorem 18.22: *Let* $\mathcal{L}$ *be a linear differential operator with constant coefficients, and let* $\mathcal{L}$ *have* **polynomial symbol** $\mathcal{P}$*.*

- *If* $f : \mathbb{R}^D \longrightarrow \mathbb{R}$ *is a function with Fourier Transform* $\widehat{f} : \mathbb{R}^D \longrightarrow \mathbb{R}$*, and* $g = \mathcal{L} f$*, then* g *has Fourier transform:* $\widehat{g}(\boldsymbol{\mu}) \;=\; \mathcal{P}(\boldsymbol{\mu}) \cdot \widehat{f}(\boldsymbol{\mu})$*, for all* $\boldsymbol{\mu} \in \mathbb{R}^D$*.*
- *If* $q : \mathbb{R}^D \longrightarrow \mathbb{R}$ *is a function with Fourier transform* $\widehat{q}$*, and* f *has Fourier transform*

$$\widehat{f}(\boldsymbol{\mu}) \;=\; \frac{\widehat{q}(\boldsymbol{\mu})}{\mathcal{P}(\boldsymbol{\mu})},$$

then f *is a solution to the Poisson-type nonhomogeneous equation "*$\mathcal{L} f \;=\; q$*."*

Let $u : \mathbb{R}^D \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *be another function, and, for all* $t \geq 0$*, let* $u_t(\mathbf{x}) = u(\mathbf{x},t)$*. Let* u_t *have Fourier transform* $\widehat{u}_t$*.*

- *Suppose* $\widehat{u}_t(\boldsymbol{\mu}) = \exp\left(-\mathcal{P}(\boldsymbol{\mu}) \cdot t\right) \cdot \widehat{f}(\boldsymbol{\mu})$*, for all* $\boldsymbol{\mu} \in \mathbb{R}^D$*. Then the function* $u(\mathbf{x},t)$ *is a solution to the first-order evolution equation*

$$\partial_t \, u(\mathbf{x},t) \;=\; \mathcal{L}\, u(\mathbf{x},t)$$

with initial conditions $u(\mathbf{x},0) = f(\mathbf{x})$*.*

- *Suppose* $\widehat{u}_t(\boldsymbol{\mu}) = \cos\left(\sqrt{\mathcal{P}(\boldsymbol{\mu})} \cdot t\right) \cdot \widehat{f}(\boldsymbol{\mu})$*, for all* $\boldsymbol{\mu} \in \mathbb{R}^D$*. Then the function* $u(\mathbf{x},t)$ *is a solution to the second-order evolution equation*

$$\partial_t^2 \, u(\mathbf{x},t) \;=\; \mathcal{L}\, u(\mathbf{x},t)$$

with **initial position** $u(\mathbf{x},0) = f(\mathbf{x})$ *and* **initial velocity** $\partial_t \, u(\mathbf{x},0) = 0$*.*

- *Suppose* $\widehat{u}_t(\boldsymbol{\mu}) = \dfrac{\sin\left(\sqrt{\mathcal{P}(\boldsymbol{\mu})} \cdot t\right)}{\sqrt{\mathcal{P}(\boldsymbol{\mu})}} \cdot \widehat{f}(\boldsymbol{\mu})$, *for all* $\boldsymbol{\mu} \in \mathbb{R}^D$. *Then the function* $u(\mathbf{x}, t)$ *is a solution to the second-order evolution equation*

$$\partial_t^2\, u(\mathbf{x}, t) \;=\; \mathcal{L}\, u(\mathbf{x}, t)$$

with **initial position** $u(\mathbf{x}, 0) = 0$ *and* **initial velocity** $\partial_t\, u(\mathbf{x}, 0) = f(\mathbf{x})$.

Proof: **Exercise 18.11** □

Exercise 18.12 Go back through this chapter and see how all of the different solution theorems for the Heat Equation, Wave Equation, and Poisson equation are special cases of this result. What about the solution for the Dirichlet problem on a half-space? How does it fit into this formalism?

18.6 Practice Problems

1. Let $f(x) \;=\; \begin{cases} 1 & \text{if } 0 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$, as in Example 17.5 on page 339
   (a) Use the Fourier method to solve the Dirichlet problem on a half-space, with boundary condition $u(x, 0) \;=\; f(x)$.
   (b) Use the Fourier method to solve the Heat equation on a line, with initial condition $u_0(x) \;=\; f(x)$.

2. Solve the two-dimensional Heat Equation, with initial conditions

$$f(x, y) \quad = \quad \begin{cases} 1 & \text{if } 0 \le x \le X \text{ and } 0 \le y \le Y; \\ 0 & \text{otherwise.} \end{cases}$$

   where $X, Y > 0$ are constants. (**Hint:** See problem # 3 on page 352 of §17.6)

3. Solve the two-dimensional Wave equation, with

$$\textbf{Initial Position:}\quad u(x, y, 0) \;=\; 0,$$
$$\textbf{Initial Velocity:}\quad \partial_t\, u(x, y, 0) \;=\; f_1(x, y) \;=\; \begin{cases} 1 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1; \\ 0 & \text{otherwise.} \end{cases}.$$

   (**Hint:** See problem # 3 on page 352 of §17.6)

4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined: $f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ (see Figure 17.9 on page 353). Solve the **Heat Equation** on the real line, with initial conditions $u(x; 0) = f(x)$. (Use the Fourier method; see problem # 4 on page 353 of §17.6)

5. Let $f(x) = x \cdot \exp\left(\dfrac{-x^2}{2}\right)$. (See problem # 5 on page 353 of §17.6.)

(a) Solve the **Heat Equation** on the real line, with initial conditions $u(x;0) = f(x)$. (Use the Fourier method.)

(b) Solve the **Wave Equation** on the real line, with initial position $u(x;0) = f(x)$ and initial velocity $\partial_t u(x,0) = 0$. (Use the Fourier method.)

6. Let $f(x) = \dfrac{2x}{(1+x^2)^2}$. (See problem # 8 on page 353 of §17.6.)

(a) Solve the **Heat Equation** on the real line, with initial conditions $u(x;0) = f(x)$. (Use the Fourier method.)

(b) Solve the **Wave Equation** on the real line, with initial position $u(x,0) = 0$ and initial velocity $\partial_t u(x,0) = f(x)$. (Use the Fourier method.)

7. Let $f(x) = \begin{cases} 1 & \text{if} \quad -4 < x < 5; \\ 0 & \quad\text{otherwise.} \end{cases}$ (See problem # 9 on page 353 of §17.6.) Use the 'Fourier Method' to solve the one-dimensional Heat Equation ($\partial_t\, u(x;t) = \triangle u(x;t)$) on the domain $\mathbb{X} = \mathbb{R}$, with initial conditions $u(x;0) = f(x)$.

8. Let $f(x) = \dfrac{x\cos(x) - \sin(x)}{x^2}$. (See problem # 10 on page 353 of §17.6.) Use the 'Fourier Method' to solve the one-dimensional Heat Equation ($\partial_t\, u(x;t) = \triangle u(x;t)$) on the domain $\mathbb{X} = \mathbb{R}$, with initial conditions $u(x;0) = f(x)$.

9. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ had Fourier transform $\widehat{f}(\mu) = \dfrac{\mu}{\mu^4+1}$.

(a) Find the solution to the one-dimensional Heat Equation $\partial_t\, u = \triangle u$, with initial conditions $u(x;0) = f(x)$ for all $x \in \mathbb{R}$.

(b) Find the solution to the one-dimensional Wave Equation $\partial_t^2\, u = \triangle u$, with

$$\begin{array}{rrcll} \text{Initial position} & u(x;0) & = & 0, & \text{for all } x \in \mathbb{R}. \\ \text{Initial velocity} & \partial_t\, u(x;0) & = & f(x), & \text{for all } x \in \mathbb{R}. \end{array}$$

(c) Find the solution to the two-dimensional Laplace Equation $\triangle u(x,y) = 0$ on the half-space $\mathbb{H} = \{(x,y)\ ;\ x \in \mathbb{R},\ y \geq 0\}$, with boundary condition: $u(x,0) = f(x)$ for all $x \in \mathbb{R}$.

(d) **Verify** your solution to question (c). That is: check that your solution satisfies the Laplace equation and the desired boundary conditions.

For the sake of simplicity, you may assume that the 'formal derivatives' of integrals converge (ie. 'the derivative of the integral is the integral of the derivatives', etc.)

10. Fix $\mu \in \mathbb{R}$, and define $f_\mu : \mathbb{R}^2 \longrightarrow \mathbb{C}$ by $f_\mu(x,y) := \exp\Big(-|\mu|\cdot y\Big)\exp\Big(-\mu\mathbf{i}x\Big)$. Show that f is harmonic on $\mathbb{R}^2$.

(This function appears in the Fourier solution to the half-plane Dirichlet problem; see Proposition 18.17 on page 365.)

Notes:

Solutions

Solutions to §2.5

1. Let $\mathbf{V}(\mathbf{x}) = \nabla f(\mathbf{x})$. Hence, $V_1(\mathbf{x}) = \partial_1 f(\mathbf{x})$, $V_2(\mathbf{x}) = \partial_2 f(\mathbf{x})$, $V_3(\mathbf{x}) = \partial_3 f(\mathbf{x})$, and $V_4(\mathbf{x}) = \partial_4 f(\mathbf{x})$. Thus,

$$\begin{aligned}\mathbf{div}\,\mathbf{V}(\mathbf{x}) &= \partial_1 V_1(\mathbf{x}) + \partial_2 V_2(\mathbf{x}) + \partial_3 V_3(\mathbf{x}) + \partial_4 V_4(\mathbf{x}) \\ &= \partial_1\partial_1 f(\mathbf{x}) + \partial_2\partial_2 f(\mathbf{x}) + \partial_3\partial_3 f(\mathbf{x}) + \partial_4\partial_4 f(\mathbf{x}) \\ &= \partial_1^2 f(\mathbf{x}) + \partial_2^2 f(\mathbf{x}) + \partial_3^2 f(\mathbf{x}) + \partial_4^2 f(\mathbf{x}) = \triangle f(\mathbf{x}).\end{aligned}$$

2.
$$\begin{aligned}\partial_x f(x,y;t) &= \exp(-34t)\cdot\partial_x \sin(3x+5y) = 3\exp(-34t)\cdot\cos(3x+5y). \\ \text{Thus, } \partial_x^2 f(x,y;t) &= \partial_x 3\exp(-34t)\cdot\cos(3x+5y) = 3\exp(-34t)\cdot\partial_x \cos(3x+5y) \\ &= -9\exp(-34t)\cdot\sin(3x+5y). \\ \text{Likewise, } \partial_y f(x,y;t) &= \exp(-34t)\cdot\partial_y \sin(3x+5y) = 5\exp(-34t)\cdot\cos(3x+5y). \\ \text{Thus, } \partial_y^2 f(x,y;t) &= \partial_y 5\exp(-34t)\cdot\cos(3x+5y) = 5\exp(-34t)\cdot\partial_y \cos(3x+5y) \\ &= -25\exp(-34t)\cdot\sin(3x+5y). \\ \text{Thus, } \triangle f(x,y;t) &= \partial_x^2 f(x,y;t) + \partial_y^2 f(x,y;t) \\ &= -9\exp(-34t)\cdot\sin(3x+5y) - 25\exp(-34t)\cdot\sin(3x+5y) \\ &= -34\exp(-34t)\cdot\sin(3x+5y). \\ \text{Finally, } \partial_t f(x,y;t) &= \sin(3x+5y)\partial_t \exp(-34t) = \sin(3x+5y)\cdot(-34)\cdot\exp(-34t)\cdot \\ &= -34\exp(-34t)\cdot\sin(3x+5y) = \triangle f(x,y;t), \quad \text{as desired.}\end{aligned}$$

3. Observe that $\partial_x u(x,y) = \dfrac{2x}{x^2+y^2}$, and $\partial_y u(x,y) = \dfrac{2y}{x^2+y^2}$.

Thus,
$$\partial_x^2 u(x,y) = \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} = \frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}.$$
and
$$\partial_y^2 u(x,y) = \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} = \frac{2x^2+2y^2-4y^2}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}.$$

Thus, $\triangle u(x,y) = \partial_x^2 u(x,y) + \partial_y^2 u(x,y) = \dfrac{2y^2-2x^2}{(x^2+y^2)^2} + \dfrac{2x^2-2y^2}{(x^2+y^2)^2} = 0$, as desired.

4. Observe that $\partial_x u(x,y,z) = \dfrac{\frac{-1}{2}2x}{(x^2+y^2+z^2)^{3/2}} = \dfrac{-x}{\|x,y,z\|^3}$.

Likewise $\partial_y u(x,y,z) = \dfrac{-y}{\|x,y,z\|^3}$ and $\partial_z u(x,y,z) = \dfrac{-z}{\|x,y,z\|^3}$.

$$\text{Thus,}\quad \partial_x^2 u(x,y,z) = \frac{-1}{\|x,y,z\|^3} + \frac{\frac{-3}{2}(2x)(-x)}{\|x,y,z\|^5} = \frac{-\left(x^2+y^2+z^2\right)}{\|x,y,z\|^5} + \frac{3x^2}{\|x,y,z\|^5} = \frac{2x^2-y^2-z^2}{\|x,y,z\|^5}.$$

$$\text{Likewise,}\quad \partial_y^2 u(x,y,z) = \frac{2y^2-x^2-z^2}{\|x,y,z\|^5} \quad\text{and}\quad \partial_z^2 u(x,y,z) = \frac{2z^2-x^2-y^2}{\|x,y,z\|^5}.$$

Thus,

$$\begin{aligned}\triangle u(x,y,z) &= \partial_x^2 u(x,y,z) + \partial_y^2 u(x,y,z) + \partial_z^2 u(x,y,z) \\ &= \frac{\left(2x^2-y^2-z^2\right)+\left(2y^2-x^2-z^2\right)+\left(2z^2-x^2-y^2\right)}{\|x,y,z\|^5} = \frac{0}{\|x,y,z\|^5} = 0,\end{aligned}$$

as desired.

5. Observe that $\partial_x u(x,y;t) = \frac{-2x}{4t}\frac{1}{4\pi t}\exp\left(\frac{-x^2-y^2}{4t}\right) = \frac{-x}{2t}u(x,y;t)$.

Thus, $\partial_x^2 u(x,y;t) = \frac{-1}{2t}u(x,y;t) + \frac{x^2}{4t^2}u(x,y;t) = \left(\frac{x^2}{4t^2} - \frac{1}{2t}\right) u(x,y;t)$.

Likewise, $\partial_y^2 u(x,y;t) = \left(\frac{y^2}{4t^2} - \frac{1}{2t}\right) u(x,y;t)$.

$$\text{Hence } \triangle u(x,y;t) = \partial_x^2 u(x,y;t) + \partial_y^2 u(x,y;t) = \left(\frac{x^2}{4t^2} + \frac{y^2}{4t^2} - 2\frac{1}{2t}\right) u(x,y;t) = \frac{-1}{t}u(x,y;t) + \frac{x^2+y^2}{4t^2}u(x,y;t).$$

Meanwhile, $\partial_t u(x,y,;t) = \frac{-1}{t}u(x,y;t) + \frac{x^2+y^2}{4t^2}u(x,y;t) = \triangle u(x,y;t)$, as desired.

6. (a)

$$\begin{aligned} \partial_x h(x,y) &= \alpha\cdot\cosh(\alpha x)\cdot\sin(\beta y), \\ \text{thus, } \partial_x^2 h(x,y) &= \alpha^2\cdot\sinh(\alpha x)\cdot\sin(\beta y) = \alpha^2\cdot h(x,y). \\ \text{Likewise, } \partial_y h(x,y) &= \beta\cdot\sinh(\alpha x)\cdot\cos(\beta y), \\ \text{thus, } \partial_y^2 h(x,y) &= -\beta^2\cdot\sinh(\alpha x)\cdot\sin(\beta y) = -\beta^2\cdot h(x,y). \\ \text{Hence, } \triangle h(x,y) &= \partial_x^2 h(x,y) + \partial_y^2 h(x,y) = \alpha^2\cdot h(x,y) - \beta^2\cdot h(x,y) \\ &= (\alpha^2-\beta^2)\cdot h(x,y) \end{aligned}$$

(b)

$$\Big(h \text{ is harmonic}\Big) \iff \Big(\triangle h = 0\Big) \iff \Big(\alpha^2-\beta^2=0\Big) \iff \Big(\alpha^2=\beta^2\Big) \iff \boxed{\Big(\alpha=\pm\beta\Big)}$$

Solutions to §3.4

1. (a) $\partial_x u(x,t) = 7\cos(7x)\cos(7t)$, and $\partial_x^2 u(x,t) = -49\sin(7x)\cos(7t)$. Likewise, $\partial_t u(x,t) = -7\sin(7x)\sin(7t)$, so that $\partial_x^2 u(x,t) = -49\sin(7x)\cos(7t) = \partial_x^2 u(x,t)$, as desired.

   (b) $\partial_x u(x,t) = 3\cos(3x)\cos(3t)$, and $\partial_x^2 u(x,t) = -9\sin(3x)\cos(3t)$. Likewise, $\partial_t u(x,t) = -3\sin(3x)\sin(3t)$, so that $\partial_x^2 u(x,t) = -9\sin(3x)\cos(3t) = \partial_x^2 u(x,t)$, as desired.

   (c) $\partial_x u(x,t) = \frac{-2}{(x-t)^3}$, and $\partial_x^2 u(x,t) = \frac{6}{(x-t)^4}$, while $\partial_t u(x,t) = \frac{(-1)^2\cdot 2}{(x-t)^3} = \frac{2}{(x-t)^3}$, and $\partial_t^2 u(x,t) = (-1)^2\frac{6}{(x-t)^4} = \frac{6}{(x-t)^4} = \partial_x^2 u(x,t)$.

   (d) $\partial_x u(x,t) = 2(x-t)-3$ and $\partial_x^2 u(x,t) = 2$, while $\partial_x u(x,t) = -2(x-t)+3$ and $\partial_x^2 u(x,t) = (-1)^2\cdot 2 = 2 = \partial_x^2 u(x,t)$.

   (e) $\partial_x v(x,t) = 2(x-t)$, and $\partial_x^2 v(x,t) = 2$. Likewise, $\partial_t v(x,t) = -2(x-t)$, and $\partial_t^2 v(x,t) = (-1)^2\cdot 2 = 2 = \partial_x^2 v(x,t)$, as desired.

2. Yes, u satisfies the wave equation. To see this, observe: $\partial_x u(x,t) = f'(x-t)$, and $\partial_x^2 u(x,t) = f''(x-t)$. Likewise, $\partial_x u(x,t) = -f'(x-t)$, and $\partial_x^2 u(x,t) = (-1)^2 f''(x-t) = f''(x-t) = \partial_x^2 u(x,t)$, as desired.

3. Since ∂_x^2 and ∂_t^2 are both linear operators, we have:

$$\partial_t^2 w(x,t) = 3\partial_t^2 u(x,t) - 2\partial_t^2 v(x,t) = 3\partial_x^2 u(x,t) - 2\partial_x^2 v(x,t) = \partial_x^2 w(x,t),$$

as desired.

4. Since ∂_x^2 and ∂_t^2 are both linear operators, we have:

$$\partial_t^2 w(x,t) = 5\partial_t^2 u(x,t) + 2\partial_t^2 v(x,t) = 5\partial_x^2 u(x,t) + 2\partial_x^2 v(x,t) = \partial_x^2 w(x,t),$$

as desired.

5. One way to do this is to observe that $\sin(x+t)$ and $\sin(x-t)$ are both solutions, by problem #2, and thus, their sum is also a solution, by problem #4. Another way is to explicitly compute derivatives:

$$\partial_x u(x,t) = \cos(x+t)-\cos(x-t) \quad\text{and}\quad \partial_x^2 u(x,t) = -\sin(x+t)+\sin(x-t),$$

while

$$\partial_t u(x,t) = \cos(x+t)+\cos(x-t) \quad\text{and}\quad \partial_t^2 u(x,t) = -\sin(x+t)+(-1)^2\sin(x-t) = -\sin(x+t)+\sin(x-t).$$

Thus, $\partial_t^2 u = \partial_x^2 u$, as desired.

6. (a)

$$\begin{array}{rrcll}
& \partial_x\ u(x,y;t) & = & \Big(\partial_x\ \sinh(3x)\Big)\cdot\cos(5y)\cdot\cos(4t) & = & 3\cosh(3x)\cdot\cos(5y)\cdot\cos(4t).\\
\text{Thus,} & \partial_x^2\ u(x,y;t) & = & 3\Big(\partial_x\ \cosh(3x)\Big)\cdot\cos(5y)\cdot\cos(4t) & = & 9\sinh(3x)\cdot\cos(5y)\cdot\cos(4t)\\
& & = & 9u(x,y;t). & & \\
\text{Likewise,} & \partial_y\ u(x,y;t) & = & \sinh(3x)\cdot\Big(\partial_y\ \cos(5y)\Big)\cdot\cos(4t) & = & -5\sinh(3x)\cdot\sin(5y)\cdot\cos(4t).\\
\text{Thus,} & \partial_y^2\ u(x,y;t) & = & -5\sinh(3x)\cdot\Big(\partial_y\ \sin(5y)\Big)\cdot\cos(4t) & = & -25\sinh(3x)\cdot\cos(5y)\cdot\cos(4t)\\
& & = & -25y(x,y;t). & & \\
\text{Thus,} & \triangle\ u(x,y;t) & = & \partial_x^2\ u(x,y;t)+\partial_y^2\ u(x,y;t) & = & 9u(x,y;t)-25u(x,y;t)\\
& & = & -16u(x,y;t). & & \\
\text{Meanwhile,} & \partial_t\ u(x,y;t) & = & \sinh(3x)\cdot\cos(5y)\cdot\Big(\partial_t\ \cos(4t)\Big) & = & -4\sinh(3x)\cdot\cos(5y)\cdot\sin(4t)\\
\text{Thus,} & \partial_t^2\ u(x,y;t) & = & -4\sinh(3x)\cdot\cos(5y)\cdot\Big(\partial_t\ \sin(4t)\Big) & = & -16\sinh(3x)\cdot\cos(5y)\cdot\cos(4t)\\
& & = & \triangle u(x,y;t), \quad \text{as required.} & &
\end{array}$$

(b) $\partial_x\ u\ =\ \cos(x)\cos(2y)\sin(\sqrt{5}t)$, and $\partial_x^2\ u\ =\ -\sin(x)\cos(2y)\sin(\sqrt{5}t)\ =-u$.

$\partial_y\ u\ =\ -2\sin(x)\sin(2y)\sin(\sqrt{5}t)$, and $\partial_y^2\ u\ =\ -4\sin(x)\sin(2y)\sin(\sqrt{5}t)\ =\ -4u$.

Thus, $\triangle u\ =\ \partial_x^2\ u+\partial_y^2\ u\ =\ -u-4u\ =\ -5u$.

$\partial_t\ u\ =\ \sqrt{5}\sin(x)\cos(2y)\cos(\sqrt{5}t)$, and $\partial_t^2\ u\ =\ -5\sin(x)\sin(2y)\sin(\sqrt{5}t)\ =\ -5u\ =\ \triangle u$, as desired.

(c) $\partial_x\ u\ =\ 3\cos(3x-4y)\cos(5t)$ and $\partial_x^2\ u\ =\ -9\sin(3x-4y)\cos(5t)\ =-9u$.

$\partial_y\ u\ =\ -4\cos(3x-4y)\cos(5t)$, and $\partial_y^2\ u\ =\ (-1)^3\cdot 16\sin(3x-4y)\cos(5t)\ =\ -16u$.

Thus, $\triangle u\ =\ \partial_x^2\ u+\partial_y^2\ u\ =\ -9u-16u=-25u$.

$\partial_t\ u\ =\ -5\sin(3x-4y)\sin(5t)$ and $\partial_t^2\ u\ =\ -25\sin(3x-4y)\cos(5t)\ =\ -25u\ =\ \triangle u$, as desired.

Solutions to §4.7

1. $\partial_t\,\omega_t(\mathbf{x})\ =\ \dfrac{-\mathbf{i}}{\hbar}\,\dfrac{m_e v_1^2}{2}\,\omega_t(\mathbf{x})$.

Also, $\partial_{x_1}\,\omega_t(\mathbf{x})\ =\ \dfrac{\mathbf{i}}{\hbar}\,m_e v_1\omega_t(\mathbf{x})$. Thus, $\partial_{x_1}^2\,\omega_t(\mathbf{x})\ =\ \left(\dfrac{\mathbf{i}}{\hbar}\,m_e v_1\right)^2\omega_t(\mathbf{x})\ =\ \dfrac{-m_e^2 v_1^2}{\hbar^2}\,\omega_t(\mathbf{x})$.

Meanwhile, $\partial_{x_2}\,\omega\equiv 0\equiv\partial_{x_3}\,\omega\ =\ 0$, because ω does not depend on the x_2 or x_3 variables.

Thus,

$$\begin{aligned}
\frac{-\hbar^2}{2\,m_e}\ \triangle\,\omega_t(\mathbf{x}) \ &=\ \frac{-\hbar^2}{2\,m_e}\,\partial_{x_1}^2\,\omega_t(\mathbf{x})\ =\ \frac{-\hbar^2}{2\,m_e}\cdot\frac{-m_e^2v_1^2}{\hbar^2}\,\omega_t(\mathbf{x})\\
&=\ \frac{m_e v_1^2}{2}\,\omega_t(\mathbf{x})\ =\ \left(\frac{\hbar}{-\mathbf{i}}\right)\left(\frac{-\mathbf{i}}{\hbar}\right)\frac{m_e v_1^2}{2}\,\omega_t(\mathbf{x})\\
&=\ \mathbf{i}\hbar\cdot\frac{-\mathbf{i}}{\hbar}\,\frac{m_e v_1^2}{2}\,\omega_t(\mathbf{x})\ =\ \mathbf{i}\hbar\cdot\partial_t\,\omega_t(\mathbf{x}),
\end{aligned}$$

as desired.

2. $\partial_t\,\omega_t(\mathbf{x})\ =\ \frac{-\mathbf{i}}{2}\,|\mathbf{v}|^2\omega_t(\mathbf{x})$

$\partial_{x_1}\,\omega_t(\mathbf{x})\ =\ \mathbf{i}v_1\omega_t(\mathbf{x})$. Thus, $\partial_{x_1}^2\,\omega_t(\mathbf{x})\ =\ (\mathbf{i}v_1)^2\omega_t(\mathbf{x})\ =\ -v_1^2\omega_t(\mathbf{x})$.

$\partial_{x_2}\,\omega_t(\mathbf{x})\ =\ \mathbf{i}v_2\omega_t(\mathbf{x})$. Thus, $\partial_x^2\,\omega_t(\mathbf{x})\ =\ (\mathbf{i}v_2)^2\omega_t(\mathbf{x})\ =\ -v_2^2\omega_t(\mathbf{x})$.

$\partial_{x_3}\,\omega_t(\mathbf{x})\ =\ \mathbf{i}v_3\omega_t(\mathbf{x})$. Thus, $\partial_{x_3}^2\,\omega_t(\mathbf{x})\ =\ (\mathbf{i}v_3)^2\omega_t(\mathbf{x})\ =\ -v_3^2\omega_t(\mathbf{x})$.

Thus, $\triangle\omega_t(\mathbf{x})\ =\ \partial_{x_1}^2\,\omega_t(\mathbf{x})\ +\ \partial_{x_2}^2\,\omega_t(\mathbf{x})\ +\ \partial_{x_3}^2\,\omega_t(\mathbf{x})\ =\ -(v_1^2+v_2^2+v_3^2)\omega_t(\mathbf{x})\ =\ |\mathbf{v}|^2\omega_t(\mathbf{x})$

Thus,

$\mathbf{i}\partial_t\,\omega\ =\ \frac{-\mathbf{i}^2}{2}\,|\mathbf{v}|^2\omega\ =\ \frac{-1}{2}\,|\mathbf{v}|^2\omega\ =\ \frac{-1}{2}\,|\mathbf{v}|^2\omega\ =\ \frac{-1}{2}\,\triangle\,\omega$, as desired.

Solutions to §5.4

1. (a) **Linear, homogeneous.** $\mathcal{L}=\partial_t\ -\ \triangle$ is linear, and the equation takes the (homogeneous) form $\mathcal{L}(u)=0$.

(b) **Linear, nonhomogeneous.** $\mathcal{L}=\triangle$ is linear, and the equation takes the (nonhomogeneous) form $\triangle(u)=q$.

(c) **Linear, homogeneous.** $\mathcal{L}=\triangle$ is linear, and the equation takes the (homogeneous) form $\triangle(u)=0$.

(d) **Nonlinear.** To see that the operator $\mathcal{L}$ is *not* linear, first recall that $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$. Thus,

$$\mathcal{L}(u) \quad = \quad \det\begin{bmatrix} \partial_x^2 u & \partial_x \partial_y u \\ \partial_x \partial_y u & \partial_y^2 u \end{bmatrix} \quad = \quad \left(\partial_x^2 u\right) \cdot \left(\partial_y^2 u\right) \; - \; (\partial_x \partial_y u)^2$$

$$\begin{aligned} \text{Thus,} \qquad \mathcal{L}(u+v) \quad &= \quad \left(\partial_x^2 (u+v)\right) \cdot \left(\partial_y^2 (u+v)\right) \; - \; (\partial_x \partial_y (u+v))^2 \\ &= \quad \left(\partial_x^2 u \; + \; \partial_x^2 v\right) \cdot \left(\partial_y^2 u \; + \; \partial_y^2 v\right) \; - \; (\partial_x \partial_y u \; + \; \partial_x \partial_y v)^2 \\ &= \quad \left(\partial_x^2 u\right) \cdot \left(\partial_y^2 u\right) \; + \; \left(\partial_x^2 v\right) \cdot \left(\partial_y^2 u\right) \; + \; \left(\partial_x^2 u\right) \cdot \left(\partial_y^2 v\right) \; + \; \left(\partial_x^2 v\right) \cdot \left(\partial_y^2 v\right) \\ & \qquad - \; (\partial_x \partial_y u)^2 \; - \; 2\,(\partial_x \partial_y u)\,(\partial_x \partial_y v) - (\partial_x \partial_y v)^2 \end{aligned}$$

while $\mathcal{L}(u) \; + \; \mathcal{L}(v) \quad = \quad \left(\partial_x^2 u\right) \cdot \left(\partial_y^2 u\right) \; - \; (\partial_x \partial_y u)^2 \; + \; \left(\partial_x^2 v\right) \cdot \left(\partial_y^2 v\right) \; - \; (\partial_x \partial_y v)^2$. Thus,

$$\mathcal{L}(u+v) \; - \; \Big(\mathcal{L}(u) + \mathcal{L}(v)\Big) \quad = \quad \left(\partial_x^2 u\right) \cdot \left(\partial_y^2 v\right) \; + \; \left(\partial_x^2 v\right) \cdot \left(\partial_y^2 u\right) \; - \; 2\,(\partial_x \partial_y u)\,(\partial_x \partial_y v) \quad \neq \quad 0.$$

(e) **Nonlinear.** To see that the operator $\mathcal{L}$ is *not* linear, observe that

$$\begin{aligned} \mathcal{L}(u+v) \quad &= \quad \partial_t (u+v) - \triangle(u+v) \; - \; q \circ (u+v) \\ &= \quad \partial_t u \; + \; \partial_t v \; - \; \triangle u \; - \; \triangle v \; - \; q \circ (u+v) \end{aligned}$$

On the other hand, $\mathcal{L}(u) + \mathcal{L}(v) \quad = \quad \partial_t u \; + \; \partial_t v \; - \; \triangle u \; - \; \triangle v \; - \; q \circ u \; - \; q \circ v$. Thus,

$$\mathcal{L}(u+v) \; - \; \Big(\mathcal{L}(u) + \mathcal{L}(v)\Big) \quad = \quad q \circ u \; + \; q \circ v \; - \; q \circ (u+v)$$

and this expression is not zero unless the function q itself is linear.

(f) **Nonlinear.** To see that the operator $\mathcal{L}$ is *not* linear, first use the chain rule to check that $\partial_x (q \circ u)(\mathbf{x}) \; = \; q' \circ u(\mathbf{x}) \cdot \partial_x u(\mathbf{x})$. Thus,

$$\begin{aligned} \mathcal{L}(u+v) \quad &= \quad \partial_t (u+v) - \partial_x (q \circ u)(\mathbf{x}) \quad = \quad \partial_t u \; + \; \partial_t v \; + \; \Big[q' \circ (u+v)\Big] \cdot (\partial_x u \; + \; \partial_x v) \\ &= \quad \partial_t u \; + \; \partial_t v \; + \; \Big[q' \circ (u+v)\Big] \cdot \partial_x u \; + \; \Big[q' \circ (u+v)\Big] \cdot \partial_x \end{aligned}$$

On the other hand, $\mathcal{L}(u) + \mathcal{L}(v) \quad = \quad \partial_t u \; + \; \partial_t v \; + \; [q' \circ u] \cdot \partial_x u \; + \; [q' \circ v] \cdot \partial_x v$. Thus,

$$\mathcal{L}(u+v) \; - \; \Big(\mathcal{L}(u) + \mathcal{L}(v)\Big) \quad = \quad \Big[q' \circ (u+v) - q' \circ u\Big] \cdot \partial_x u \; + \; \Big[q' \circ (u+v) - q' \circ v\Big] \cdot \partial_x v \quad \neq \quad 0$$

(g) **Linear, homogeneous.** The equation takes the (homogeneous) form $\mathcal{L}(u) = 0$, where $\mathcal{L}(u) = \triangle - \lambda \cdot u$ is a linear operator To see that $\mathcal{L}$ is linear, observe that, for any functions u and v,

$$\mathcal{L}(u+v) \quad = \quad \triangle(u+v) \; - \; \lambda \cdot (u+v) \quad = \quad \triangle u \; + \; \triangle v \; - \; \lambda \cdot u \; - \; \lambda \cdot v \quad = \quad \mathcal{L}(u) + \mathcal{L}(v).$$

(h) **Linear, homogeneous.** $\mathcal{L} = \partial_t \; - \partial_x^3$ is linear, and the equation takes the (homogeneous) form $\mathcal{L}(u) = 0$.

(i) **Linear, homogeneous.** $\mathcal{L} = \partial_t \; - \partial_x^4$ is linear, and the equation takes the (homogeneous) form $\mathcal{L}(u) = 0$.

(j) **Linear, homogeneous.** The equation takes the (homogeneous) form $\mathcal{L}(u) = 0$, where $\mathcal{L} = \partial_t \; - \mathbf{i} \triangle - q \cdot u$. To see that $\mathcal{L}$ is linear, observe that, for any functions u and v,

$$\begin{aligned} \mathcal{L}(u+v) \quad &= \quad \partial_t (u+v) \; - \; \mathbf{i} \triangle (u+v) \; - \; q \cdot u(u+v) \\ &= \quad \partial_t u \; + \; \partial_t v \; - \; \mathbf{i} \triangle u \; - \; \mathbf{i} \triangle v \; - \; q \cdot u \; - \; q \cdot v \quad = \quad \mathcal{L}(u) + \mathcal{L}(v) \end{aligned}$$

(k) **Nonlinear.** To see that the operator $\mathcal{L}$ is *not* linear, observe that

$$\begin{aligned} \mathcal{L}(u+v) \quad &= \quad \partial_t (u+v) - (u+v) \cdot \partial_x (u+v) \\ &= \quad \partial_t u \; + \; \partial_t v \; - \; u \cdot \partial_x u \; - \; v \cdot \partial_x u \; - \; u \cdot \partial_x v \; - \; v \cdot \partial_x v. \end{aligned}$$

However, $\mathcal{L}(u) + \mathcal{L}(v) \quad = \quad \partial_t u \; - \; u \cdot \partial_x u \; + \; \partial_t v \; - \; v \cdot \partial_x v$.
Thus, $\mathcal{L}(u+v) \; - \; \mathcal{L}(u) - \mathcal{L}(v) \quad = \quad -v \cdot \partial_x u \; - \; u \cdot \partial_x v \quad \neq \quad 0$.

(l) **Nonlinear.** To see that $\mathcal{L}$ is not linear, recall that the Triangle inequality says $|a+b| \leq |a| + |b|$, with strict inequality if a and b have opposite sign. Thus,

$$\mathcal{L}(u+v) \quad = \quad |\partial_x (u+v)| \quad = \quad |\partial_x u \; + \; \partial_x v| \quad \leq \quad |\partial_x u| \; + \; |\partial_x v| \quad = \quad \mathcal{L}(u) + \mathcal{L}(v),$$

with strict inequality if $\partial_x u$ and $\partial_x v$ have opposite sign.

2. (a) $\partial_x^2 u(x,y) = -u(x,y) = \partial_y^2 u(x,y)$. Thus, $\triangle u(x,y) = \partial_x^2 u(x,y)+\partial_y^2 u(x,y) = -2f(x,y)$. Hence $\boxed{u \text{ is an eigenfunction}}$ of $\triangle$, with eigenvalue $\boxed{-2}$.

(b) $\partial_x^2 u(x,y) = -\sin(x)$, while $\partial_y^2 u(x,y) = -\sin(y)$. Thus, $\triangle u(x,y) = \partial_x^2 u(x,y) + \partial_y^2 u(x,y) = -\sin(x) - \sin(y) = -u(x,y)$. Hence $\boxed{u \text{ is an eigenfunction}}$ of $\triangle$, with eigenvalue $\boxed{-1}$.

(c) $\partial_x^2 u(x,y) = -4\cos(2x)$, while $\partial_y^2 u(x,y) = -\sin(y)$. Thus, $\triangle u(x,y) = \partial_x^2 u(x,y) + \partial_y^2 u(x,y) = -4\sin(x) - \sin(y) \neq \lambda \cdot u(x,y)$, for any $\lambda \in \mathbb{R}$. Hence $\boxed{u \text{ is not an eigenfunction}}$ of $\triangle$.

(d) $\partial_x u(x,y) = 3\cos(3x)\cdot\cos(4y)$, and $\partial_x^2 u(x,y) = -9\sin(3x)\cdot\cos(4y) = -9u(x,y)$. Likewise $\partial_y u(x,y) = -4\sin(3x)\cdot\sin(4y)$, and $\partial_y^2 u(x,y) = -16\sin(3x)\cdot\cos(4y) = -16u(x,y)$.

Thus, $\triangle u = \partial_x^2 u + \partial_x^2 u = -9u - 16u = -25u$, so u is an **eigenfunction**, with eigenvalue $\boxed{-25}$.

(e) $\partial_x u(x,y) = 3\cos(3x)$, and $\partial_x^2 u(x,y) = -9\sin(3x)$. Likewise $\partial_y u(x,y) = -4\sin(4y)$, and $\partial_y^2 u(x,y) = -16\cos(4y)$. Thus, $\triangle u(x,y) = \partial_x^2 u(x,y) + \partial_x^2 u(x,y) = -9\sin(3x) - 16\cos(4y)$, so u is **not** an eigenfunction.

(f) $\partial_x u(x,y) = 3\cos(3x)$, and $\partial_x^2 u(x,y) = -9\sin(3x)$. Likewise $\partial_y u(x,y) = -3\sin(3y)$, and $\partial_y^2 u(x,y) = -9\cos(3y)$. Thus, $\triangle u(x,y) = \partial_x^2 u(x,y) + \partial_x^2 u(x,y) = -9\sin(3x) - 9\cos(3y) = -9u(x,y)$, so u is an **eigenfunction**, with eigenvalue $\boxed{-9.}$

(g) $\partial_x u(x,y) = 3\cos(3x)\cdot\cosh(4y)$, and $\partial_x^2 u(x,y) = -9\sin(3x)\cdot\cosh(4y) = -9u(x,y)$. Likewise $\partial_y u(x,y) = 4\sin(3x)\cdot\sinh(4y)$, and $\partial_y^2 u(x,y) = 16\sin(3x)\cdot\cosh(4y) = 16u(x,y)$.

Thus, $\triangle u = \partial_x^2 u + \partial_x^2 u = -9u + 16u = 7u$, so u is an **eigenfunction**, with eigenvalue $\boxed{7}$.

(h) $\partial_x u(x,y) = 3\cosh(3x)\cdot\cosh(4y)$, and $\partial_x^2 u(x,y) = 9\sinh(3x)\cdot\cosh(4y) = 9u(x,y)$. Likewise $\partial_y u(x,y) = 4\sinh(3x)\cdot\sinh(4y)$, and $\partial_y^2 u(x,y) = 16\sinh(3x)\cdot\cosh(4y) = 16u(x,y)$.

Thus, $\triangle u = \partial_x^2 u + \partial_x^2 u = 9u + 16u = 25u$, so u is an **eigenfunction**, with eigenvalue $\boxed{25}$.

(i) $\partial_x u(x,y) = 3\cosh(3x)$, and $\partial_x^2 u(x,y) = 9\sinh(3x)$. Likewise $\partial_y u(x,y) = 4\sinh(4y)$, and $\partial_y^2 u(x,y) = 16\cosh(4y)$. Thus, $\triangle u(x,y) = \partial_x^2 u(x,y) + \partial_x^2 u(x,y) = 9\sin(3x) + 16\cosh(4y)$, so u is **not** an eigenfunction.

(j) $\partial_x u(x,y) = 3\cosh(3x)$, and $\partial_x^2 u(x,y) = 9\sinh(3x)$. Likewise $\partial_y u(x,y) = 3\sinh(3y)$, and $\partial_y^2 u(x,y) = 9\cosh(3y)$. Thus, $\triangle u(x,y) = \partial_x^2 u(x,y) + \partial_x^2 u(x,y) = 9\sinh(3x) + 9\cosh(3y) = 9u(x,y)$, so u is an **eigenfunction**, with eigenvalue $\boxed{9}$.

(k) $\partial_x u(x,y) = 3\cos(3x+4y)$, and $\partial_x^2 u(x,y) = -9\sin(3x+4y) = -9u(x,y)$. Likewise $\partial_y u(x,y) = 4\cos(3x+4y)$, and $\partial_y^2 u(x,y) = -16\sin(3x+4y) = -16u(x,y)$.

Thus, $\triangle u = \partial_x^2 u + \partial_x^2 u = -9u - 16u = -25u$, so u is an **eigenfunction**, with eigenvalue $\boxed{-25}$.

(l) $\partial_x u(x,y) = 3\cosh(3x+4y)$, and $\partial_x^2 u(x,y) = 9\sinh(3x+4y) = 9u(x,y)$. Likewise $\partial_y u(x,y) = 4\cosh(3x+4y)$, and $\partial_y^2 u(x,y) = 16\sinh(3x+4y) = 16u(x,y)$.

Thus, $\triangle u = \partial_x^2 u + \partial_x^2 u = 9u + 16u = 25u$, so u is an **eigenfunction**, with eigenvalue $\boxed{25}$.

(m) $\partial_x u(x,y) = 3\sin^2(x)\cos(x)\cdot\cos(y)$, and

$$\partial_x^2 u(x,y) = \left(-6\sin(x)\cos^2(x) - 3\sin^3(x)\right)\cdot\cos(y)$$

Likewise $\partial_y u(x,y) = -4\sin^3(x)\cdot\cos^3(y)\sin(y)$, and

$$\begin{aligned} \partial_y^2 u(x,y) &= \sin^3(x)\cdot\left(-12\sin^2(y)\cos^2(y) - 4\cos^4(y)\right) \\ \text{Thus,}\quad \triangle u(x,y) &= \left(-6\sin(x)\cos^2(x) - 3\sin^3(x)\right)\cdot\cos(y) \\ &\quad + \sin^3(x)\cdot\left(-12\sin^2(y)\cos^2(y) - 4\cos^4(y)\right), \end{aligned}$$

so u is **not** an eigenfunction.

(n) $\partial_x u(x,y) = 3e^{3x}\cdot e^{4y}$, and $\partial_x^2 u(x,y) = 9e^{3x}\cdot e^{4y} = 9u(x,y)$. Likewise $\partial_y u(x,y) = 4e^{3x}\cdot e^{4y}$, and $\partial_y^2 u(x,y) = 16e^{3x}\cdot e^{4y} = 16u(x,y)$.

Thus, $\triangle u = \partial_x^2 u + \partial_x^2 u = 9u + 16u = 25u$, so u is an **eigenfunction**, with eigenvalue $\boxed{25}$.

(o) $\partial_x u(x,y) = 3e^{3x}$, and $\partial_x^2 u(x,y) = 9e^{3x}$. Likewise $\partial_y u(x,y) = 4e^{4y}$, and $\partial_y^2 u(x,y) = 16e^{4y}$. Thus, $\triangle u(x,y) = \partial_x^2 u(x,y) + \partial_x^2 u(x,y) = 9e^{3x} + 16e^{4y}$, so u is **not** an eigenfunction.

(p) $\partial_x u(x,y) = 3e^{3x}$, and $\partial_x^2 u(x,y) = 9e^{3x}$. Likewise $\partial_y u(x,y) = 3e^{4y}$, and $\partial_y^2 u(x,y) = 9e^{4y}$. Thus, $\triangle u(x,y) = \partial_x^2 u(x,y) + \partial_x^2 u(x,y) = 9e^{3x} + 9e^{4y} = 9u(x,y)$, so u is an **eigenfunction**, with eigenvalue $\boxed{9}$.

Solutions to §6.3

1. **Evolution equation of order 2.** The order of $\triangle$ is 2. The isolated ∂_t term makes it an evolution equation.

2. **Nonevolution equation of order 2.** The order of $\triangle$ is 2. There is no ∂_t term, so this is not an evolution equation.

3. **Nonevolution equation of order 2.** The order of $\triangle$ is 2. There is no ∂_t term, so this is not an evolution equation.

4. **Nonvolution equation of order 2.** The operators ∂_x^2, $\partial_x \partial_y$, and ∂_y^2 all have order 2. There is no ∂_t term, so this is not an evolution equation.

5. **Evolution equation of order 2.** The order of $\triangle$ is 2. There is an isolated ∂_t term, so this is an evolution equation.

6. **Evolution equation of order 1.** The orders of ∂_x and ∂_t are both 1. There is an isolated ∂_t term, so this is an evolution equation.

7. **Nonevolution equation of order 2.** The order of $\triangle$ is 2. There is no ∂_t term, so this is not an evolution equation.

8. **Evolution equation of order 3.** The order of ∂_x^3 is 3. The isolated ∂_t term makes it an evolution equation.

9. **Evolution equation of order 4.** The order of ∂_x^4 is 4. The isolated ∂_t term makes it an evolution equation.

10. **Evolution equation of order 2.** The order of $\triangle$ is 2. The isolated ∂_t term makes it an evolution equation.

11. **Evolution equation of order 1.** The order of ∂_x is 1. There is an isolated ∂_t term, so this is an evolution equation.

12. **Nonevolution equation of order 1.** There is no time derivative, so this is not an evolution equation. The operator ∂_x has order 1.

Solutions to §6.7

1. First note that $\partial\,[0,\pi] \;=\; \{0,\pi\}$.

   (a) $\sin(3\cdot 0) \;=\; 0 \;=\; \sin(3\pi)$, so $\sin(3x)$ is zero on $\{0,\pi\}$, so we have HDBC.
   $\partial_x \sin(3x) \;=\; 3\cos(3x)$, and $\cos(3\cdot 0) \;=\; 1$, while $\cos(3\cdot\pi) \;=\; -1$. Thus, $\sin(3x)$ satisfies neither HNBC nor PBC.

   (b) To see that u satisfies homog. Dirichlet BC, observe:
   $u(0) \;=\; \sin(0) + 3\sin(0) - 4\sin(0) \;=\; 0+0-0 = 0$, and
   $u(\pi) \;=\; \sin(\pi) + 3\sin(2\pi) - 4\sin(7\pi) \;=\; 0+0-0 = 0$.
   To see that u does *not* satisfy homog. Neuman BC, observe:
   $u'(0) \;=\; \cos(0) + 6\cos(0) - 28\cos(0) \;=\; 1+6-28 = -21 \neq 0$, and
   $u'(\pi) \;=\; \cos(\pi) + 6\cos(2\pi) - 28\cos(7\pi)\; 1+6+28 = 37 \neq 0$.
   Finally, $u(0) = u(\pi)$. However, to see that u does *not* satisfy Periodic BC, observe that $u'(0) = 21 \neq -37 = u'(\pi)$.

   (c) To see that u does *not* satisfy homog. Dirichlet BC, observe:
   $u(0) \;=\; \cos(0) + 3\sin(0) - 2\cos(0) \;=\; 1+0-2 \;=\; -1 \neq 0$, and
   $u(\pi) \;=\; \cos(\pi) + 3\sin(3\pi) - 2\cos(6\pi) \;=\; -1-0-2 = -3 \neq 0$.
   To see that u does *not* satisfy homog. Neuman BC, observe:
   $u'(0) \;=\; -\sin(0) + 9\cos(0) + 12\sin(0) \;=\; 9 \;\neq\; 0$, and
   $u'(\pi) \;=\; -\sin(\pi) + 9\cos(3\pi) + 12\sin(6\pi) \;=\; -9 \;\neq\; 0$.
   Finally, to see that u does *not* satisfy Periodic BC, observe that $u(0) = -1 \neq -3 = u(\pi)$.

   (d) To see that u satisfies homog. Dirichlet BC, observe:
   $u(0) \;=\; 3 + \cos(0) - 4\cos(0) = 3+1-4 \;=\; 0$, and $u(\pi) \;=\; 3 + \cos(2\pi) - 4\cos(6\pi) = 3+1-4 \;=\; 0$.
   To see that u satisfies homog. Neuman BC, observe:
   $u'(0) \;=\; -2\sin(0) - 24\sin(0) \;=\; 0$, and $u'(\pi) \;=\; -2\sin(2\pi) - 24\sin(6\pi) \;=\; 0$.
   Finally, to see that u satisfies Periodic BC, observe $u(0) = 0 = u(\pi)$ and that $u'(0) = 0 = u'(\pi)$.

   (e) To see that u does *not* satisfy homog. Dirichlet BC, observe:
   $u(0) \;=\; 5 + \cos(0) - 4\cos(0) = 5+1-4 \;=\; 2$, and $u(\pi) \;=\; \cos(2\pi) - 4\cos(6\pi) = 5+1-4 \;=\; 2$.
   To see that u satisfies homog. Neuman BC, observe:
   $u'(0) \;=\; -2\sin(0) - 24\sin(0) \;=\; 0$, and $u'(\pi) \;=\; -2\sin(2\pi) - 24\sin(6\pi) \;=\; 0$.
   Finally, to see that u satisfies Periodic BC, observe $u(0) = 2 = u(\pi)$ and that $u'(0) = 0 = u'(\pi)$.

2. First note that $\partial\,[-\pi,\pi] \;=\; \{-\pi,\pi\}$.

   (a) To see that u satisfies homog. Dirichlet BC, observe:
   $u(-\pi) \;=\; \sin(-\pi) + 5\sin(-2\pi) - 2\sin(-3\pi) \;=\; 0$, and $u(\pi) \;=\; \sin(\pi) + 5\sin(2\pi) - 2\sin(3\pi) \;=\; 0$.
   To see that u does *not* satisfy homog. Neuman BC, observe:
   $u'(-\pi) \;=\; \cos(-\pi) + 10\cos(-2\pi) - 6\cos(-3\pi) \;=\; -1+10+6 \;=\; 15 \;\neq\; 0$, and
   $u'(\pi) \;=\; \cos(\pi) + 10\cos(2\pi) - 6\cos(3\pi) \;=\; -1+10+6 \;=\; 15 \;\neq\; 0$.
   Finally, to see that u satisfies Periodic BC, observe $u(0) = 0 = u(\pi)$ and that $u'(0) = 15 = u'(\pi)$.

(b) To see that u does *not* satisfy homog. Dirichlet BC, observe:
$u(-\pi) \;=\; 3\cos(-\pi) - 3\sin(-2\pi) - 4\cos(-2\pi) \;=\; -3 - 0 + 4 = 1 \neq 0$, and
$u(\pi) \;=\; 3\cos(\pi) - 3\sin(2\pi) - 4\cos(2\pi) \;=\; -3 - 0 + 4 = 1 \neq 0$.
To see that u does *not* satisfy homog. Neuman BC, observe:
$u'(-\pi) \;=\; -3\sin(-\pi) - 6\cos(-2\pi) + 8\sin(-2\pi) \;=\; -6 \neq 0$, and
$u'(\pi) \;=\; -3\sin(\pi) - 6\cos(2\pi) + 8\sin(2\pi) \;=\; -6 \neq 0$.
Finally, to see that u satisfies Periodic BC, observe $u'(0) = -3 = u'(\pi)$ and that $u'(0) = -6 = u'(\pi)$.

(c) To see that u satisfies homog. Dirichlet BC, observe:
$u(-\pi) \;=\; 6 + \cos(-\pi) - 3\cos(-2\pi) = 6 - 1 - 3 = 2 \neq 0$, and
$u(\pi) \;=\; 6 + \cos(\pi) - 3\cos(2\pi) = 6 - 1 - 3 = 2 \neq 0$.
To see that u satisfies homog. Neuman BC, observe:
$u'(-\pi) \;=\; -\sin(\pi) - 6\sin(2\pi) \;=\; 0 + 0 \;=\; 0$, and $u'(\pi) \;=\; -\sin(\pi) - 6\sin(2\pi) \;=\; 0 + 0 \;=\; 0$.
Finally, to see that u satisfies Periodic BC, observe $u'(0) = 2 = u'(\pi)$ and that $u'(0) = 0 = u'(\pi)$.

3. First note that

$$\begin{aligned} \partial\,[0,\pi]^2 &= \Big(\{0\}\times[0,\pi]\Big) \sqcup \Big(\{\pi\}\times[0,\pi]\Big) \sqcup \Big([0,\pi]\times\{0\}\Big) \sqcup \Big([0,\pi]\times\{0\}\Big). \\ &= \Big\{(x,y)\in[0,\pi]^2\;;\ \text{either } x=0 \text{ or } x=\pi \text{ or } y=0 \text{ or } y=\pi\Big\}. \end{aligned}$$

(a) f satisfies homogeneous Dirichlet BC, because

$$\begin{array}{rclclcl} f(0,y) &=& \sin(0)\sin(y) &=& 0\cdot\sin(y) &=& 0; \\ f(\pi,y) &=& \sin(\pi)\sin(y) &=& 0\cdot\sin(y) &=& 0; \\ f(x,0) &=& \sin(x)\sin(0) &=& \sin(x)\cdot 0 &=& 0; \\ f(x,\pi) &=& \sin(x)\sin(\pi) &=& \sin(x)\cdot 0 &=& 0; \end{array}$$

f does *not* satisfy homogeneous Neumann BC, because $\partial_x f(x,y) \;=\; \cos(x)\sin(y)$, so $\partial_\perp f(0,\frac{\pi}{2}) \;=\; -\partial_x f(0,\frac{\pi}{2}) \;=\; -\cos(0)\cdot\sin(\frac{\pi}{2}) \;=\; -1\cdot 1 \;=\; -1 \neq 0$.

(b) g does *not* satisfy homogeneous Dirichlet BC, because $g(0,\frac{\pi}{2}) \;=\; \sin(0) + \sin(\frac{\pi}{2}) \;=\; 0 + 1 \;=\; 1 \neq 0$.
g does *not* satisfy homogeneous Neumann BC, because $\partial_x g(x,y) \;=\; \cos(x)$, so $\partial_\perp g(0,\frac{\pi}{2}) \;=\; -\partial_x g(0,\frac{\pi}{2}) \;=\; -\cos(0) \;=\; -1 \neq 0$.

(c) h does *not* satisfy homogeneous Dirichlet BC, because $h(0,0) \;=\; \cos(0)\cdot\cos(0) \;=\; 1\cdot 1 \;=\; 1 \neq 0$.
h *does* satisfy homogeneous Neumann BC, because:

$$\begin{array}{rclclcl} \partial_\perp h(0,y) &=& -\partial_x g(0,y) &=& 2\sin(0) &=& 0; \\ \partial_\perp h(\pi,y) &=& \partial_x g(\pi,y) &=& -2\sin(2\pi) &=& 0; \\ \partial_\perp h(x,0) &=& -\partial_y g(x,0) &=& \sin(0) &=& 0; \\ \partial_\perp h(x,\pi) &=& \partial_y g(x,\pi) &=& -\sin(\pi) &=& 0. \end{array}$$

(d) To see that u satisfies homog. Dirichlet BC, observe:
$u(0,y) \;=\; \sin(0)\sin(3y) \;=\; 0$; $\quad u(\pi,y) \;=\; \sin(5\pi)\sin(3y) \;=\; 0$;
$u(x,0) \;=\; \sin(5x)\sin(0) \;=\; 0$; $\quad u(x,\pi) \;=\; \sin(5x)\sin(3\pi) \;=\; 0$.
To see that u does *not* satisfy homog Neumann. BC, observe:
$\partial_x u(0,y) \;=\; 5\cos(0)\sin(3y) \;=\; 5\sin(3y) \neq 0$; $\quad \partial_x u(\pi,y) \;=\; 5\cos(5\pi)\sin(3y) \;=\; -5\sin(3y) \neq 0$;
$\partial_y u(x,0) \;=\; 3\sin(5x)\cos(0) \;=\; 3\sin(5x) \neq 0$; $\quad \partial_y u(x,\pi) \;=\; 3\sin(5x)\cos(3\pi) \;=\; -3\sin(5x) \neq 0$.
Finally, $u(0,y) \;=\; u(\pi,y)$ and $u(x,0) \;=\; u(x,\pi)$. However, to see that u does *not* satisfy periodic BC, observe: $\partial_x u(0,y) \;=\; 5\sin(3y) \;\neq\; -5\sin(3y) \;=\; \partial_x u(\pi,y)$. Likewise, $\partial_y u(0,y) \;=\; \sin 3(5x) \;\neq\; -3\sin(5x) \;=\; \partial_y u(\pi,y)$.

(e) To see that u does *not* satisfy homog. Dirichlet BC, observe:
$u(0,y) \;=\; \cos(0)\cos(7y) \;=\; \cos(7y) \;\neq\; 0$; $\quad u(\pi,y) \;=\; \cos(-2\pi)\cos(7y) \;=\; \cos(7y) \;\neq\; 0$;
$u(x,0) \;=\; \cos(-2x)\cos(0) \;=\; \cos(-2x) \;\neq\; 0$; $\quad u(x,\pi) \;=\; \cos(-2x)\cos(7\pi) \;=\; -\cos(2x) \;\neq\; 0$.
To see that u satisfies homog. Neumann BC, observe:
$\partial_x u(0,y) \;=\; 2\sin(0)\cos(7y) \;=\; 0$; $\quad \partial_x u(\pi,y) \;=\; 2\sin(-2\pi)\cos(7y) \;=\; 0$;
$\partial_y u(x,0) \;=\; -7\cos(-2x)\sin(0) \;=\; 0$; $\quad \partial_y u(x,\pi) \;=\; -7\cos(-2x)\sin(7\pi) \;=\; 0$.
Finally, to see that u does *not* satisfy periodic BC, observe that $u(x,0) \;=\; -u(x,\pi)$.

4. First note that $\partial\,\mathbb{D} \;=\; \{(r,\theta)\;;\ r=1\}$.

(a) Evaluating u on the boundary, we get: $u(1,\theta) = (1-1^2) = 0$. Thus, u satisfies HDBC.
Also, $\partial_\perp u(r,\theta) \;=\; 2r$, so $\partial_\perp u(1,\theta) \;=\; -2$. Thus, u does not satisfy HNBC.

(b) To see that u satisfies homog. Dirichlet BC, observe: $u(1,\theta) \;=\; 1 - 1 = 0$.
To see that u does *not* satisfy homog. Neuman BC, observe: $\partial_r u(r,\theta) \;=\; -3r^2$, Thus, $\partial_r u(1,\theta) \;=\; -3 \neq 0$.

(c) To see that u does *not* satisfy homog. Dirichlet BC, observe: $u(r,\theta) \;=\; 3 + (1-1) = 3 \neq 0$.
To see that u satisfies homog. Neuman BC, observe: $\partial_r u(r,\theta) \;=\; -4r\cdot(1-r^2)$, Thus, $\partial_r u(1,\theta) \;=\; -4\cdot 0 \;=\; 0$.

(d) To see that u satisfies homog. Dirichlet BC, observe: $u(r,\theta) \;=\; \sin(\theta)\cdot(1-1) \;=\; \sin(\theta)\cdot 0 \;=\; 0$.
To see that u satisfies homog. Neuman BC, observe: $\partial_r u(r,\theta) \;=\; -4\sin(\theta)\cdot r\cdot(1-r^2)$, Thus, $\partial_r u(1,\theta) \;=\; -4\sin(\theta)\cdot 0 \;=\; 0$.

(e) u satisfies homogeneous *Dirichlet* BC because, for any $\theta \in [0, 2\pi]$, $\cos(1,\theta)(e - e^1) = \cos(2\theta)(e - e) = 0$.
u does not satisfy homogeneous *Neumann* BC because $\partial_r u(1,\theta) = \cos(2\theta)e^r$. Thus, for any $\theta \in [0, 2\pi]$, $\partial_\perp u(1,\theta) = \partial_r u(1,\theta) = \cos(2\theta)e^1 = e\cos(2\theta)$, which is not necessarily zero. For example, $\partial_\perp(1,0) = \cos(0)e = e \neq 0$.

5. First note that $\partial\mathbb{B} = \{(1,\theta,\varphi)\ ;\ r = 1\}$.

(a) To see that u satisfies homog. Dirichlet BC, observe: $u(1,\theta,\varphi) = (1-1)^2 = 0^2 = 0$.
To see that u satisfies homog. Neuman BC, observe: $\partial_r\, u(r,\theta,\phi) = -2\cdot(1-r)$, Thus, $\partial_r\, u(1,\theta,\varphi) = -2\cdot 0 = 0$.

(b) To see that u does *not* satisfies homog. Dirichlet BC, observe: $u(1,\theta,\varphi) = (1-1)^3 + 5 = 5 \neq 0$.
To see that u satisfies homog. Neuman BC, observe: $\partial_r\, u(0,\theta,\phi) = -3\cdot(1-r)2$, Thus, $\partial_r\, u(1,\theta,\varphi) = -3\cdot 02 = 0$.

Solutions to §7.7

1. (a) $\|f_n\|_2^2 = \int_0^1 f^2(x)\ dx = \int_0^1 e^{-2nx}\ dx = \frac{-1}{2n}e^{-2nx}\Big|_{x=0}^{x=1}\ dx = \frac{-1}{2n}\left(e^{-2n} - 1\right) = \frac{1}{2n}\left(1 - e^{-2n}\right)$. Thus, $\|f_n\|_2 = \frac{1}{\sqrt{2n}}\left(1 - e^{-2n}\right)^{1/2}$.

(b) Yes, becase $\lim_{n\to 0}\|f_n\|_2 = \lim_{n\to 0}\frac{1}{\sqrt{2n}}\left(1 - e^{-2n}\right)^{1/2} = 0$.

(c) $\|f_n\|_\infty = 1$ for any $n \in \mathbb{N}$. To see this, observe that, $0 < f_n(x) < 1$ for all $x \in (0,1]$. But if x is sufficiently close to 0, then $f_n(x) = e^{-nx}$ is arbitrarily close to 1.

(d) No, because $\|f_n\|_\infty = 1$ for all n.

(e) Yes. For any $x > 0$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} e^{-nx} = 0$.

2. (a) Yes. For any $x \in [0,1]$, find some $N \in \mathbb{N}$ so that $\frac{2}{N} < x$. Then $\frac{2}{n} < x$ for all $n > N$, so $f_n(x) = 0$ for all $n > N$.

(b) $\|f_n\|_2^2 = \int_0^1 f_n^2(x)\ dx = \int_{1/n}^{2/n} (\sqrt{n})^2\ dx = \int_{1/n}^{2/n} n\ dx = \frac{n}{n} = 1$. Thus, $\|f_n\|_2 = 1$.

(c) No, because $\|f_n\|_2 = 1$ for all n.

(d) $\|f_n\|_\infty = \sqrt{n}$ for any $n \in \mathbb{N}$. To see this, observe that , observe that, $0 \le f_n(x) \le \sqrt{n}$ for all $x \in (0,1]$, and also, $f_n\left(\frac{3}{2n}\right) = \sqrt{n}$.

(e) No, because $\|f_n\|_\infty = \sqrt{n}$ for all n.

3. (a) $\|f_n\|_\infty = \frac{1}{\sqrt{n}}$ for any $n \in \mathbb{N}$. To see this, observe that , observe that, $0 \le f_n(x) \le \frac{1}{\sqrt{n}}$ for all $x \in (0,1]$, and also, $f_n(1) = \frac{1}{\sqrt{n}}$.

(b) Yes, because $\|f_n\|_\infty = \frac{1}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0$.

(c) Yes, because uniform convergence implies pointwise convergence. To be specific, for any $\epsilon > 0$, let $N > \frac{1}{\epsilon^2}$. Then for any $n > N$, and any $x \in \mathbb{R}$, it is clear that $|f(x)| < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \epsilon$.

(d) $\|f_n\|_2^2 = \int_{-\infty}^{\infty} f_n^2(x)\ dx = \int_0^n \left(\frac{1}{\sqrt{n}}\right)^2\ dx = \int_0^n \frac{1}{n}\ dx = \frac{n}{n} = 1$. Thus, $\|f_n\|_2 = 1$.

(e) No, because $\|f_n\|_2 = 1$ for all n.

4. (a) Yes. For any $x \in (0,1]$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{1}{\sqrt[3]{nx}} = \lim_{n\to\infty}\frac{1}{\sqrt[3]{n}}\cdot\frac{1}{\sqrt[3]{x}} = \frac{1}{\sqrt[3]{x}}\cdot\lim_{n\to\infty}\sqrt[3]{\frac{1}{n}} = \frac{1}{\sqrt[3]{x}}\cdot\sqrt[3]{\lim_{n\to\infty}\frac{1}{n}} = \frac{1}{\sqrt[3]{x}}\cdot\sqrt[3]{0} = 0$.

(b) $\|f_n\|_2^2 = \int_0^1 |f_n(x)|^2\ dx = \int_0^1 \left|\frac{1}{\sqrt[3]{nx}}\right|^2\ dx = \int_0^1 \frac{1}{(nx)^{2/3}}\ dx = \frac{3}{n}(nx)^{1/3}\Big|_{x=0}^{x=1} = \frac{3}{n}\left(n^{1/3} - 0\right) = \frac{3n^{1/3}}{n} = \frac{3}{n^{2/3}}$.

Thus, $\|f_n\|_2 = \sqrt{\frac{3}{n^{2/3}}} = \boxed{\frac{\sqrt{3}}{n^{1/3}}.}$

(c) Yes, because $\lim_{n\to\infty}\|f_n\|_2 = \lim_{n\to\infty}\sqrt{\frac{3}{n}} = \sqrt{3\cdot\lim_{n\to\infty}\frac{1}{n}} = \sqrt{3\cdot 0} = 0$.

(d) Fix $n \in \mathbb{N}$. Since the function $f_n(x)$ is *decreasing* in x, we know that the value of $f_n(x)$ is *largest* when x is close to zero. Thus, $\|f_n\|_\infty = \sup_{0<x\le 1}\left|f_n(x)\right| = \sup_{0<x\le 1} f_n(x) = \lim_{x\to 0} f_n(x) = \lim_{x\to 0}\frac{1}{\sqrt[3]{nx}} = \sqrt[3]{\frac{1}{n}\cdot\lim_{x\to 0}\frac{1}{x}} = \sqrt[3]{\frac{1}{n}\cdot\infty} = \boxed{\infty.}$

(e) No, because $\lim_{n\to\infty} \|f_n\|_\infty = \lim_{n\to\infty} \infty = \infty \neq 0$.

5. (a) No. Let $x = 0$. Then $\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} \frac{1}{((n\cdot 0)+1)^2} = \lim_{n\to\infty} \frac{1}{1} = 1 \neq 0$.

(b) $\|f_n\|_2^2 = \int_0^1 |f_n(x)|^2\, dx = \int_0^1 \left|\frac{1}{(nx+1)^2}\right|^2 dx = \int_0^1 \frac{1}{(nx+1)^4}\, dx = \frac{1}{n}\int_1^{n+1} \frac{1}{y^4}\, dy = \frac{1}{n}\frac{-1}{3y^3}\Big|_1^{n+1} = \frac{1}{3n}\left(1 - \frac{1}{(n+1)^3}\right)$.

Thus, $\|f_n\|_2 = \boxed{\sqrt{\frac{1}{3n}\left(1 - \frac{1}{(n+1)^3}\right)}.}$

(c) Yes, because

$$\lim_{n\to\infty} \|f_n\|_2 = \lim_{n\to\infty} \sqrt{\frac{1}{3n}\left(1-\frac{1}{(n+1)^3}\right)} = \sqrt{\lim_{n\to\infty}\frac{1}{3n} - \lim_{n\to\infty}\frac{1}{3n(n+1)^3}} = \sqrt{0-0} = 0.$$

(d) Fix $n \in \mathbb{N}$. Since the function $f_n(x)$ is *decreasing* in x, we know that the value of $f_n(x)$ is *largest* when x is zero. Thus, $\|f_n\|_\infty = \max_{0\le x\le 1} \left|f_n(x)\right| = f_n(0) = \boxed{1.}$

(e) No, because $\lim_{n\to\infty} \|f_n\|_\infty = \lim_{n\to\infty} 1 = 1 \neq 0$.

6. (a) Recall that $\sin(a)\cdot\sin(b) = -\frac{1}{2}\Big(\cos(a+b) - \cos(a-b)\Big)$. Thus,

$$\begin{aligned}\langle f, g\rangle &= \frac{1}{|\mathbb{X}|}\int_{\mathbb{X}} f(x)\cdot g(x)\, dx = \frac{1}{\pi}\int_0^\pi \sin(3x)\cdot\sin(2x)\, dx = \frac{-1}{2\pi}\int_0^\pi \cos(3x+2x) - \cos(3x-2x)\, dx \\ &= \frac{-1}{2\pi}\left(\int_0^\pi \cos(5x)\, dx - \int_0^\pi \cos(x)\, dx\right) = \frac{-1}{2\pi}\left(\frac{1}{5}\sin(5x)\Big|_{x=0}^{x=\pi} - \sin(x)\Big|_{x=0}^{x=\pi}\right) \\ &= \frac{-1}{2\pi}\left(\frac{1}{5}(0-0) + (0-0)\right) = \boxed{0.}\end{aligned}$$

(b) Recall that $\sin(a)\cdot\sin(b) = -\frac{1}{2}\Big(\cos(a+b) - \cos(a-b)\Big)$. Thus,

$$\begin{aligned}\langle f, g\rangle &= \frac{1}{|\mathbb{X}|}\int_{\mathbb{X}} f(x)\cdot g(x)\, dx = \int_0^\pi \sin(nx)\cdot\sin(mx)\, dx = \frac{-1}{2\pi}\int_0^\pi \cos(nx+mx) - \cos(nx-mx)\, dx \\ &= \frac{-1}{2\pi}\left(\int_0^\pi \cos\left[(n+m)x\right]\, dx - \int_0^\pi \cos\left[(n-m)x\right]\, dx\right) \\ &= \frac{-1}{2\pi}\left(\frac{1}{n+m}\sin\left[(n+m)x\right]\Big|_{x=0}^{x=\pi} - \frac{1}{n-m}\sin\left[(n-m)x\right]\Big|_{x=0}^{x=\pi}\right) \\ &= \frac{-1}{2\pi}\left(\frac{1}{n+m}(0-0) - \frac{1}{n-m}(0-0)\right) = \boxed{0.}\end{aligned}$$

(c) Recall that $\sin(a)\cdot\sin(b) = -\frac{1}{2}\Big(\cos(a+b) - \cos(a-b)\Big)$. Thus,

$$\begin{aligned}\langle f, g\rangle &= \frac{1}{|\mathbb{X}|}\int_{\mathbb{X}} f(x)\cdot g(x)\, dx = \frac{1}{\pi}\int_0^\pi \sin(nx)\cdot\sin(nx)\, dx = \frac{-1}{2\pi}\int_0^\pi \cos(nx+nx) - \cos(nx-nx)\, dx \\ &= \frac{-1}{2\pi}\left(\int_0^\pi \cos(2nx)\, dx - \int_0^\pi \cos(0)\, dx\right) = \frac{-1}{2\pi}\left(\frac{1}{2n}\sin(2nx)\Big|_{x=0}^{x=\pi} - \int_0^\pi 1\, dx\right) \\ &= \frac{-1}{2\pi}\left(\frac{1}{2n}(0-0) - \pi\right) = \frac{-1}{2\pi}(-\pi) = \boxed{\frac{1}{2}.} \quad \text{Thus, } \|f\|_2 = \sqrt{\frac{1}{2}} = \boxed{\frac{1}{\sqrt{2}}.}\end{aligned}$$

(d) Recall that $\cos(a)\cdot\cos(b) = \frac{1}{2}\Big(\cos(a+b) + \cos(a-b)\Big)$. Thus,

$$\begin{aligned}\langle f, g\rangle &= \frac{1}{|\mathbb{X}|}\int_{\mathbb{X}} f(x)\cdot g(x)\, dx = \frac{1}{\pi}\int_0^\pi \cos(2x)\cdot\cos(3x)\, dx = \frac{1}{2\pi}\int_0^\pi \cos(2x+3x) + \cos(2x-3x)\, dx \\ &= \frac{1}{2\pi}\left(\int_0^\pi \cos(5x)\, dx + \int_0^\pi \cos(-x)\, dx\right) = \frac{1}{2\pi}\left(\frac{1}{5}\sin(5x)\Big|_{x=0}^{x=\pi} - \sin(x)\Big|_{x=0}^{x=\pi}\right) \\ &= \frac{1}{2\pi}\left(\frac{1}{5}(0-0) - (0-0)\right) = \boxed{0.}\end{aligned}$$

(e) Recall that $\cos(a)\cdot\cos(b) = \frac{1}{2}\Big(\cos(a+b)+\cos(a-b)\Big)$. Thus,

$$\begin{aligned}\langle f,g\rangle &= \frac{1}{|\mathbb{X}|}\int_{\mathbb{X}} f(x)\cdot g(x)\ dx = \frac{1}{\pi}\int_0^\pi \cos(nx)\cdot\cos(mx)\ dx = \frac{1}{2\pi}\int_0^\pi \cos(nx+mx)+\cos(nx-mx)\ dx \\ &= \frac{1}{2\pi}\left(\int_0^\pi \cos\left[(n+m)x\right]\ dx + \int_0^\pi \cos\left[(n-m)x\right]\ dx\right) \\ &= \frac{1}{2\pi}\left(\frac{1}{n+m}\sin\left[(n+m)x\right]\Big|_{x=0}^{x=\pi} + \frac{1}{n-m}\sin\left[(n-m)x\right]\Big|_{x=0}^{x=\pi}\right) \\ &= \frac{1}{2\pi}\left(\frac{1}{n+m}(0-0) + \frac{1}{n-m}(0-0)\right) = \boxed{0.}\end{aligned}$$

(f) Recall that $\sin(a)\cdot\cos(b) = \frac{1}{2}\Big(\sin(a+b)+\sin(a-b)\Big)$. Thus,

$$\begin{aligned}\langle f,g\rangle &= \frac{1}{|\mathbb{X}|}\int_{\mathbb{X}} f(x)\cdot g(x)\ dx = \frac{1}{\pi}\int_0^\pi \sin(3x)\cdot\cos(2x)\ dx = \frac{1}{2\pi}\int_0^\pi \sin(3x+2x)+\sin(3x-2x)\ dx \\ &= \frac{1}{2\pi}\left(\int_0^\pi \sin(5x)\ dx + \int_0^\pi \sin(x)\ dx\right) = \frac{1}{2\pi}\left(\frac{-1}{5}\cos(5x)\Big|_{x=0}^{x=\pi} - \cos(x)\Big|_{x=0}^{x=\pi}\right) \\ &= \frac{1}{2\pi}\left(\frac{-1}{5}[(-1)-1]-[(-1)-1]\right) = \frac{1}{2\pi}\left(\frac{2}{5}+2\right) = \boxed{\frac{6}{5\pi}.}\end{aligned}$$

7\. (a) Recall that $\sin(a)\cdot\sin(b) = -\frac{1}{2}\Big(\cos(a+b)-\cos(a-b)\Big)$. Thus,

$$\begin{aligned}\langle f,g\rangle &= \frac{1}{|\mathbb{X}|}\int_{\mathbb{X}} f(x)\cdot g(x)\ dx = \frac{1}{2\pi}\int_{-\pi}^\pi \sin(nx)\cdot\sin(mx)\ dx = \frac{-1}{4\pi}\int_{-\pi}^\pi \cos(nx+mx)-\cos(nx-mx)\ dx \\ &= \frac{-1}{4\pi}\left(\int_{-\pi}^\pi \cos\left[(n+m)x\right]\ dx - \int_{-\pi}^\pi \cos\left[(n-m)x\right]\ dx\right) \\ &= \frac{-1}{4\pi}\left(\frac{1}{n+m}\sin\left[(n+m)x\right]\Big|_{x=-\pi}^{x=\pi} - \frac{1}{n-m}\sin\left[(n-m)x\right]\Big|_{x=-\pi}^{x=\pi}\right) \\ &= \frac{-1}{4\pi}\left(\frac{1}{n+m}(0-0) - \frac{1}{n-m}(0-0)\right) = \boxed{0.}\end{aligned}$$

(b) Recall that $\sin(a)\cdot\sin(b) = -\frac{1}{2}\Big(\cos(a+b)-\cos(a-b)\Big)$. Thus,

$$\begin{aligned}\langle f,g\rangle &= \frac{1}{|\mathbb{X}|}\int_{\mathbb{X}} f(x)\cdot g(x)\ dx = \frac{1}{2\pi}\int_{-\pi}^\pi \sin(nx)\cdot\sin(nx)\ dx = \frac{-1}{4\pi}\int_{-\pi}^\pi \cos(nx+nx)-\cos(nx-nx)\ dx \\ &= \frac{-1}{4\pi}\left(\int_{-\pi}^\pi \cos(2nx)\ dx - \int_{-\pi}^\pi \cos(0)\ dx\right) = \frac{-1}{4\pi}\left(\frac{1}{2n}\sin(2nx)\Big|_{x=-\pi}^{x=\pi} - \int_{-\pi}^\pi 1\ dx\right) \\ &= \frac{-1}{4\pi}\left(\frac{1}{2n}(0-0) - 2\pi\right) = \frac{-1}{4\pi}(-2\pi) = \boxed{\frac{1}{2}.} \qquad \text{Thus, } \|f\|_2 = \sqrt{\frac{1}{2}} = \boxed{\frac{1}{\sqrt{2}}.}\end{aligned}$$

(c) Recall that $\cos(a)\cdot\cos(b) = \frac{1}{2}\Big(\cos(a+b)+\cos(a-b)\Big)$. Thus,

$$\begin{aligned}\langle f,g\rangle &= \frac{1}{|\mathbb{X}|}\int_{\mathbb{X}} f(x)\cdot g(x)\ dx = \frac{1}{2\pi}\int_{-\pi}^\pi \cos(nx)\cdot\cos(mx)\ dx = \frac{1}{4\pi}\int_{-\pi}^\pi \cos(nx+mx)+\cos(nx-mx)\ dx \\ &= \frac{1}{4\pi}\left(\int_{-\pi}^\pi \cos\left[(n+m)x\right]\ dx + \int_{-\pi}^\pi \cos\left[(n-m)x\right]\ dx\right) \\ &= \frac{1}{4\pi}\left(\frac{1}{n+m}\sin\left[(n+m)x\right]\Big|_{x=-\pi}^{x=\pi} + \frac{1}{n-m}\sin\left[(n-m)x\right]\Big|_{x=-\pi}^{x=\pi}\right) \\ &= \frac{1}{4\pi}\left(\frac{1}{n+m}(0-0) + \frac{1}{n-m}(0-0)\right) = \boxed{0.}\end{aligned}$$

(d) Recall that $\sin(a)\cdot\cos(b) = \frac{1}{2}\Big(\sin(a+b)+\sin(a-b)\Big)$. Thus,

$$\begin{aligned}\langle f,g\rangle &= \frac{1}{|\mathbb{X}|}\int_{\mathbb{X}} f(x)\cdot g(x)\ dx = \frac{1}{2\pi}\int_0^\pi \sin(3x)\cdot\cos(2x)\ dx = \frac{1}{4\pi}\int_0^\pi \sin(3x+2x)+\sin(3x-2x)\ dx \\ &= \frac{1}{4\pi}\left(\int_0^\pi \sin(5x)\ dx + \int_0^\pi \sin(x)\ dx\right) = \frac{1}{4\pi}\left(\frac{-1}{5}\cos(5x)\Big|_{x=-\pi}^{x=\pi} - \cos(x)\Big|_{x=-\pi}^{x=\pi}\right) \\ &= \frac{1}{4\pi}\left(\frac{-1}{5}(-1-(-1))+(1-(-1))\right) = \frac{1}{4\pi}(0+0) = \boxed{0.}\end{aligned}$$

Solutions to §8.4

1. Let $I = \int_0^\pi e^{\alpha x} \sin(nx)\ dx$. Then

$$\begin{aligned} I &= \frac{-1}{n}\left(e^{\alpha x}\cos(nx)\Big|_{x=0}^{x=\pi} - \alpha\int_0^\pi e^{\alpha x}\cos(nx)\ dx\right) \\ &= \frac{-1}{n}\left(e^{\alpha\pi}\cos(n\pi) - e^0\cos(0) - \frac{\alpha}{n}\left[e^{\alpha x}\sin(nx)\Big|_{x=0}^{x=\pi} - \alpha\int_0^\pi e^{\alpha x}\sin(nx)\ dx\right]\right) \\ &= \frac{-1}{n}\left(e^{\alpha\pi}(-1)^n - 1 - \frac{\alpha}{n}\left[(0-0) - \alpha\cdot I\right]\right) = \frac{-1}{n}\left(e^{\alpha\pi}(-1)^n - 1 + \frac{\alpha^2}{n}I\right) \\ &= \frac{1+(-1)^{n+1}e^{\alpha\pi}}{n} - \frac{\alpha^2}{n^2}I \end{aligned}$$

Thus, $I + \frac{\alpha^2}{n^2}I = \frac{1+(-1)^{n+1}e^{\alpha\pi}}{n}$. In other words, $\frac{n^2+\alpha^2}{n^2}I = \frac{1+(-1)^{n+1}e^{\alpha\pi}}{n}$. Thus, In other words, $I = \left(\frac{n^2}{n^2+\alpha^2}\right)\cdot\left(\frac{1+(-1)^{n+1}e^{\alpha\pi}}{n}\right) = \frac{n\left(1+(-1)^{n+1}e^{\alpha\pi}\right)}{n^2+\alpha^2}$. Hence

$$B_n = \frac{2}{\pi}I = \frac{2n\left(1+(-1)^{n+1}e^{\alpha\pi}\right)}{\pi(n^2+\alpha^2)}$$

Thus, the Fourier Sine series is: $\sum_{n=1}^\infty B_n\sin(nx) = \boxed{\frac{2}{\pi}\sum_{n=1}^\infty \frac{n\left(1+(-1)^{n+1}e^{\alpha\pi}\right)}{n^2+\alpha^2}\sin(nx).}$

2. $A_0 = \frac{1}{\pi}\int_0^\pi \sinh(x)\ dx = \frac{1}{\pi}\cosh(x)\Big|_{x=0}^{x=\pi} = \frac{\cosh(\pi)-1}{\pi}$.

Next, let $I = \int_0^\pi \sinh(x)\cdot\cos(nx)\ dx$. Then, applying integration by parts,

$$\begin{aligned} I &= \frac{1}{n}\left(\sinh(x)\cdot\sin(nx)\Big|_{x=0}^{x=\pi} - \int_0^\pi \cosh(x)\cdot\sin(nx)\ dx\right) = (0-0) - \frac{1}{n}\int_0^\pi \cosh(x)\cdot\sin(nx)\ dx \\ &= \frac{1}{n^2}\left(\cosh(x)\cdot\cos(nx)\Big|_{x=0}^{x=\pi} - \int_0^\pi \sinh(x)\cdot\cos(nx)\ dx\right) = \frac{1}{n^2}\left(\cosh(\pi)\cdot\cos(n\pi) - 1 - I\right) \end{aligned}$$

Thus, $I = \frac{1}{n^2}\left((-1)^n\cdot\cosh(\pi) - 1 - I\right)$. Hence, $(n^2+1)I = (-1)^n\cdot\cosh(\pi) - 1$.

Hence, $I = \frac{(-1)^n\cdot\cosh(\pi)-1}{n^2+1}$. Thus, $A_n = \frac{2}{\pi}I = \frac{2}{\pi}\frac{(-1)^n\cdot\cosh(\pi)-1}{n^2+1}$.

Thus, the Fourier cosine series is:

$$\sinh(x) \underset{L2}{\approx} A_0 + \sum_{n=1}^\infty A_n\cos(nx) = \boxed{\frac{\cosh(\pi)-1}{\pi} + \frac{2\cosh(\pi)-2}{\pi}\sum_{n=1}^\infty \frac{(-1)^n\cdot\cos(nx)}{n^2+1}}.$$

3. Let $I = \int_0^\pi \cosh(\alpha x)\sin(nx)\ dx$. Then

$$\begin{aligned} I &= \frac{-1}{n}\left(\cosh(\alpha x)\cos(nx)\Big|_{x=0}^{x=\pi} - \alpha\int_0^\pi \sinh(\alpha x)\cos(nx)\ dx\right) \\ &= \frac{-1}{n}\left(\cosh(\alpha\pi)\cos(n\pi) - \cosh(0)\cos(0) - \frac{\alpha}{n}\left[\sinh(\alpha x)\sin(nx)\Big|_{x=0}^{x=\pi} - \alpha\int_0^\pi \cosh(\alpha x)\sin(nx)\ dx\right]\right) \\ &= \frac{-1}{n}\left(\cosh(\alpha\pi)(-1)^n - 1 - \frac{\alpha}{n}\left[(0-0) - \alpha\cdot I\right]\right) = \frac{-1}{n}\left(\cosh(\alpha\pi)(-1)^n - 1 + \frac{\alpha^2}{n}I\right) \\ &= \frac{1+(-1)^{n+1}\cosh(\alpha\pi)}{n} - \frac{\alpha^2}{n^2}I \end{aligned}$$

Thus, $I + \frac{\alpha^2}{n^2}I = \frac{1+(-1)^{n+1}\cosh(\alpha\pi)}{n}$. In other words, $\frac{n^2+\alpha^2}{n^2}I = \frac{1+(-1)^{n+1}\cosh(\alpha\pi)}{n}$. Thus, In other words, $I = \left(\frac{n^2}{n^2+\alpha^2}\right)\cdot\left(\frac{1+(-1)^{n+1}\cosh(\alpha\pi)}{n}\right) = \frac{n\left(1+(-1)^{n+1}\cosh(\alpha\pi)\right)}{n^2+\alpha^2}$. Hence

$$B_n = \frac{2}{\pi}I = \frac{2n\left(1+(-1)^{n+1}\cosh(\alpha\pi)\right)}{\pi(n^2+\alpha^2)}$$

Thus, the Fourier Sine series is: $\sum_{n=1}^{\infty} B_n \sin(nx) \;=\; \boxed{\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{n\left(1+(-1)^{n+1}\cosh(\alpha\pi)\right)}{n^2+\alpha^2}\sin(nx).}$

4. First, $A_0 \;=\; \frac{1}{\pi}\int_0^{\pi} f(x)\,dx \;=\; \frac{1}{\pi}\int_0^{\pi} x\,dx \;=\; \frac{1}{\pi}\frac{x^2}{2}\Big|_{x=0}^{x=\pi} \;=\; \frac{1}{\pi}\frac{\pi^2}{2} \;=\; \frac{\pi}{2}.$ Next,

$$\begin{aligned}
A_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx \;=\; \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx \;=\; \frac{2}{n\pi}\left(x\sin(nx)\Big|_{x=0}^{x=\pi} - \int_0^{\pi}\sin(nx)\,dx\right)\\
&= \frac{2}{n\pi}\left(\pi\sin(n\pi) - 0\sin(0) + \frac{1}{n}\cos(nx)\Big|_{x=0}^{x=\pi}\right) \;=\; \frac{2}{n\pi}(0-0) + \frac{2}{n^2\pi}\left(\cos(n\pi)-\cos(0)\right)\\
&= \frac{2}{n^2\pi}\left((-1)^n - 1\right) \;=\; \frac{-2}{n^2\pi}\begin{cases} 2 & \text{if } n \text{ odd}\\ 0 & \text{if } n \text{ even}\end{cases}
\end{aligned}$$

Thus, the cosine series is given: $f(x) \;=\; \boxed{\frac{\pi}{2} - \frac{4}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^2}\cos(nx).}$

5. (a) First note that $A_0 \;=\; \frac{1}{\pi}\int_0^{\pi} g(x)\,dx \;=\; \frac{1}{\pi}\int_0^{\pi/2} 1\,dx \;=\; \frac{1}{\pi}\frac{\pi}{2} \;=\; \frac{1}{2}.$

Next, for any $n > 0$,

$$\begin{aligned}
A_n &= \frac{2}{\pi}\int_0^{\pi} g(x)\cos(nx)\,dx \;=\; \frac{2}{\pi}\int_0^{\pi/2}\cos(nx)\,dx \;=\; \frac{2}{n\pi}\sin(nx)\Big|_{x=0}^{x=\pi/2}\\
&= \frac{2}{n\pi}\left(\sin\left(\frac{n\pi}{2}\right) - \sin(0)\right) \;=\; \frac{2}{n\pi}\begin{cases} (-1)^k & \text{if } n \text{ is odd and } n = 2k+1\\ 0 & \text{if } n \text{ is even}\end{cases}
\end{aligned}$$

Hence, $A_n = \begin{cases} \frac{2}{\pi}\frac{(-1)^k}{2k+1} & \text{if } n \text{ is odd and } n = 2k+1;\\ 0 & \text{if } n \text{ is even.}\end{cases}$ Thus, the Fourier cosine series of $g(x)$ is: $\boxed{\frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\left((2k+1)x\right)}$

(b) For any $n > 0$,

$$\begin{aligned}
B_n &= \frac{2}{\pi}\int_0^{\pi} g(x)\sin(nx)\,dx \;=\; \frac{2}{\pi}\int_0^{\pi/2}\sin(nx)\,dx \;=\; \frac{-2}{n\pi}\cos(nx)\Big|_{x=0}^{x=\pi/2}\\
&= \frac{-2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos(0)\right) \;=\; \frac{-2}{n\pi}\begin{cases} (-1)^k - 1 & \text{if } n \text{ is even and } n = 2k\\ -1 & \text{if } n \text{ is odd}\end{cases}
\end{aligned}$$

Hence, $B_n = \begin{cases} \frac{2}{n\pi} & \text{if } n \text{ is odd}\\ 0 & \text{if } n \equiv 0 \pmod 4\\ \frac{4}{n\pi} & \text{if } n \equiv 2 \pmod 4\end{cases}$

Thus, the Fourier sine series of $g(x)$ is: $\boxed{\frac{2}{\pi}\sum_{k=0}^{\infty}\left(\frac{\sin[(4k+1)x]}{4k+1} + \frac{2\sin[(4k+2)x]}{4k+2} + \frac{\sin[(4k+3)x]}{4k+3}\right).}$

6. First note that $A_0 \;=\; \frac{1}{\pi}\int_0^{\pi} g(x)\,dx \;=\; 3\frac{1}{\pi}\int_0^{\pi/2} 1\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 1\,dx \;=\; \frac{3}{\pi}\frac{\pi}{2} + \frac{1}{\pi}\frac{\pi}{2} \;=\; 2.$

Next, for any $n > 0$,

$$\begin{aligned}
A_n &= \frac{2}{\pi}\int_0^{\pi} g(x)\cos(nx)\,dx \;=\; 3\frac{2}{\pi}\int_0^{\pi/2}\cos(nx)\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}\cos(nx)\,dx\\
&= \frac{6}{n\pi}\sin(nx)\Big|_{x=0}^{x=\pi/2} + \frac{2}{n\pi}\sin(nx)\Big|_{x=\pi/2}^{x=\pi}\\
&= \frac{6}{n\pi}\left(\sin\left(\frac{n\pi}{2}\right) - \sin(0)\right) + \frac{2}{n\pi}\left(\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right)\\
&= \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right) \;=\; \frac{4}{n\pi}\begin{cases} (-1)^k & \text{if } n \text{ is odd and } n = 2k+1\\ 0 & \text{if } n \text{ is even}\end{cases}
\end{aligned}$$

Hence, $A_n = \begin{cases} \frac{4}{\pi}\frac{(-1)^k}{2k+1} & \text{if } n \text{ is odd and } n = 2k+1;\\ 0 & \text{if } n \text{ is even.}\end{cases}$ Thus, the Fourier cosine series of $g(x)$ is: $\boxed{2 + \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\left((2k+1)x\right)}$

7. First,

$$\begin{aligned}
\int_0^{\pi/2} x\sin(nx)\,dx &= \frac{-1}{n}\left[x\cos(nx)\Big|_0^{\pi/2} - \int_0^{\pi/2}\cos(nx)\,dx\right]\\
&= \frac{-1}{n}\left[\frac{\pi}{2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n}\sin(nx)\Big|_0^{\pi/2}\right] \;=\; \frac{-\pi}{2n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right).
\end{aligned}$$

$$\text{Next,} \int_{\pi/2}^{\pi} x\sin(nx)\,dx = \frac{-1}{n}\left[x\cos(nx)\Big|_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\cos(nx)\,dx\right]$$
$$= \frac{-1}{n}\left[\pi\cos(n\pi) - \frac{\pi}{2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n}\sin(nx)\Big|_{\pi/2}^{\pi}\right]$$
$$= \frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) - \frac{(-1)^n\pi}{n} - \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right).$$

$$\text{Finally,} \int_{\pi/2}^{\pi}\pi\sin(nx)\,dx = \frac{-\pi}{n}\cos(nx)\Big|_{\pi/2}^{\pi} = \frac{-\pi}{n}\left[(-1)^n - \cos\left(\frac{n\pi}{2}\right)\right]$$
$$= \frac{(-1)^{n+1}\pi}{n} + \frac{\pi}{n}\cos\left(\frac{n\pi}{2}\right).$$

Putting it all together, we have:

$$\int_0^{\pi} f(x)\sin(nx)\,dx = \int_0^{\pi/2} x\sin(nx)\,dx + \int_{\pi/2}^{\pi}\pi\sin(nx)\,dx - \int_{\pi/2}^{\pi} x\sin(nx)\,dx$$
$$= \frac{-\pi}{2n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right) + \frac{(-1)^{n+1}\pi}{n} + \frac{\pi}{n}\cos\left(\frac{n\pi}{2}\right)$$
$$-\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) + \frac{(-1)^n\pi}{n} + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)$$
$$= \frac{2}{n^2}\sin\left(\frac{n\pi}{2}\right)$$
$$= \frac{2}{n^2}\begin{cases}(-1)^k & \text{if } n \text{ is odd, and } n=2k+1;\\ 0 & \text{if } n \text{ is even.}\end{cases}$$

Thus,

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \begin{cases}\frac{4}{n^2\pi}(-1)^k & \text{if } n \text{ is odd, } n=2k+1;\\ 0 & \text{if } n \text{ is even.}\end{cases}$$

Thus, the Fourier sine series is $f(x) = \boxed{\frac{4}{\pi}\sum_{\substack{n=1\\ n\text{ odd;}\\ n=2k+1}}^{\infty}\frac{(-1)^k}{n^2}\sin(nx).}$

8. $$B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi/2} x\cdot\sin(nx)\,dx$$
$$= \frac{-2}{n\pi}\left(x\cos(nx)\Big|_{x=0}^{x=\pi/2} - \int_0^{\pi/2}\cos(nx)\,dx\right)$$
$$= \frac{-2}{n\pi}\left(\frac{\pi}{2}\cos\left(\frac{n\pi}{2}\right) - 0 - \frac{1}{n}\sin(nx)\Big|_{x=0}^{x=\pi/2}\right)$$
$$= \frac{-1}{n}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{n^2\pi}\left(\sin\left(\frac{n\pi}{2}\right) - 0\right)$$
$$= \frac{-1}{n}\begin{cases}(-1)^k & \text{if } n=2k \text{ is even}\\ 0 & \text{if } n \text{ is odd}\end{cases} + \frac{1}{2n^2\pi}\begin{cases}0 & \text{if } n \text{ is even}\\ (-1)^k & \text{if } n=2k+1 \text{ is odd}\end{cases}$$
$$= \begin{cases}\frac{(-1)^{k+1}}{n} & \text{if } n=2k \text{ is even}\\ \frac{2\cdot(-1)^k}{n^2\pi} & \text{if } n=2k+1 \text{ is odd}\end{cases}.$$

Thus,the Fourier sine series is:

$$\sum_{n=1}^{\infty} B_n\sin(nx) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k}\sin(2kx) + \sum_{k=0}^{\infty}\frac{2\cdot(-1)^k}{(2k+1)^2\pi}\sin\Big((2k+1)x\Big)$$

Solutions to §10.3

1. $B_{nm} = \frac{4}{\pi}\int_0^{\pi}\int_0^{\pi} x^2\cdot y\sin(nx)\sin(my)\,dx\,dy = \left(\frac{2}{\pi}\int_0^{\pi} x^2\sin(nx)\,dx\right)\cdot\left(\frac{2}{\pi}\int_0^{\pi} y\sin(my)\,dy\right)$

$\overline{\overline{(2)\&(3)}}$ $\left[(-1)^{n+1}\frac{2\pi}{n} + \frac{4}{\pi n^3}\Big((-1)^n - 1\Big)\right]\cdot\left[(-1)^{m+1}\frac{2}{m}\right] = \frac{4\pi(-1)^{n+m}}{nm} + \frac{8(-1)^m}{\pi n^3 m}\Big(1-(-1)^n\Big)$. Thus, $B_{nm} = \begin{cases}\frac{4\pi(-1)^{n+m}}{nm} & \text{if } n \text{ is even}\\ \frac{4\pi(-1)^{n+m}}{nm} + \frac{16(-1)^m}{\pi n^3 m} & \text{if } n \text{ is odd}\end{cases}.$

Thus, the Fourier Sine Series is:

$$\sum_{n,m=1}^{\infty} B_{nm}\sin(nx)\sin(my) = \boxed{4\pi\sum_{n,m=1}^{\infty}\frac{(-1)^{n+m}}{nm}\sin(nx)\sin(my) + \frac{16}{\pi}\sum_{\substack{n,m=1\\ n\text{ even}}}^{\infty}\frac{(-1)^m}{n^3 m}\sin(nx)\sin(my).}$$

2.

$$\begin{aligned} B_{nm} &= \frac{4}{\pi}\int_0^\pi\int_0^\pi (x+y)\sin(nx)\sin(my)\ dx\ dy \\ &= \frac{4}{\pi}\int_0^\pi\int_0^\pi x\sin(nx)\sin(my)\ dx\ dy \ + \ \frac{4}{\pi}\int_0^\pi\int_0^\pi y\sin(nx)\sin(my)\ dx\ dy \\ &= \left(\frac{2}{\pi}\int_0^\pi x\sin(nx)\ dx\right)\cdot\left(\frac{2}{\pi}\int_0^\pi \sin(my)\ dy\right) \ + \ \left(\frac{2}{\pi}\int_0^\pi \sin(nx)\ dx\right)\cdot\left(\frac{2}{\pi}\int_0^\pi y\sin(my)\ dy\right) \\ &= \left((-1)^{n+1}\frac{2}{n}\right)\cdot\left(\frac{2}{\pi}\frac{1-(-1)^m}{m}\right) \ + \ \left(\frac{2}{\pi}\frac{1-(-1)^n}{n}\right)\cdot\left((-1)^{m+1}\frac{2}{m}\right) \\ &= \frac{-8}{\pi nm}\left[\left(\begin{cases} (-1)^n & \text{if } m \text{ is odd;} \\ 0 & \text{if } m \text{ is even.}\end{cases}\right) + \left(\begin{cases} (-1)^m & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.}\end{cases}\right)\right] \end{aligned}$$

Thus, the Fourier Sine Series is:

$$\sum_{n,m=1}^{\infty} B_{nm}\sin(nx)\sin(my) \quad = \quad \boxed{\frac{-8}{\pi}\left(\sum_{\substack{n,m=1\\ m\text{ odd}}}^{\infty}\frac{(-1)^n}{nm}\sin(nx)\sin(my) \ + \ \sum_{\substack{n,m=1\\ n\text{ odd}}}^{\infty}\frac{(-1)^m}{nm}\sin(nx)\sin(my)\right).}$$

3.

$$\begin{aligned} B_{nm} &= \frac{4}{\pi}\int_0^\pi\int_0^\pi \cos(Nx)\cdot\cos(My)\cdot\sin(nx)\cdot\sin(my)\ dx\ dy \\ &= \left(\frac{2}{\pi}\int_0^\pi \cos(Nx)\sin(nx)\ dx\right)\cdot\left(\frac{2}{\pi}\int_0^\pi \cos(My)\sin(my)\ dy\right) \\ &\underset{(*)}{=\!=} \left(\begin{cases} 0 & \text{if } n+N \text{ is even} \\ \dfrac{4n}{\pi(n^2-N^2)} & \text{if } n+N \text{ is odd.}\end{cases}\right)\cdot\left(\begin{cases} 0 & \text{if } m+M \text{ is even} \\ \dfrac{4m}{\pi(m^2-M^2)} & \text{if } m+M \text{ is odd.}\end{cases}\right) \\ &= \begin{cases} 0 & \text{if either } (n+N) \text{ or } (m+M) \text{ is even;} \\ \dfrac{16\cdot n\cdot m}{\pi^2(n^2-N^2)(m^2-M^2)} & \text{if } (n+N) \text{ and } (m+M) \text{ are both odd.}\end{cases} \end{aligned}$$

where $(*)$ is by Example 8.2(b) on page 146. Thus, the Fourier Sine Series is:

$$\sum_{n,m=1}^{\infty} B_{nm}\sin(nx)\sin(my) \quad = \quad \boxed{\frac{16}{\pi^2}\sum_{\substack{n,m=1\\ \text{both odd}}}^{\infty}\frac{n\cdot m}{(n^2-N^2)(m^2-M^2)}\sin(nx)\sin(my).}$$

4. Setting $\alpha = N$ in Example 8.3 on page 148, we obtain the one-dimensional Fourier sine series of $\sinh(Ny)$:

$$\frac{2\sinh(N\pi)}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{N^2+n^2}\sin(ny).$$

Thus, the 2-dimensional Fourier Sine series of $\sin(Nx)\cdot\sinh(Ny)$ is

$$\sin(Nx)\cdot\frac{2\sinh(N\pi)}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{N^2+n^2}\sin(ny) \ = \ \boxed{\frac{2\sinh(N\pi)}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{N^2+n^2}\sin(Nx)\cdot\sin(ny)}$$

Solutions to §11.4

1. Recall from the solution to problem #5 of §8.4 that:

$$\text{The Fourier cosine series of } g(x) \text{ is:} \quad \frac{1}{2} \ + \ \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\Big((2k+1)x\Big)$$

$$\text{The Fourier sine series of } g(x) \text{ is:} \quad \frac{-2}{\pi}\sum_{k=0}^{\infty}\left(\frac{\sin[(4k+1)x]}{4k+1} \ + \ \frac{2\sin[(4k+2)x]}{4k+2} \ + \ \frac{\sin[(4k+3)x]}{4k+3}\right).$$

(a) $u(x,t) =$

$$\frac{-2}{\pi}\sum_{k=0}^{\infty}\left(\frac{\exp\Big(-(4k+1)^2t\Big)\sin[(4k+1)x]}{4k+1} \ + \ \frac{2\exp\Big(-(4k+2)^2t\Big)\sin[(4k+2)x]}{4k+2} \ + \ \frac{\exp\Big(-(4k+3)^2t\Big)\sin[(4k+3)x]}{4k+3}\right).$$

(b) $u(x,t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\left(-(2k+1)^2 t\right)\cdot\cos\left((2k+1)x\right).$

(c) $w(x,t) =$

$$\frac{-2}{\pi}\sum_{k=0}^{\infty}\left(\frac{\sin((4k+1)t)\sin[(4k+1)x]}{(4k+1)^2} + \frac{2\sin((4k+2)t)\sin[(4k+2)x]}{(4k+2)^2} + \frac{\sin((4k+3)t)\sin[(4k+3)x]}{(4k+3)^2}\right).$$

2. (a) The sine series *is* $\boxed{\sin(3x).}$

(b) First we compute the Fourier cosine coefficients:

$$A_0 = \frac{1}{\pi}\int_0^{\pi}\sin(3x)\,dx = \frac{-1}{3\pi}\cos(3x)\Big|_{x=0}^{x=\pi} = \frac{-1}{3\pi}\left(\cos(3\pi)-\cos(0)\right) = \frac{-1}{3\pi}(-1-1) = \frac{2\pi}{3}.$$

$$A_n = \frac{2}{\pi}\int_0^{\pi}\cos(nx)\sin(3x)\,dx \underset{(*)}{=} \frac{2}{\pi}\begin{cases}0 & \text{if } 3+n \text{ is even}\\ \dfrac{2\cdot 3}{\pi(3^2-n^2)} & \text{if } 3+n \text{ is odd.}\end{cases} = \frac{2}{\pi}\begin{cases}0 & \text{if } n \text{ is odd}\\ \dfrac{6}{\pi(9-n^2)} & \text{if } n \text{ is even.}\end{cases}$$

where $(*)$ is by Theorem 7.7(c) on page 117. Thus, the cosine series is: $A_0 + \sum_{n=1}^{\infty}A_n\cos(nx) = \boxed{\frac{2\pi}{3} + \frac{12}{\pi}\sum_{\substack{n=1\\ n \text{ even}}}^{\infty}\frac{1}{\pi(9-n^2)}\cos(nx).}$

(c) i. From part (a) and Proposition 11.1 on page 191, we get: $u(x;t) = \boxed{\sin(3x)\cdot e^{-9t}.}$

ii. From part (b) and Proposition 11.3 on page 192, we get:

$$u(x;t) = \boxed{\frac{2\pi}{3} + \frac{12}{\pi}\sum_{\substack{n=1\\ n \text{ even}}}^{\infty}\frac{1}{\pi(9-n^2)}\cos(nx)\cdot e^{-n^2 t}.}$$

(d) From part (a) and Proposition 11.8 on page 197, we get: $v(x;t) = \boxed{\frac{1}{3}\sin(3x)\cdot\sin(3t).}$

3. (a) $u(x;t) = \sum_{n=0}^{\infty}A_n\cos(nx)\exp(-n^2t) = \boxed{\sum_{n=0}^{\infty}\frac{1}{2^n}\cos(nx)\exp(-n^2t).}$

(b) **Heat Equation:**

$$\begin{aligned}\partial_t\, u(x;t) &= \partial_t\left(\sum_{n=0}^{\infty}\frac{1}{2^n}\cos(nx)\exp(-n^2t)\right) = \sum_{n=0}^{\infty}\frac{1}{2^n}\cos(nx)\cdot\partial_t\left(\exp(-n^2t)\right)\\ &= \sum_{n=0}^{\infty}\frac{1}{2^n}\cos(nx)\cdot(-n^2)\cdot\exp(-n^2t) = \sum_{n=0}^{\infty}\frac{1}{2^n}(-n^2)\cdot\cos(nx)\cdot\exp(-n^2t)\\ &= \sum_{n=0}^{\infty}\frac{1}{2^n}\cdot\partial_x^2\left(\cos(nx)\right)\cdot\exp(-n^2t) = \partial_x^2\left(\sum_{n=0}^{\infty}\frac{1}{2^n}\cos(nx)\cdot\exp(-n^2t)\right)\\ &= \triangle u(x;t), \quad\text{as desired.}\end{aligned}$$

Boundary Conditions:

$$\begin{aligned}\partial_x\, u(x;t) &= \partial_x\left(\sum_{n=0}^{\infty}\frac{1}{2^n}\cos(nx)\exp(-n^2t)\right) = \sum_{n=0}^{\infty}\frac{1}{2^n}\,\partial_x\left(\cos(nx)\right)\exp(-n^2t)\\ &= -\sum_{n=0}^{\infty}\frac{n}{2^n}\sin(nx)\exp(-n^2t)\end{aligned}$$

Hence,

$$\begin{aligned}\partial_\perp\, u(0;t) = -\partial_x\, u(0;t) &= \sum_{n=0}^{\infty}\frac{n}{2^n}\sin(n\cdot 0)\exp(-n^2t) = \sum_{n=0}^{\infty}\frac{n}{2^n}0\exp(-n^2t)\\ &= 0, \quad\text{as desired,}\end{aligned}$$

$$\begin{aligned}\text{and}\quad \partial_\perp\, u(\pi;t) = \partial_x\, u(\pi;t) &= -\sum_{n=0}^{\infty}\frac{n}{2^n}\sin(n\pi)\exp(-n^2t) = -\sum_{n=0}^{\infty}\frac{n}{2^n}0\exp(-n^2t)\\ &= 0, \quad\text{as desired.}\end{aligned}$$

Initial conditions:

$$u(x;0) = \sum_{n=0}^{\infty} \frac{1}{2^n}\cos(nx)\exp(-n^2\cdot 0) = \sum_{n=0}^{\infty} \frac{1}{2^n}\cos(nx)\cdot(1) = \sum_{n=0}^{\infty} \frac{1}{2^n}\cos(nx)\cdot(1) = f(x) \quad \text{as desired.}$$

(c) $u(x;t) = \sum_{n=1}^{\infty} B_n \sin(nx)\cos(nt) = \boxed{\sum_{n=1}^{\infty} \frac{1}{n!}\sin(nx)\cos(nt).}$

4. (a) For any $n \in \mathbb{N}$, $B_n := \frac{2}{\pi}\int_0^\pi x\cdot\sin(nx)\,dx = \boxed{\frac{2(-1)^{n+1}}{n}}.$ To see this, note that

$$\int_0^\pi x\cdot\sin(nx)\,dx = \frac{-1}{n}\left(x\cdot\cos(nx)\Big|_{x=0}^{x=\pi} - \int_0^\pi \cos(nx)\,dx\right) = \frac{-1}{n}\left(\pi\cdot\cos(n\pi) - \underbrace{0\cdot\cos(0) - \frac{1}{n}\sin(nx)\Big|_{x=0}^{x=\pi}}_{0}\right) = \frac{-1}{n}(-1)^n\pi = \frac{(-1)^{n+1}\pi}{n}$$

thus, the Fourier sine series is $\sum_{n=1}^{\infty} B_n\sin(nx) = \boxed{2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx).}$

(b) $\boxed{\text{Yes.}}$ f is a continuous function on $(0,\pi)$, so the Fourier series converges pointwise on $(0,\pi)$, by Theorem 10.1(b)

(c) $\boxed{\text{No.}}$ f does *not* satisfy homogeneous Dirichlet boundary conditions (because $f(\pi) = \pi \neq 0$) so the Fourier sine series cannot converge uniformly to f, by Theorem 10.1(d).

Alternately, notice that $\sum_{n=1}^{\infty}|B_n| = 2\sum_{n=1}^{\infty}\frac{1}{n} = \infty$. Thus, the sum of the absolute values of the Fourier coefficients is divergent; hence the Fourier series cannot converge uniformly, by Theorem 10.1(c).

(d) First note that $A_0 := \frac{1}{\pi}\int_0^\pi x\,dx = \frac{1}{2\pi}x^2\Big|_{x=0}^{x=\pi} = \frac{\pi}{2}.$

Also, for any $n \geq 1$, $A_n := \frac{2}{\pi}\int_0^\pi x\cdot\cos(nx)\,dx = \boxed{\begin{cases} \frac{-4}{\pi n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even}\end{cases}}.$ To see this, note that

$$\int_0^\pi x\cdot\cos(nx)\,dx = \frac{1}{n}\left(x\cdot\sin(nx)\Big|_{x=0}^{x=\pi} - \int_0^\pi \sin(nx)\,dx\right) = \frac{1}{n}\left(\pi\cdot\underbrace{\sin(n\pi)}_{0} - 0\cdot\underbrace{\sin(0)}_{0} + \frac{1}{n}\cos(nx)\Big|_{x=0}^{x=\pi}\right) = \frac{1}{n^2}\Big((-1)^n - 1\Big) = \begin{cases} \frac{-2}{n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even}\end{cases}.$$

thus, the Fourier cosine series is $\sum_{n=0}^{\infty} A_n\cos(nx) = \boxed{\frac{\pi}{2} - \frac{4}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{\cos(nx)}{n^2}.}$

(e) **[i]** By Proposition 13.1, the unique solution is $u(x,t) := \boxed{2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2t}\sin(nx).}$

[ii] By Proposition 13.2, the unique solution is $u(x,t) := \boxed{\frac{\pi}{2} - \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{-4}{\pi n^2}e^{-n^2t}\cos(nx).}$

(f) **Initial Conditions:** If $t = 0$, then

$$u(x;0) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\underbrace{e^{-n^2 0}}_{=1}\sin(nx) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx) = f(x).$$

Boundary Conditions: Setting $x = 0$, we get

$$u(0;t) \;=\; 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2t}\underbrace{\sin(0)}_{=0} \;=\; 0,$$

for all $t > 0$. Likewise, Setting $x = \pi$, we get

$$u(\pi;t) \;=\; 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2t}\underbrace{\sin(n\pi)}_{=0} \;=\; 0,$$

for all $t > 0$.

Heat Equation:

$$\begin{aligned}\partial_t\, u(x;t) &= \partial_t\left(2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2t}\sin(nx)\right) \;=\; 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\partial_t\, e^{-n^2t}\right)\sin(nx)\\ &= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(-n^2)e^{-n^2t}\sin(nx) \;=\; 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2t}\left(\partial_x^2\sin(nx)\right)\\ &= \partial_x^2\left(2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2t}\sin(nx)\right) \;=\; \partial_x^2\, u(x;t).\end{aligned}$$

(g) By Propositon 13.7, the unique solution is $u(x,t) := \boxed{2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)\cos(nt).}$

Solutions to §12.5

1. (a) $u(x,y) \;=\; \dfrac{4\sinh(5x)\sin(5y)}{\sinh(5\pi)}$.

(b) First let's check the Laplace equation:

$$\begin{aligned}\triangle u(x,y) &= \frac{4}{\sinh(5\pi)}\cdot\triangle\sinh(5x)\sin(5y)\\ &= \frac{4}{\sinh(5\pi)}\cdot\left(\partial_x^2\sinh(5x)\sin(5y)+\partial_y^2\sinh(5x)\sin(5y)\right)\\ &= \frac{4}{\sinh(5\pi)}\cdot\left(25\sinh(5x)\sin(5y)-25\sinh(5x)\sin(5y)\right)\\ &= \frac{4}{\sinh(5\pi)}\cdot\left(0\right) \;=\; 0.\end{aligned}$$

Now boundary conditions:

$$\begin{aligned}u(x,0) &= \frac{4\sinh(5x)\sin(5\cdot 0)}{\sinh(5\pi)} &&= \frac{4\sinh(5x)\cdot 0}{\sinh(5\pi)} &&= 0\\ u(x,\pi) &= \frac{4\sinh(5x)\sin(5\cdot\pi)}{\sinh(5\pi)} &&= \frac{4\sinh(5x)\cdot 0}{\sinh(5\pi)} &&= 0\\ u(0,y) &= \frac{4\sinh(5\cdot 0)\sin(5y)}{\sinh(5\pi)} &&= \frac{4\cdot 0\cdot\sin(5y)}{\sinh(5\pi)} &&= 0.\\ u(\pi,y) &= \frac{4\sinh(5\pi)\sin(5y)}{\sinh(5\pi)} &&= 4\sin(5y) &&= f(y).\end{aligned}$$

2. (a) $u(x,y;t) \;=\; \frac{1}{5}\sin(3x)\sin(4y)\sin(5t)$.

(b)
$$\begin{aligned}u(x,y,0) &= \frac{1}{5}\sin(3x)\sin(4y)\sin(5\cdot 0) &&= \frac{1}{5}\sin(3x)\sin(4y)\cdot 0 \;=\; 0.\\ \partial_t\, u(x,y,t) &= \frac{1}{5}\sin(3x)\sin(4y)\cdot\partial_t\,\sin(5t) &&= \frac{1}{5}\sin(3x)\sin(4y)\cdot 5\cos(5t)\\ &= \sin(3x)\sin(4y)\cos(5t).\end{aligned}$$

Hence,
$$\begin{aligned}\partial_t\, u(x,y,0) &= \sin(3x)\sin(4y)\cos(5\cdot 0) &&= \sin(3x)\sin(4y)\cdot 1\\ &= f_1(x,y), \quad \text{as desired.}\end{aligned}$$

3. (a) $h(x,y) = \boxed{\dfrac{\sinh(2x)\sin(2y)}{\sinh(2\pi)}.}$

(b) $h(x,y) = \boxed{\dfrac{\sinh(4\pi - 4x)\sin(4y)}{\sinh(4\pi)} + \dfrac{\sin(3x)\sinh(3y)}{\sinh(3\pi)}.}$

4. The 2-dimensional Fourier sine series of $q(x,y)$ *is* $\sin(x)\cdot\sin(3y) + 7\sin(4x)\cdot\sin(2y)$. Thus,

$$u(x,y) = \frac{\sin(x)\cdot\sin(3y)}{1+3^2} + \frac{7\sin(4x)\cdot\sin(2y)}{4^2+2^2} = \boxed{\frac{-\sin(x)\cdot\sin(3y)}{10} - \frac{7\sin(4x)\cdot\sin(2y)}{20}}$$

5. The 2-dimensional Fourier cosine series of $f(x,y)$ *is* $\cos(5x)\cdot\cos(y)$. Thus,

$$u(x,y;t) = \cos(5x)\cdot\cos(y)\cdot\exp(-(5^2+1)t) = \boxed{\cos(5x)\cdot\cos(y)\cdot\exp(-26t)}$$

6. (a) For Neumann boundary conditions, we use the two-dimensional Fourier cosine series for f, which is just $\cos(2x)\cos(3y)$. Thus, $u(x,y;t) = \boxed{\cos(2x)\cos(3y)\cdot e^{-13t}.}$

(b) For Dirichlet boundary conditions, we use the two-dimensional Fourier sine series for f. Setting $N = 2$ and $M = 3$ in the solution to problem # 3 on page 188 of §10.3, we obtain:

$$f(x,y) = \frac{16}{\pi^2}\sum_{\substack{n,m=1\\ n\text{ odd}\\ m\text{ even}}}^{\infty} \frac{n\cdot m}{(n^2-4)(m^2-9)}\sin(nx)\sin(my).$$

Thus, the solution is:

$$u(x,y;t) = \boxed{\frac{16}{\pi^2}\sum_{\substack{n,m=1\\ n\text{ odd}\\ m\text{ even}}}^{\infty} \frac{n\cdot m}{(n^2-4)(m^2-9)}\sin(nx)\sin(my)\cos\left(\sqrt{n^2+m^2}\cdot t\right).}$$

(c) For Neumann boundary conditions, we use the two-dimensional Fourier cosine series for f, which is just $\cos(2x)\cos(3y)$. Thus, $u(x,y;t) = \boxed{\dfrac{-\cos(2x)\cos(3y)}{13}.}$

(d) For Dirichlet boundary conditions, we use the two-dimensional Fourier sine series for f from problem #3 of §10.3. We obtain:

$$u(x,y) = \boxed{\frac{-16}{\pi^2}\sum_{\substack{n,m=1\\ \text{both odd}}}^{\infty} \frac{n\cdot m}{(n^2-4)(m^2-9)(n^2+m^2)}\sin(nx)\sin(my).}$$

(e) Let $u(x,y)$ be the solution to the Poisson problem with *homogeneous* Dirichlet boundary conditions. We solved for $u(x,y)$ in problem (6d), obtaining:

$$u(x,y) = \frac{-16}{\pi^2}\sum_{\substack{n,m=1\\ \text{both odd}}}^{\infty} \frac{n\cdot m\cdot\sin(nx)\sin(my)}{(n^2-4)(m^2-9)(n^2+m^2)}.$$

Let $h(x,y)$ be the solution to the *Laplace* equation, with the specified *inhomogeneous* Dirichlet boundary conditions. We solved for $h(x,y)$ in problem (3a), obtaining: $h(x,y) = \dfrac{\sinh(2x)\sin(2y)}{\sinh(2\pi)}$.

We obtain the complete solution by summing:

$$v(x,y) = h(x,y) + u(x,y) = \boxed{\frac{\sinh(2x)\sin(2y)}{\sinh(2\pi)} - \frac{16}{\pi^2}\sum_{\substack{n,m=1\\ \text{both odd}}}^{\infty} \frac{n\cdot m\cdot\sin(nx)\sin(my)}{(n^2-4)(m^2-9)(n^2+m^2)}.}$$

7. (a) For Dirichlet boundary conditions, we use the two-dimensional Fourier sine series for f. Setting $N = 3$ in the solution to problem #4 of §10.3, we get: $\dfrac{2\sinh(3\pi)}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{9+n^2}\sin(3x)\cdot\sin(ny)$.

Thus, $u(x,y;t) = \boxed{\dfrac{2\sinh(3\pi)}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{9+n^2}\sin(3x)\cdot\sin(ny)\exp(-(9+n^2)t).}$

(b) The one-dimensional Fourier sine series of $T(x)$ *is* $\sin(3x)$. Thus, $w(x,y) = \boxed{\dfrac{\sin(3x)\cdot\sinh(3y)}{\sinh(3\pi)}}$.

(c) The equilibrium solution from part (b) is $u(x,y) = \dfrac{\sin(3x)\cdot\sinh(3y)}{\sinh(3\pi)}$. The initial conditions are $h(x,y) = 0$. Define $g(x,y) = h(x,y) - u(x,y) = \dfrac{-\sin(3x)\cdot\sinh(3y)}{\sinh(3\pi)} = \dfrac{-f(x,y)}{\sinh(3\pi)}$, where $f(x,y)$ is as in part (a). From part (a), the Heat Equation with initial conditions $v(x,y;0) = g(x,y)$ has solution:

$$\begin{aligned} v(x,y;t) &= \frac{-2\sinh(3\pi)}{\pi\sinh(3\pi)}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{9+n^2}\sin(3x)\cdot\sin(ny)\exp(-(9+n^2)t) \\ &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{9+n^2}\sin(3x)\cdot\sin(ny)\exp(-(9+n^2)t) \end{aligned}$$

Putting it together, the solution is:

$$\begin{aligned} u(x,y;t) &= w(x,y) + v(x,y;t) \\ &= \boxed{\frac{\sin(3x)\cdot\sinh(3y)}{\sinh(3\pi)} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{9+n^2}\sin(3x)\cdot\sin(ny)\exp(-(9+n^2)t).} \end{aligned}$$

Solutions to §14.9

1. $$\begin{aligned} \triangle\Phi_n(r,\theta) &= \partial_r^2\,\Phi_n(r,\theta) + \tfrac{1}{r}\partial_r\,\Phi_n(r,\theta) + \tfrac{1}{r^2}\partial_\theta^2\,\Phi_n(r,\theta) \\ &= n(n-1)r^{n-2}\cos(n\theta) + \tfrac{n}{r}r^{n-1}\cos(n\theta) + \tfrac{1}{r^2}(-n^2)r^n\cos(n\theta) \\ &= \left(n(n-1)r^{n-2} + nr^{n-2} + (-n^2)r^{n-2}\right)\cos(n\theta) \\ &= \left((n^2-n)+n-n^2\right)r^{n-2}\cdot\cos(n\theta) = (0)\cdot\cos(n\theta) = 0. \end{aligned}$$

2. $$\begin{aligned} \triangle\Psi_n(r,\theta) &= \partial_r^2\,\Psi_n(r,\theta) + \tfrac{1}{r}\partial_r\,\Psi_n(r,\theta) + \tfrac{1}{r^2}\partial_\theta^2\,\Psi_n(r,\theta) \\ &= n(n-1)r^{n-2}\sin(n\theta) + \tfrac{n}{r}r^{n-1}\sin(n\theta) + \tfrac{1}{r^2}(-n^2)r^n\sin(n\theta) \\ &= \left(n(n-1)r^{n-2} + nr^{n-2} + (-n^2)r^{n-2}\right)\sin(n\theta) \\ &= \left((n^2-n)+n-n^2\right)r^{n-2}\cdot\sin(n\theta) = (0)\cdot\sin(n\theta) = 0. \end{aligned}$$

3. $$\begin{aligned} \triangle\phi_n(r,\theta) &= \partial_r^2\,\phi_n(r,\theta) + \tfrac{1}{r}\partial_r\,\phi_n(r,\theta) + \tfrac{1}{r^2}\partial_\theta^2\,\phi_n(r,\theta) \\ &= n(n+1)r^{-n-2}\cos(n\theta) - \tfrac{n}{r}r^{-n-1}\cos(n\theta) + \tfrac{1}{r^2}(-n^2)r^{-n}\cos(n\theta) \\ &= \left(n(n+1)r^{-n-2} - nr^{-n-2} + (-n^2)r^{-n-2}\right)\cos(n\theta) \\ &= \left((n^2+n)-n-n^2\right)r^{-n-2}\cdot\cos(n\theta) = (0)\cdot\cos(n\theta) = 0. \end{aligned}$$

4. $$\begin{aligned} \triangle\psi_n(r,\theta) &= \partial_r^2\,\psi_n(r,\theta) + \tfrac{1}{r}\partial_r\,\psi_n(r,\theta) + \tfrac{1}{r^2}\partial_\theta^2\,\psi_n(r,\theta) \\ &= n(n+1)r^{-n-2}\sin(n\theta) - \tfrac{n}{r}r^{-n-1}\sin(n\theta) + \tfrac{1}{r^2}(-n^2)r^{-n}\sin(n\theta) \\ &= \left(n(n+1)r^{-n-2} - nr^{-n-2} + (-n^2)r^{-n-2}\right)\sin(n\theta) \\ &= \left((n^2+n)-n-n^2\right)r^{-n-2}\cdot\sin(n\theta) = (0)\cdot\sin(n\theta) = 0. \end{aligned}$$

5. $$\begin{aligned} \triangle\phi_0(r,\theta) &= \partial_r^2\,\phi_0(r,\theta) + \tfrac{1}{r}\partial_r\,\phi_0(r,\theta) + \tfrac{1}{r^2}\partial_\theta^2\,\phi_0(r,\theta) \\ &= \partial_r^2\,\log|r| + \tfrac{1}{r}\partial_r\,\log|r| + \tfrac{1}{r^2}\partial_\theta^2\,\log|r| \\ &= \tfrac{-1}{r^2} + \tfrac{1}{r}\cdot\tfrac{1}{r} + 0 = \tfrac{-1}{r^2} + \tfrac{1}{r^2} = 0. \end{aligned}$$

6. (a) By Proposition 14.2 on page 239, the unique bounded solution is $u(r,\theta) = \boxed{r^3\cdot\cos(3\theta) + 2r^5\cdot\sin(5\theta).}$

   (b) By Proposition 14.6 on page 244, the unique bounded solution is $u(r,\theta) = \boxed{\dfrac{\cos(3\theta)}{r^3} + \dfrac{2\sin(5\theta)}{r^5}.}$

   (c) By Proposition 14.8 on page 246, the solutions all have the form: $u(r,\theta) = \boxed{C - \dfrac{\cos(3\theta)}{3\cdot r^3} - \dfrac{2\sin(5\theta)}{5\cdot r^5},}$ where C is a constant.

7. (a) By Proposition 14.2 on page 239, the unique bounded solution is $u(r,\theta) = \boxed{2r\cos(\theta) - 6r^2\sin(2\theta).}$

   (b) By Proposition 14.4 on page 241, the solutions all have the form $u(r,\theta) = \boxed{C + 2r\cos(\theta) - 3r^2\sin(2\theta),}$ where C is any constant.

8. (a) Proposition 14.2 on page 239, the *unique* bounded solution is $u(r,\theta) = \boxed{4\cdot r^5\cos(5\theta).}$

(b) First the boundary conditions:

$$u(1,\theta) \quad = \quad 4 \cdot (1)^5 \cos(3\theta) \quad = \quad 4\cos(5\theta),$$

as desired. Next the Laplacian:

$$\begin{aligned}
\triangle u(r,\theta) &= \partial_r^2\, u(r,\theta) \;+\; \frac{1}{r}\partial_r\, u(r,\theta) \;+\; \frac{1}{r^2}\partial_\theta^2\, u(r,\theta) \\
&= 4 \cdot \partial_r^2\, r^5 \cos(5\theta) \;+\; \frac{4}{r}\partial_r\, r^5 \cos(5\theta) \;+\; \frac{4}{r^2}\partial_\theta^2\, r^5 \cos(5\theta) \\
&= 4 \cdot 5 \cdot 4 \cdot r^3 \cos(5\theta) \;+\; \frac{4}{r} 5r^4 \cos(5\theta) \;+\; \frac{4}{r^2} r^5 (-25)\cos(5\theta) \\
&= 4 \cdot 5 \cdot 4 \cdot r^3 \cos(5\theta) \;+\; 4 \cdot 5r^3 \cos(5\theta) \;+\; 4 \cdot (-25) r^3 \cos(5\theta) \\
&= (80 + 20 - 100) \cdot r^3 \cos(5\theta) \;=\; (0) \cdot r^3 \cos(5\theta) \;=\; 0, \qquad \text{as desired.}
\end{aligned}$$

9. (a) By Proposition 14.8 on page 246, the 'decaying gradient' solutions all have the form $u(r,\theta) = \boxed{C \;+\; 5\log|r| \;-\; \frac{4\sin(3\theta)}{3r^3},}$ where C is any constant. Thus, the solution is *not* unique.

(b) $\partial_r\; u(r,\theta) \quad = \quad \partial_r\, 5\log|r| \;-\; \partial_r \frac{4\sin(3\theta)}{3r^3} \quad = \quad \frac{5}{r} \;+\; \frac{4\sin(3\theta)}{r^4}.$

Thus, $\partial_r\, u(1,\theta) \quad = \quad \frac{5}{1} \;+\; \frac{4\sin(3\theta)}{1^4} \quad = \quad 5 + 4\sin(3\theta)$, as desired.

10. (a)
$$\begin{aligned}
u(r,\theta) &= \sum_{n=1}^{\infty} \frac{A_n}{n} r^n \cos(n\theta) \;+\; \sum_{n=1}^{\infty} \frac{B_n}{n} r^n \sin(n\theta) \;+\; C \\
&= \boxed{\frac{2r^5\cos(5\theta)}{5} \;+\; \frac{r^3\sin(3\theta)}{3} \;+\; C,}
\end{aligned}$$

where C is any constant. The solution is thus not unique.

(b) $u(r,\theta) \quad = \quad \sum_{n=1}^{\infty} A_n \frac{cos(n\theta)}{r^n} \;+\; \sum_{n=1}^{\infty} B_n \frac{sin(n\theta)}{r^n} \quad = \quad \boxed{2\frac{\cos(5\theta)}{r^5} \;+\; \frac{\sin(3\theta)}{r^3}.}$

11.
$$\begin{aligned}
\triangle\Phi_{n,\lambda} &= \partial_r^2\, \Phi_{n,\lambda}(r,\theta) \;+\; \tfrac{1}{r}\partial_r\, \Phi_{n,\lambda}(r,\theta) \;+\; \tfrac{1}{r^2}\partial_\theta^2\, \Phi_{n,\lambda}(r,\theta) \\
&= \partial_r^2 \Big(\mathcal{J}_n(\lambda\cdot r)\cdot\cos(n\theta)\Big) \;+\; \tfrac{1}{r}\partial_r \Big(\mathcal{J}_n(\lambda\cdot r)\cdot\cos(n\theta)\Big) \;+\; \tfrac{1}{r^2}\partial_\theta^2 \Big(\mathcal{J}_n(\lambda\cdot r)\cdot\cos(n\theta)\Big) \\
&= \cos(n\theta)\cdot\partial_r^2\, \mathcal{J}_n(\lambda\cdot r) \;+\; \tfrac{1}{r}\cos(n\theta)\cdot\partial_r\, \mathcal{J}_n(\lambda\cdot r) \;+\; \tfrac{1}{r^2}\mathcal{J}_n(\lambda\cdot r)\cdot\partial_\theta^2\, \cos(n\theta) \\
&= \cos(n\theta)\cdot\partial_r^2\, \mathcal{J}_n(\lambda\cdot r) \;+\; \tfrac{1}{r}\cos(n\theta)\cdot\partial_r\, \mathcal{J}_n(\lambda\cdot r) \;+\; \tfrac{1}{r^2}\mathcal{J}_n(\lambda\cdot r)\cdot(-n^2)\cos(n\theta) \\
&= \frac{\cos(n\theta)}{r^2}\cdot\Big(r^2\cdot\partial_r^2\, \mathcal{J}_n(\lambda\cdot r) \;+\; r\cdot\partial_r\, \mathcal{J}_n(\lambda\cdot r) \;-\; n^2\cdot\mathcal{J}_n(\lambda\cdot r)\Big) \\
&\overline{\overline{\text{(C)}}}\; \frac{\cos(n\theta)}{r^2}\cdot\Big(\lambda^2 r^2\cdot\mathcal{J}_n''(\lambda\cdot r) \;+\; \lambda r\cdot\mathcal{J}_n'(\lambda\cdot r) \;-\; n^2\cdot\mathcal{J}_n(\lambda\cdot r)\Big) \qquad (*)
\end{aligned}$$

where **(C)** is the Chain Rule. Now, recall that $\mathcal{J}_n$ is a solution of Bessel's equation:

$$x^2\mathcal{J}''(x) \;+\; x\mathcal{J}'(x) \;+\; (x^2 - n^2)\cdot\mathcal{J}(x) \quad = \quad 0$$

Hence, substituting $x = \lambda r$, we get:

$$(\lambda r)^2\mathcal{J}''(\lambda r) \;+\; \lambda r\cdot\mathcal{J}'(\lambda r) \;+\; (\lambda r)^2\cdot\mathcal{J}(\lambda r) - n^2\cdot\mathcal{J}(\lambda r) \quad = \quad 0,$$

or, equivalently

$$\lambda^2 r^2\cdot\mathcal{J}''(\lambda r) \;+\; \lambda r\cdot\mathcal{J}'(\lambda r) \;-\; n^2\cdot\mathcal{J}(\lambda r) \quad = \quad -\lambda^2 r^2\cdot\mathcal{J}(\lambda r) \qquad (18.9)$$

Substituting (18.9) into $(*)$ yields

$$\triangle\Phi_{n,\lambda}(r,\theta) \quad = \quad \frac{\cos(n\theta)}{r^2}\Big(-\lambda^2 r^2\cdot\mathcal{J}_n(\lambda\cdot r)\Big) \quad = \quad -\lambda^2\cdot\cos(n\theta)\cdot\mathcal{J}_n(\lambda\cdot r) \quad = \quad -\lambda^2\cdot\Phi_{n,\lambda}(r,\theta),$$

as desired.

12. Similar to #12.

13. Similar to #12.

14. Similar to #12.

Solutions to §16.8

1. Fix $x \in \mathbb{R}$. Then

$$\begin{aligned} f*(g*h)(x) &= \int_{-\infty}^{\infty} f(y)(g*h)(x-y)\ dy \;=\; \int_{-\infty}^{\infty} f(y)\left(\int_{-\infty}^{\infty} g(z)h\left[(x-y)-z\right]\ dz\right)\ dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)g(z)h\left[x-(y+z)\right]\ dz\ dy \;\underset{(*)}{=}\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)g(w-y)h(x-w)\ dw\ dy \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(y)g(w-y)\ dy\right)h(x-w)\ dw \;=\; \int_{-\infty}^{\infty}(f*g)(w)\cdot h(x-w)\ dw \\ &= (f*g)*h(x) \end{aligned}$$

Here, $(*)$ is the change of variables $z := w - y$; hence $w = z + y$ and $dw = dz$.

2. Fix $x \in \mathbb{R}$. Then

$$\begin{aligned} f*(g+h)(x) &= \int_{-\infty}^{\infty} f(y)\cdot(rg+h)(x-y)\ dy \\ &= \int_{-\infty}^{\infty} f(y)\cdot\Big(rg(x-y)+h(x-y)\Big)\ dy \\ &= \int_{-\infty}^{\infty} f(y)\cdot rg(x-y)+f(y)\cdot h(x-y)\ dy \\ &= r\int_{-\infty}^{\infty} f(y)\cdot g(x-y)\ dy \;+\; \int_{-\infty}^{\infty} f(y)\cdot h(x-y)\ dy \\ &= rf*g(x)+f*h(x). \end{aligned}$$

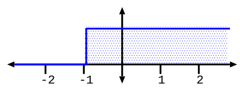
3.

$$\begin{aligned} (f_{\triangleright d}*g)(x) &= \int_{-\infty}^{\infty} f_{\triangleright d}(x)\cdot g(x-y)\ dy \;=\; \int_{-\infty}^{\infty} f(x-d)\cdot g(x-y)\ dy \\ &\underset{(*)}{=} \int_{-\infty}^{\infty} f(x-d)\cdot g(x-d-z)\ dz \;=\; f*g(x-d). \end{aligned}$$

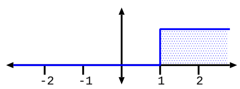
Here, $(*)$ is the change of variables $z = y - d$, so that $dz = dy$.

4. (a) Let $\mathcal{H}(x)$ be the Heaviside step function. As in Prop.45(c) (p.77), define:

$$\mathcal{H}_{\triangleleft(-1)}(x) \;=\; \mathcal{H}(x+1) \;=\; \begin{cases} 0 & \text{if} \quad x < -1 \\ 1 & \text{if} \quad -1 \leq x \end{cases}$$



$$\mathcal{H}_{\triangleleft 1}(x) \;=\; \mathcal{H}(x-1) \;=\; \begin{cases} 0 & \text{if} \quad x < 1 \\ 1 & \text{if} \quad 1 \leq x \end{cases}$$



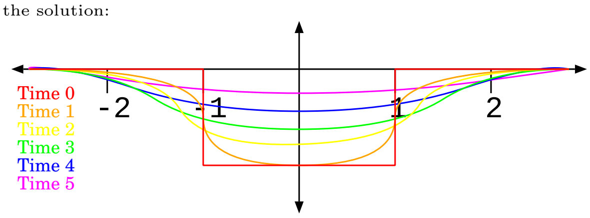
Then $\mathcal{I} = \mathcal{H}_{\triangleleft 1} - \mathcal{H}_{\triangleleft(-1)}$.

Thus,

$$\begin{aligned} u(x;t) &\underset{\text{(P16.11)}}{=} \mathcal{I}*\mathcal{G}_t(x) \;=\; \left(\mathcal{H}_{\triangleleft 1} - \mathcal{H}_{\triangleleft(-1)}\right)*\mathcal{G}_t(x) \\ &\underset{\text{(16.13a,b)}}{=} \mathcal{H}_{\triangleleft 1}*\mathcal{G}_t(x) - \mathcal{H}_{\triangleleft(-1)}*\mathcal{G}_t(x) \underset{\text{(16.13c)}}{=} \mathcal{H}*\mathcal{G}_t(x-1) - \mathcal{H}*\mathcal{G}_t(x+1) \\ &\underset{\text{(X16.12)}}{=} \boxed{\Phi\left(\frac{x-1}{\sqrt{2t}}\right) - \Phi\left(\frac{x+1}{\sqrt{2t}}\right).} \end{aligned}$$

Here, $\mathcal{G}_t(x) = \dfrac{1}{2\sqrt{\pi t}}\exp\left(\dfrac{-x^2}{4t}\right)$ is the Gauss-Weierstrass Kernel, and $\Phi(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{x}\exp\left(\dfrac{-x^2}{2}\right)\ dx$ is the sigmoid function. **(P16.11)** is by Corollary 16.11 on page 310; **(16.13a,b)** is by Prop. 16.13(a) and Prop. 16.13(b) (p. 312); **(16.13c)** is by Prop. 16.13(c); **(X16.12)** is by Example 16.12 on page 311.

Here is a sketch of the solution:



(b) Recall the Heaviside step function $h_0(x) = \begin{cases} 1 & \text{if} \quad 0 \le x \\ 0 & \text{if} \quad x < 0 \end{cases}$. Observe that $u(x) = h_0(x) - h_1(x)$, where $h_1(x) = h(x-1)$.

Let $u_0(x;t)$ be the solution to the Heat equation with initial conditions $u_0(x;0) = h_0(x)$. By Example 35 on p. 67 of the notes, we know that $u_0(x;t) = \Phi\left(\frac{x}{\sqrt{2t}}\right)$, where Φ is the sigmoid function.

Let $u_1(x;t)$ be the solution to the Heat equation with initial conditions $u_1(x;0) = h_1(x) = h_0(x-1)$. By simply translating Example 35, we have $u_1(x;t) = \Phi\left(\frac{x-1}{\sqrt{2t}}\right)$.

Let $u(x;t) = u_0(x;t) - u_1(x;t)$. By linearity, u is also a solution to the Heat Equation, with initial conditions $u(x;0) = h_0(x) - h_1(x) = \mathcal{I}(x)$. Thus, u is the solution to our initial value problem, and

$$u(x,t) = u_0(x;t) - u_1(x;t) = \boxed{\Phi\left(\frac{x}{\sqrt{2t}}\right) - \Phi\left(\frac{x-1}{\sqrt{2t}}\right)}$$

(c) Observe that $\mathcal{I}(x) = -h_{(-1)} + 2h_0(x) - h_1(x)$, where $h_{(-1)}(x) = h(x+1)$ and $h_1(x) = h(x-1)$.

Let $u_0(x;t)$ be the solution to the Heat equation with initial conditions $u_0(x;0) = h_0(x)$. As in question 1(a), we have $u_0(x;t) = \Phi\left(\frac{x}{\sqrt{2t}}\right)$.

Let $u_1(x;t)$ be the solution to the Heat equation with initial conditions $u_1(x;0) = h_1(x)$. As in question 1(a), we have $u_1(x;t) = \Phi\left(\frac{x-1}{\sqrt{2t}}\right)$.

Let $u_{(-1)}(x;t)$ be the solution to the Heat equation with initial conditions $u_{(-1)}(x;0) = h_{(-1)}(x)$. By similar reasoning, $u_{(-1)}(x;t) = \Phi\left(\frac{x+1}{\sqrt{2t}}\right)$.

Let $u(x;t) = -u_{(-1)} + 2u_0(x;t) - u_1(x;t)$. By linearity, u is also a solution to the Heat Equation, with initial conditions $u(x;0) = h_{(-1)} + 2h_0(x) - h_1(x) = f(x)$. Thus, u is the solution to our initial value problem, and

$$u(x,t) = -u_{(-1)} + 2u_0(x;t) - u_1(x;t) = \boxed{\Phi\left(\frac{x+1}{\sqrt{2t}}\right) + 2\Phi\left(\frac{x}{\sqrt{2t}}\right) - \Phi\left(\frac{x-1}{\sqrt{2t}}\right)}$$

5. (a) Applying the Chain Rule, we get:

$$\begin{aligned} \partial_t v(x;t) &= \tfrac{1}{2}\Big(f'(x+t) - f'(x-t)\Big). \\ \text{Thus,} \quad \partial_t^2 v(x;t) &= \tfrac{1}{2}\Big(f''(x+t) + f''(x-t)\Big). \\ \text{Likewise,} \quad \partial_x v(x;t) &= \tfrac{1}{2}\Big(f'(x+t) + f'(x-t)\Big). \\ \text{Thus,} \quad \partial_x^2 v(x;t) &= \tfrac{1}{2}\Big(f''(x+t) + f''(x-t)\Big). \end{aligned}$$

We conclude that $\partial_t^2 v(x;t) = \frac{1}{2}\Big(f''(x+t) + f''(x-t)\Big) = \partial_x^2 v(x;t)$.

(b) $v(x;0) = \frac{1}{2}\Big(f(x+0) + f(x-0)\Big) = \frac{1}{2}\Big(f(x) + f(x)\Big) = \frac{2}{2}f(x) = \boxed{f(x).}$

(c) From part (a) we know: $\partial_t v(x;t) = \frac{1}{2}\Big(f'(x+t) - f'(x-t)\Big)$. Thus, $\partial_t v(x;0) = \frac{1}{2}\Big(f'(x+0) - f'(x-0)\Big) = \frac{1}{2}\Big(f'(x) - f'(x)\Big) = \frac{1}{2}0 = \boxed{0.}$

6. (a) Let $F(x) = \int_0^x f_1(y)\,dy$ be an antiderivative of f_1. Then: $v(x,t) = \frac{1}{2}\Big(F(x+t) - F(x-t)\Big)$.

$$\begin{aligned} \text{Thus,} \ \partial_t v(x,t) &= \frac{1}{2}\Big(\partial_t F(x+t) - \partial_t F(x-t)\Big) \overline{\overline{\text{(CR)}}} \frac{1}{2}\Big(F'(x+t) + F'(x-t)\Big) \\ &\overline{\overline{\text{(FTC)}}} \frac{1}{2}\Big(f_1(x+t) + f_1(x-t)\Big). \qquad (18.10) \\ \text{Thus,} \quad \partial_t^2 v(x,t) &= \frac{1}{2}\Big(\partial_t f_1(x+t) + \partial_t f_1(x-t)\Big) \\ &\overline{\overline{\text{(CR)}}} \frac{1}{2}\Big(f_1'(x+t) - f_1'(x-t)\Big). \qquad (18.11) \end{aligned}$$

(CR) is the *Chain Rule*, and **(FTC)** is the *Fundamental Theorem of Calculus*.
Likewise

$$\begin{aligned} \partial_x v(x,t) &= \frac{1}{2}\Big(\partial_x F(x+t) - \partial_x F(x-t)\Big) \overline{\overline{\text{(CR)}}} \frac{1}{2}\Big(F'(x+t) - F'(x-t)\Big) \\ &\overline{\overline{\text{(FTC)}}} \frac{1}{2}\Big(f_1(x+t) - f_1(x-t)\Big). \\ \text{Thus,} \quad \partial_x^2 v(x,t) &= \frac{1}{2}\Big(\partial_x f_1(x+t) - \partial_x f_1(x-t)\Big) \\ &\overline{\overline{\text{(CR)}}} \frac{1}{2}\Big(f_1'(x+t) - f_1'(x-t)\Big). \qquad (18.12) \end{aligned}$$

(CR) is the *Chain Rule*, and **(FTC)** is the *Fundamental Theorem of Calculus*.
Comparing equations (18.11) and (18.12), we see that $\partial_t^2 v(x,t) = \partial_x^2 v(x,t)$, so v satisfies the Wave Equation.

(b) Also, setting $t = 0$ in equation (18.10), we get

$$\partial_t\, v(x,0) \quad = \quad \frac{1}{2}\Big(f_1(x+0) + f_1(x-0)\Big) \quad = \quad \frac{1}{2}\Big(f_1(x) + f_1(x)\Big) \quad = \quad f_1(x),$$

so $v(x,t) = f_1(x)$

(c) Setting $t = 0$ in the definition of $v(x,t)$, we get:

$$v(x,0) \quad = \quad \frac{1}{2}\int_{x-0}^{x+0} f_1(y)\ dy \quad = \quad \frac{1}{2}\int_{x}^{x} f_1(y)\ dy \quad = \quad 0.$$

(because an integral over a point is zero). Thus, $v(x,t)$ has the desired initial position.

7. (a) We apply the d'Alembert Traveling Wave solution (Lemma 16.21 on page 321). Define

$$w_L(x,t) \quad = \quad f_0(x+t) \quad = \quad \begin{cases} 1 & \text{if} \quad -t \le x < 1-t \\ 0 & \text{otherwise} \end{cases}$$

$$w_R(x,t) \quad = \quad f_0(x-t) \quad = \quad \begin{cases} 1 & \text{if} \quad t \le x < 1+t \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and} \quad w(x,t) \quad = \quad \frac{1}{2}\Big(w_L(x,t) \ + \ w_R(x,t)\Big) \quad = \quad \boxed{\begin{cases} 0 & \text{if} \quad x < -t \\ \frac{1}{2} & \text{if} \quad -t \le x < t \\ 1 & \text{if} \quad t \le x < 1-t \quad \text{(only possible if } t < \frac{1}{2}) \\ \frac{1}{2} & \text{if} \quad t = x = 1-t \quad \text{(only possible if } t = \frac{1}{2}) \\ 0 & \text{if} \quad 1-t \le x < t \quad \text{(only possible if } t > \frac{1}{2}) \\ \frac{1}{2} & \text{if} \quad 1-t \le x < 1+t \\ 0 & \text{if} \quad t \le x \end{cases}}$$

This solution does *not* satisfy homogeneous Dirichlet BC on $[0,\pi]$.

(b) We apply the d'Alembert Traveling Wave solution (Lemma 16.21 on page 321).

$$\begin{aligned} w_L(x,t) &= f_0(x+t) = \sin(3x+3t) = \sin(3x)\cos(3t) + \cos(3x)\sin(3t) \\ w_R(x,t) &= f_0(x-t) = \sin(3x-3t) = \sin(3x)\cos(3t) - \cos(3x)\sin(3t) \\ \text{and} \quad w(x,t) &= \frac{1}{2}\Big(w_L(x,t) \ + \ w_R(x,t)\Big) = \frac{1}{2}\Big(2\cdot\sin(3x)\cos(3t)\Big) = \boxed{\sin(3x)\cos(3t).} \end{aligned}$$

This solution satisfies homogeneous Dirichlet BC on $[0,\pi]$.

(c) We apply the d'Alembert Ripple Solution (Lemma 16.23 on page 323). Let

$$\begin{aligned} v(x;t) &= \frac{1}{2}\int_{x-t}^{x+t} f_1(y)\ dy = \frac{1}{2}\int_{x-t}^{x+t}\sin(5y)\ dy = \frac{-1}{10}\cos(5y)\Big|_{y=x-t}^{y=x+t} \\ &= \frac{-1}{10}\Big(\cos(5x+5t) - \cos(5x-5t)\Big) = \frac{1}{10}\Big(\cos(5x-5t) - \cos(5x+5t)\Big) \\ &= \frac{1}{10}\Big[\Big(\cos(5x)\cos(5t) + \sin(5x)\sin(5t)\Big) - \Big(\cos(5x)\cos(5t) - \sin(5x)\sin(5t)\Big)\Big] \\ &= \boxed{\frac{1}{5}\sin(5x)\sin(5t).} \end{aligned}$$

This solution satisfies homogeneous Dirichlet BC on $[0,\pi]$.

(d) We apply the d'Alembert Traveling Wave solution (Lemma 16.21 on page 321).

$$\begin{aligned} w_L(x,t) &= f_0(x+t) = \cos(2x+2t) = \cos(2x)\cos(2t) - \sin(2x)\sin(2t) \\ w_R(x,t) &= f_0(x-t) = \cos(2x-2t) = \cos(2x)\cos(2t) + \sin(2x)\sin(2t) \\ \text{and} \quad w(x,t) &= \frac{1}{2}\Big(w_L(x,t) \ + \ w_R(x,t)\Big) = \frac{1}{2}\Big(2\cdot\cos(2x)\cos(2t)\Big) = \boxed{\cos(2x)\cos(2t).} \end{aligned}$$

This solution does *not* satisfy homogeneous Dirichlet BC on $[0,\pi]$.

(e) We apply the d'Alembert Ripple Solution (Lemma 16.23 on page 323). Let

$$\begin{aligned} v(x;t) &= \frac{1}{2}\int_{x-t}^{x+t} f_1(y)\ dy = \frac{1}{2}\int_{x-t}^{x+t}\cos(4y)\ dy = \frac{1}{8}\sin(4y)\Big|_{y=x-t}^{y=x+t} \\ &= \frac{1}{8}\Big(\sin(4x+4t) - \sin(4x-4t)\Big) \\ &= \frac{1}{8}\Big[\Big(\sin(4x)\cos(4t) + \cos(4x)\sin(4t)\Big) - \Big(\sin(4x)\cos(4t) - \cos(4x)\sin(4t)\Big)\Big] \\ &= \boxed{\frac{1}{4}\cos(4x)\sin(4t).} \end{aligned}$$

This solution does *not* satisfy homogeneous Dirichlet BC on $[0,\pi]$.

(f) We apply the d'Alembert Traveling Wave solution (Lemma 16.21 on page 321).

$$\begin{aligned} w_L(x,t) &= f_0(x+t) = \sqrt[3]{x+t} \\ w_R(x,t) &= f_0(x-t) = \sqrt[3]{x-t} \\ \text{and} \quad w(x,t) &= \frac{1}{2}\Big(w_L(x,t) + w_R(x,t)\Big) = \boxed{\frac{1}{2}\left(\sqrt[3]{x+t} + \sqrt[3]{x-t}\right)} \end{aligned}$$

This solution does *not* satisfy homogeneous Dirichlet BC on $[0, \pi]$.

(g) We apply the d'Alembert Ripple Solution (Lemma 16.23 on page 323). Let

$$\begin{aligned} v(x;t) &= \frac{1}{2}\int_{x-t}^{x+t} f_1(y)\ dy = \frac{1}{2}\int_{x-t}^{x+t} y^{1/3}\ dy = \frac{3}{8}y^{4/3}\Big|_{y=x-t}^{y=x+t} \\ &= \boxed{\frac{3}{8}\Big((x+t)^{4/3} - (x-t)^{4/3}\Big).} \end{aligned}$$

This solution does *not* satisfy homogeneous Dirichlet BC on $[0, \pi]$.

(h) We apply the d'Alembert Ripple Solution (Lemma 16.23 on page 323).

$$\begin{aligned} w(x;t) &\overline{\overline{\text{(L54)}}} \frac{1}{2}\int_{x-t}^{x+t} f_1(x)\ dx \ \overline{\overline{(\#2)}}\ \frac{1}{2}\int_{x-t}^{x+t} \frac{\sinh(x)}{\cosh(x)}\ dx = \frac{1}{2}\log\Big(\cosh(y)\Big)\Big|_{y=x-t}^{y=x+t} \\ &= \boxed{\frac{1}{2}\Big[\log\Big(\cosh(x+t)\Big) - \log\Big(\cosh(x-t)\Big)\Big].} \end{aligned}$$

8. (b) Fix $s > 0$, and let $f(x;t) = \mathcal{G}_{s+t}(x)$ for all $t > 0$ and all $x \in \mathbb{R}$. Then $f(x;t)$ is a solution to the Heat Equation. Also, $f(x;t)$ has initial conditions: $f(x;0) = \mathcal{G}_{s+0}(x) = \mathcal{G}_s(x)$. In other words, $f(x;t)$ is the solution to the *Initial Value Problem* for the Heat Equation, with initial conditions $\mathcal{I}(x) = \mathcal{G}_s(x)$. Thus, by Corollary 16.11 on page 310, we know that, for all $t > 0$, $f(x;t) = \mathcal{I} * \mathcal{G}_t = \mathcal{G}_s * \mathcal{G}_t$. In other words, $\mathcal{G}_{s+t} = \mathcal{G}_s * \mathcal{G}_t$.

9. By Theorem 16.15 on page 315, we know that $u_t = h * \mathcal{G}_t$ is the solution to the 2-dimensional Heat Equation ($\partial_t u = \triangle u$), with initial conditions given by $u_0(x,y) = h(x,y)$, for all $(x,y) \in \mathbb{R}^2$.

   But h is harmonic —ie. $\triangle h = 0$. Hence h is an *equilibrium* of the Heat Equation, so $u_t(x,y) = h(x,y)$ for all $t \geq 0$.

10. [Omitted]

11. [Omitted]

12. (a) Define $U : \mathbb{D} \longrightarrow \mathbb{R}$ by $U(\mathbf{x}) = 1$ for all $\mathbf{x} \in \mathbb{D}$. Then U is clearly harmonic (because it is constant), and satisfies the desired boundary conditions. Thus, U is *a* solution to the Laplace equation with the required boundary conditions. However, we know that the solution to this problem is unique; hence U is *the* solution to the problem. Hence, if $u : \mathbb{D} \longrightarrow \mathbb{R}$ is any solution, then we must have $u \equiv U$ —ie. $u(\mathbf{x}) = 1$ for all $\mathbf{x} \in \mathbb{D}$.

    (b) Let $u(\mathbf{x})$ be as in part (a). Then for any $\mathbf{x} \in \mathbb{D}$,

$$1 \ \overline{\overline{(\dagger)}}\ u(\mathbf{x}) \ \overline{\overline{(*)}}\ \frac{1}{2\pi}\int_{\mathbb{S}} b(\mathbf{s})\mathcal{P}(\mathbf{x},\mathbf{s})\ d\mathbf{s} \ \overline{\overline{(\ddagger)}}\ \frac{1}{2\pi}\int_{\mathbb{S}} 1\mathcal{P}(\mathbf{x},\mathbf{s})\ d\mathbf{s} = \frac{1}{2\pi}\int_{\mathbb{S}} \mathcal{P}(\mathbf{x},\mathbf{s})\ d\mathbf{s}.$$

    here, (†) is because $u(\mathbf{x}) = 1$ for all $\mathbf{x} \in \mathbb{D}$; $(*)$ is by the Poisson Integral Formula for the disk; (‡) is because $b(\mathbf{s}) = 1$ for all $\mathbf{s} \in \mathbb{S}$,

    (c) We know, from the Poisson Integral Formula for the disk (Proposition 16.29 on page 331), that, for any $\mathbf{x} \in \mathbb{D}$,

$$u(\mathbf{x}) = \frac{1}{2\pi}\int_{\mathbb{S}} b(\mathbf{s})\mathcal{P}(\mathbf{x},\mathbf{s})\ d\mathbf{s}$$

    However, for all $\mathbf{s} \in \mathbb{S}$, we have $m \leq b(\mathbf{s}) \leq M$. Thus,

$$\begin{aligned} m &\ \overline{\overline{(*)}}\ \frac{m}{2\pi}\int_{\mathbb{S}} m\mathcal{P}(\mathbf{x},\mathbf{s})\ d\mathbf{s} = \frac{1}{2\pi}\int_{\mathbb{S}} m\mathcal{P}(\mathbf{x},\mathbf{s})\ d\mathbf{s} \leq \underbrace{\frac{1}{2\pi}\int_{\mathbb{S}} b(\mathbf{s})\mathcal{P}(\mathbf{x},\mathbf{s})\ d\mathbf{s}}_{u(\mathbf{x})} \\ &\leq \frac{1}{2\pi}\int_{\mathbb{S}} M\mathcal{P}(\mathbf{x},\mathbf{s})\ d\mathbf{s} = \frac{M}{2\pi}\int_{\mathbb{S}} m\mathcal{P}(\mathbf{x},\mathbf{s})\ d\mathbf{s} \ \overline{\overline{(*)}}\ M. \end{aligned}$$

    where $(*)$ is by part (b).

13. [Omitted]

Solutions to §17.6

1. $$\widehat{f}(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\exp(-\mu\cdot x\cdot \mathbf{i})\,dx = \frac{1}{2\pi}\int_0^1 \exp(-\mu\cdot x\cdot \mathbf{i})\,dx$$
$$= \frac{1}{-2\pi\mu\mathbf{i}}\exp\Big(-\mu\cdot x\cdot\mathbf{i}\Big)_{x=0}^{x=1} = \frac{1}{-2\pi\mu\mathbf{i}}\Big(e^{-\mu\mathbf{i}}-1\Big) = \boxed{\frac{1-e^{-\mu\mathbf{i}}}{2\pi\mu\mathbf{i}}}.$$

2. (a) $g(x) = f(x+\tau)$, where $f(x)$ is as in Example 17.5 on page 339. We know that $\widehat{f}(\mu) = \dfrac{1-e^{-\mu\mathbf{i}}}{2\pi\mu\mathbf{i}}$; thus, it follows from Theorem 17.12 on page 342 that $\widehat{g}(\mu) = e^{\tau\mu\mathbf{i}}\cdot\dfrac{1-e^{-\mu\mathbf{i}}}{2\pi\mu\mathbf{i}} = \boxed{\dfrac{e^{\tau\mu\mathbf{i}}-e^{(\tau-1)\mu\mathbf{i}}}{2\pi\mu\mathbf{i}}}$.

   (b) $g(x) = f(x/\sigma)$, where $f(x)$ is as in Example 17.5 on page 339. We know that $\widehat{f}(\mu) = \dfrac{1-e^{-\mu\mathbf{i}}}{2\pi\mu\mathbf{i}}$; thus, it follows from Theorem 17.14 on page 343 that $\widehat{g}(\mu) = \sigma\cdot\dfrac{1-e^{-\sigma\mu\mathbf{i}}}{2\pi\sigma\mu\mathbf{i}} = \boxed{\dfrac{1-e^{-\sigma\mu\mathbf{i}}}{2\pi\mu\mathbf{i}}}$.

3. $$\widehat{f}(\mu,\nu) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\cdot\exp\Big(-(\mu x+\nu y)\cdot\mathbf{i}\Big)\,dx\,dy$$
$$= \frac{1}{4\pi^2}\int_0^X\int_0^Y \exp(-\mu x\mathbf{i})\cdot\exp(-\nu y\mathbf{i})\,dx\,dy$$
$$= \frac{1}{4\pi^2}\left(\int_0^X \exp(-\mu x\mathbf{i})\,dx\right)\cdot\left(\int_0^Y \exp(-\nu y\mathbf{i})dy\right)$$
$$= \frac{1}{4\pi^2}\frac{-1}{\mu\mathbf{i}}\exp\Big(-\mu x\mathbf{i}\Big)_{x=0}^{x=X}\cdot\frac{-1}{\nu\mathbf{i}}\exp\Big(-\nu y\mathbf{i}\Big)_{y=0}^{y=Y}$$
$$= \frac{-1}{4\pi^2\mu\nu}\Big(e^{-\mu X\mathbf{i}}-1\Big)\cdot\Big(e^{-\nu Y\mathbf{i}}-1\Big)$$
$$= \boxed{\frac{\Big(1-e^{-\mu X\mathbf{i}}\Big)\cdot\Big(e^{-\nu Y\mathbf{i}}-1\Big)}{4\pi^2\mu\nu}}.$$
Thus, the Fourier inversion formula says, that, if $0 < x < X$ and $0 < y < Y$, then
$$\lim_{R\to\infty}\int_{\mathbb{D}(R)}\frac{\Big(1-e^{-\mu X\mathbf{i}}\Big)\cdot\Big(e^{-\nu Y\mathbf{i}}-1\Big)}{4\pi^2\mu\nu}\exp\Big((\mu x+\nu y)\cdot\mathbf{i}\Big)\,d\mu\,d\nu = 1$$
while, if $(x,y)\notin[0,X]\times[0,Y]$, then
$$\lim_{R\to\infty}\int_{\mathbb{D}(R)}\frac{\Big(1-e^{-\mu X\mathbf{i}}\Big)\cdot\Big(e^{-\nu Y\mathbf{i}}-1\Big)}{4\pi^2\mu\nu}\exp\Big((\mu x+\nu y)\cdot\mathbf{i}\Big)\,d\mu\,d\nu = 0.$$
At points on the *boundary* of the box $[0,X]\times[0,Y]$, however, the Fourier inversion integral will converge to neither of these values.

4. For any $\mu\in\mathbb{R}$,
$$\widehat{f}(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\cdot\exp(-\mu\mathbf{i}x)\,dx = \frac{1}{2\pi}\int_0^1 x\cdot\exp(-\mu\mathbf{i}x)\,dx$$
$$= \frac{-1}{2\mu\mathbf{i}\pi}\left(x\cdot\exp(-\mu\mathbf{i}x)\Big|_{x=0}^{x=1} - \int_0^1\exp(-\mu\mathbf{i}x)\,dx\right) = \frac{-1}{2\mu\mathbf{i}\pi}\left(\exp(-\mu\mathbf{i}) + \frac{1}{\mu\mathbf{i}}\exp(-\mu\mathbf{i}x)\Big|_{x=0}^{x=1}\right)$$
$$= \frac{-1}{2\mu\mathbf{i}\pi}\left(e^{-\mu\mathbf{i}} + \frac{e^{-\mu\mathbf{i}}-1}{\mu\mathbf{i}}\right) = \frac{1}{2\mu^2\pi}\Big((\mu\mathbf{i})e^{-\mu\mathbf{i}} + e^{-\mu\mathbf{i}}-1\Big) = \boxed{\frac{1}{2\pi\mu^2}\Big((\mu\mathbf{i}+1)\,e^{-\mathbf{i}\mu}-1\Big)}.$$

5. If $f(x) = x\cdot\exp\left(\frac{-x^2}{2}\right)$, then $f(x) = -g'(x)$, where $g(x) = \exp\left(\frac{-x^2}{2}\right)$. Thus, by Proposition 17.16**(a)** (p.343) $\widehat{f}(\mu) = -\mathbf{i}\mu\widehat{g}(\mu)$. However,
$$g(x) = \frac{\sqrt{2\pi}}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right) = \sqrt{2\pi}\cdot G(x)$$
where $G(x) = \dfrac{1}{\sqrt{2\pi}}\exp\left(\dfrac{-x^2}{2}\right)$ is the Gaussian distribution with variance $\sigma = 1$. Applying Proposition 17.17**(b)** (p. 345), we get:
$$\widehat{G}(\mu) = \frac{1}{2\pi}\exp\left(\frac{-\mu^2}{2}\right)$$

thus, $\widehat{g}(\mu) \;=\; \sqrt{2\pi}\cdot\widehat{G}(\mu) \;=\; \frac{\sqrt{2\pi}}{2\pi}\exp\left(\frac{-\mu^2}{2}\right) \;=\; \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-\mu^2}{2}\right).$

Thus, $\widehat{f}(\mu) \;=\; -\mathbf{i}\mu\widehat{g}(\mu) \;=\; \boxed{\frac{-\mu\mathbf{i}}{\sqrt{2\pi}}\exp\left(\frac{-\mu^2}{2}\right)}.$

6. Let $h(x) = \exp(-\alpha|x|)$. Then from Example 17.8 on page 340, we know: $\widehat{h}(\mu) \;=\; \frac{\alpha}{\pi}\frac{1}{\alpha^2+\mu^2} \;=\; \frac{\alpha}{\pi}g(-\mu).$

The function $g(x)$ is *absolutely integrable*. Thus, by the Strong Fourier Inversion Formula (Theorem 17.6 on page 339), we have, for any real number y,

$$\begin{aligned}\exp(-\alpha|y|) &= \int_{-\infty}^{\infty}\widehat{h}(\mu)\exp(\mathbf{i}\mu y)\,d\mu \;=\; \int_{-\infty}^{\infty}\frac{\alpha}{\pi}g(-\mu)\cdot\exp(\mathbf{i}\mu y)\,d\mu \;\underset{(*)}{=\!=}\; \frac{2\alpha}{2\pi}\int_{-\infty}^{\infty}g(\nu)\cdot\exp(-\mathbf{i}\nu y)\,d\nu\\ &= 2\alpha\cdot\widehat{g}(y),\end{aligned}$$

(where $(*)$ is just the substitution $\nu = -\mu$). In other words, $\widehat{g}(y) \;=\; \frac{1}{2\alpha}\exp(-\alpha|y|)$. Set $y = \mu$ to conclude: $\widehat{g}(\mu) \;=\; \boxed{\frac{1}{2\alpha}\exp(-\alpha|\mu|)}.$

7. Let $f(x) \;=\; \frac{1}{y^2+x^2}$. Setting $\alpha = y$ in Example 17.9 on page 341, we obtain: $\widehat{f}(\mu) \;=\; \frac{1}{2y}e^{-y\cdot|\mu|}$. Now observe that $\mathcal{K}_y(x) \;=\; \frac{y}{\pi}\cdot f(x)$. Thus, $\widehat{\mathcal{K}}_y(\mu) \;=\; \frac{y}{\pi}\cdot\widehat{f}(\mu) \;=\; \frac{y}{2\pi y}e^{-y\cdot|\mu|} \;=\; \boxed{\frac{1}{2\pi}e^{-y\cdot|\mu|}}.$

8. If $f(x) = \frac{2x}{(1+x^2)^2}$, then $f(x) = -g'(x)$, where $g(x) = \frac{1}{1+x^2}$. Thus, by Proposition 17.16**(a)** (p.343), $\widehat{f}(\mu) \;=\; -\mathbf{i}\mu\widehat{g}(\mu)$.

Setting $\alpha = 1$ in Example 17.9 on page 341, we have: $\widehat{g}(\mu) \;=\; \frac{-1}{2}\exp(-|\mu|)$. Thus, $\widehat{f}(\mu) \;=\; \boxed{\frac{-\mathbf{i}\mu}{2}\exp(-|\mu|)}.$

9.

$$\begin{aligned}\widehat{f}(\mu) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)\exp(-\mathbf{i}\mu x)\,dx \;=\; \frac{1}{2\pi}\int_{-4}^{5}\exp(-\mathbf{i}\mu x)\,dx \;=\; \frac{-1}{2\pi\mathbf{i}\mu}\exp(-\mathbf{i}\mu x)\Big|_{x=-4}^{x=5}\\ &= \frac{e^{\mathbf{i}\mu 4}-e^{-\mathbf{i}\mu 5}}{2\pi\mathbf{i}\mu} \;=\; \frac{e^{-\frac{1}{2}\mathbf{i}\mu}}{\pi\mu}\cdot\frac{e^{\mathbf{i}\mu\frac{9}{2}}-e^{-\mathbf{i}\mu\frac{9}{2}}}{2\mathbf{i}} \;=\; \boxed{\frac{e^{-\frac{1}{2}\mathbf{i}\mu}}{\pi\mu}\cdot\sin\left(\frac{9}{2}\right)}.\end{aligned}$$

10. Let $g(x) \;=\; \frac{\sin(x)}{x}$. Thus, $f(x) \;=\; g'(x)$, so Theorem 17.16(a) (p.343) says that $\widehat{f}(\mu) \;=\; (\mathbf{i}\mu)\cdot\widehat{g}(\mu)$. Now, let $h(x) \;=\; \begin{cases}1 & \text{if} \quad -1\le x\le 1\\ 0 & \text{otherwise}\end{cases}$. Then Example 17.4 on page 339 says that $\widehat{h}(\mu) \;=\; \frac{\sin(\mu)}{\pi\mu} \;=\; \frac{1}{\pi}g(\mu)$. Thus, the Fourier Inversion Formula (Theorem 17.2 on page 338) says that, for any $\mu\in\mathbb{R}$,

$$\begin{aligned}h(\mu) &= \int_{-\infty}^{\infty}\widehat{h}(\nu)\exp(\mathbf{i}\mu\nu)\,d\nu \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty}g(\nu)\exp(\mathbf{i}\mu\nu)\,d\nu\\ &\underset{(*)}{=\!=} \frac{1}{\pi}\int_{\infty}^{-\infty}g(-x)\exp(-\mathbf{i}\mu x)\,(-dx) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty}g(-x)\exp(-\mathbf{i}\mu x)\,dx\\ &\underset{(\dagger)}{=\!=} \frac{2}{2\pi}\int_{-\infty}^{\infty}g(x)\exp(-\mathbf{i}\mu x)\,dx \;\underset{(\ddagger)}{=\!=}\; 2\widehat{g}(\mu).\end{aligned}$$

$(*)$ is the change of variables $x = -\nu$, so that $dx = -d\nu$. $(\dagger)$ is because $g(x) = g(-x)$. $(\ddagger)$ is just the definition of the Fourier transform.

We conclude that $\widehat{g}(\mu) \;=\; \frac{1}{2}h(\mu)$. Thus,

$$\widehat{f}(\mu) \;=\; (\mathbf{i}\mu)\cdot\widehat{g}(\mu) \;=\; \frac{\mathbf{i}\mu}{2}h(\mu) \;=\; \boxed{\begin{cases}\frac{\mathbf{i}\mu}{2} & \text{if} \quad -1<\mu<1\\ 0 & \text{otherwise}\end{cases}}.$$

11. [omitted]

12.
$$\begin{aligned}
\widehat{h}(\mu) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x)\cdot\exp(-\mathbf{i}\mu x)\ dx \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(y)\cdot g(x-y)\ dy\right]\ \exp(-\mathbf{i}\mu x)\ dx\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)\cdot g(x-y)\cdot\exp\left(-\mathbf{i}\mu x\right)\ dy\ dx\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)\cdot g(x-y)\cdot\exp\left(-\mathbf{i}\mu y\right)\cdot\exp\Big(-\mathbf{i}\mu(x-y)\Big)\ dx\ dy\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\cdot\exp\left(-\mathbf{i}\mu y\right)\cdot\left[\int_{-\infty}^{\infty} g(x-y)\cdot\exp\Big(-\mathbf{i}\mu(x-y)\Big)\ dx\right]\ dy\\
&\underset{\text{(c)}}{=\!=} \frac{2\pi}{2\pi}\int_{-\infty}^{\infty} f(y)\cdot\exp\left(-\mathbf{i}\mu y\right)\cdot\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} g(z)\cdot\exp\left(-\mathbf{i}\mu z\right)\ dz\right]\ dy\\
&= \frac{2\pi}{2\pi}\int_{-\infty}^{\infty} f(y)\cdot\exp\left(-\mathbf{i}\mu y\right)\cdot\widehat{g}(\mu)\ dy \quad = \quad 2\pi\cdot\widehat{g}(\mu)\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\cdot\exp\left(-\mathbf{i}\mu y\right)\ dy\\
&= 2\pi\cdot\widehat{g}(\mu)\cdot\widehat{f}(\mu).
\end{aligned}$$

Here, **(c)** is the change of variables: $z = x - y$, so $dz \ = \ dx$.

13. [omitted]

14. [omitted]

15. [omitted]

16. [omitted]

17. Let $\mathcal{E}_\mu(x) := \exp(-\mathbf{i}\mu x)$. Then $\mathcal{E}'_\mu(x) \ = \ -\mathbf{i}\mu\exp(-\mathbf{i}\mu x) = -\mathbf{i}\mu\mathcal{E}_\mu(x)$. Thus,

$$\begin{aligned}
\widehat{g}(\mu) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f'(x)\mathcal{E}_\mu(x)\ dx \underset{(*)}{=\!=} \frac{1}{2\pi}\underbrace{\lim_{R\to\infty} f(x)\mathcal{E}_\mu(x)\Big|_{x=-R}^{x=R}}_{=0(\dagger)} - \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\mathcal{E}'_\mu(x)dx\\
&= 0 - \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)(-\mathbf{i}\mu)\mathcal{E}_\mu(x)dx \quad = \quad \frac{\mathbf{i}\mu}{2\pi}\int_{-\infty}^{\infty} f(x)\mathcal{E}_\mu(x)dx \quad = \quad \mathbf{i}\mu\cdot\widehat{f}(\mu)
\end{aligned}$$

as desired. Here $(*)$ is integration by parts, and $(\dagger)$ is because $\lim_{x\to\pm\infty} f(x) = 0$.

18. From problem #12 we know that

$$\begin{aligned}
\widehat{\mathcal{G}_t * \mathcal{G}_s}(\mu) &\underset{(\#1)}{=\!=} 2\pi\cdot\widehat{\mathcal{G}}_t(\mu)\cdot\widehat{\mathcal{G}}_s(\mu) \quad = \quad 2\pi\cdot\frac{1}{2\pi}e^{-\mu^2 t}\cdot\frac{1}{2\pi}e^{-\mu^2 s} \quad = \quad \frac{1}{2\pi}e^{-\mu^2 t-\mu^2 s}\\
&= \frac{1}{2\pi}e^{-\mu^2(t+s)} \quad = \quad \widehat{\mathcal{G}}_{t+s}(\mu). \qquad (18.13)
\end{aligned}$$

Hence,

$$\mathcal{G}_t * \mathcal{G}_s(x) \underset{\text{(INV)}}{=\!=} \int_{-\infty}^{\infty}\widehat{\mathcal{G}_t * \mathcal{G}_s}(\mu)\cdot\exp(\mathbf{i}\mu x)\ d\mu \underset{\text{(18.13)}}{=\!=} \int_{-\infty}^{\infty}\widehat{\mathcal{G}}_{t+s}(\mu)\cdot\exp(\mathbf{i}\mu x)\ d\mu \underset{\text{(INV)}}{=\!=} \mathcal{G}_{t+s}(x)$$

here, **(INV)** is the Fourier Inversion Formula, and **(18.13)** is by eqn.(18.13).

Solutions to §18.6

1. We already know from Example 17.5 on page 339 that $\widehat{f}(\mu) \ = \ \dfrac{1-e^{-\mu\mathbf{i}}}{2\pi\mu\mathbf{i}}$.

   (a) Proposition 18.17 on page 365 says that $u(x,y) \ = \ \boxed{\dfrac{1}{2\pi\mathbf{i}}\displaystyle\int_{-\infty}^{\infty}\frac{1-e^{-\mu\mathbf{i}}}{\mu}\cdot e^{-|\mu|\cdot y}\cdot\exp\Big(-\mu\mathbf{i}x\Big)\ d\mu.}$

   (b) Proposition 18.1 on page 355 says that

$$u(x,t) \quad = \quad \int_{-\infty}^{\infty}\widehat{f}(\mu)\cdot\exp(\mu x\mathbf{i})\cdot e^{-\mu^2 t}\ d\mu \quad = \quad \boxed{\int_{-\infty}^{\infty}\frac{1-e^{-\mu\mathbf{i}}}{2\pi\mu\mathbf{i}}\exp(\mu x\mathbf{i})\cdot e^{-\mu^2 t}\ d\mu,}$$

   by which, of course, we really mean $u(x,t) \ = \ \lim_{M\to\infty}\displaystyle\int_{-M}^{M}\frac{1-e^{-\mu\mathbf{i}}}{2\pi\mu\mathbf{i}}\exp(\mu x\mathbf{i})\cdot e^{-\mu^2 t}\ d\mu.$

2. In the solution to problem # 3 on page 352 of §17.6, the Fourier transform of $f(x,y)$ is given:

$$\widehat{f}(\mu,\nu) \quad = \quad \frac{\left(1-e^{-\mu X\mathbf{i}}\right)\cdot\left(e^{-\nu Y\mathbf{i}}-1\right)}{4\pi^2\mu\nu}$$

Thus, Proposition 18.4 on page 356 says that the corresponding solution to the two-dimensional Heat equation is:

$$\begin{aligned} u(x,y,t) &= \int_{\mathbb{R}^2} \widehat{f}(\mu,\nu)\cdot\exp\Big((\mu x+\nu y)\cdot\mathbf{i}\Big)\cdot e^{-(\mu^2+\nu^2)t}\ d\mu\ d\nu \\ &= \boxed{\int_{\mathbb{R}^2} \frac{\left(1-e^{-\mu X\mathbf{i}}\right)\cdot\left(e^{-\nu Y\mathbf{i}}-1\right)}{4\pi^2\mu\nu}\cdot\exp\Big((\mu x+\nu y)\cdot\mathbf{i}\Big)\cdot e^{-(\mu^2+\nu^2)t}\ d\mu\ d\nu} \end{aligned}$$

3. Setting $X=Y=1$ in the solution to problem # 3 on page 352 of §17.6, we get the Fourier transform of $f_1(x,y)$:

$$\widehat{f}_1(\mu,\nu) \quad = \quad \frac{\left(1-e^{-\mu\mathbf{i}}\right)\cdot\left(e^{-\nu\mathbf{i}}-1\right)}{4\pi^2\mu\nu}$$

Thus, Proposition 18.11 on page 359 says that the corresponding solution to the two-dimensional wave equation is:

$$\begin{aligned} u(x,y,t) &= \int_{\mathbb{R}^2} \frac{\widehat{f}_1(\mu,\nu)}{\sqrt{\mu^2+\nu^2}}\sin\left(\sqrt{\mu^2+\nu^2}\ \cdot t\right)\cdot\exp\Big((\mu x+\nu y)\cdot\mathbf{i}\Big)\ d\mu\ d\nu \\ &= \boxed{\frac{1}{4\pi^2}\int_{\mathbb{R}^2} \frac{\left(1-e^{-\mu\mathbf{i}}\right)\cdot\left(e^{-\nu\mathbf{i}}-1\right)}{\mu\nu\cdot\sqrt{\mu^2+\nu^2}}\sin\left(\sqrt{\mu^2+\nu^2}\ \cdot t\right)\cdot\exp\Big((\mu x+\nu y)\cdot\mathbf{i}\Big)\ d\mu\ d\nu.} \end{aligned}$$

4. From the solution to problem # 4 on page 353 of §17.6, we know that $\widehat{f}(\mu) = \frac{1}{2\pi\mu^2}\Big((\mu\mathbf{i}+1)\,e^{-\mathbf{i}\mu}-1\Big)$. Thus,

$$u(x,y) \quad = \quad \int_{-\infty}^{\infty}\widehat{f}(\mu)\cdot\exp(\mu x\mathbf{i})e^{-\mu^2t}\ d\mu \quad = \quad \boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\mu\mathbf{i}+1}{\mu^2}e^{-\mathbf{i}\mu}-\frac{1}{\mu^2}\right)\cdot\exp(\mu x\mathbf{i})e^{-\mu^2t}\ d\mu.}$$

5. From the solution to problem # 5 on page 353 of §17.6, we know that $\widehat{f}(\mu) = -\mathbf{i}\mu\widehat{g}(\mu) = \frac{-\mu\mathbf{i}}{\sqrt{2\pi}}\exp\left(\frac{-\mu^2}{2}\right)$.

   (a) Applying Proposition 18.1 on page 355, we have

$$u(x,t) \quad = \quad \int_{-\infty}^{\infty}\widehat{f}(\mu)\cdot\exp(\mu x\mathbf{i})\cdot e^{-\mu^2t}\ d\mu \quad = \quad \boxed{\frac{-\mathbf{i}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mu\exp\left(\frac{-\mu^2}{2}\right)\cdot\exp(\mu x\mathbf{i})\cdot e^{-\mu^2t}\ d\mu.}$$

   (b) Applying Proposition 18.9 on page 359, with $f_1(x)=0$ and $f_0(x)=x\cdot\exp\left(\frac{-x^2}{2}\right)$, we get:

$$u(x,t) \quad = \quad \int_{-\infty}^{\infty}\left(\widehat{f}_0(\mu)\cos(\mu t)+\frac{\widehat{f}_1(\mu)}{\mu}\sin(\mu t)\right)\cdot\exp(\mu x\mathbf{i})\ d\mu \quad = \quad \boxed{\frac{-\mathbf{i}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mu\cdot\exp\left(\frac{-\mu^2}{2}\right)\cos(\mu t)\cdot\exp(\mu x\mathbf{i})\ d\mu.}$$

6. From the solution to problem # 8 on page 353 of §17.6, we know that $\widehat{f}(\mu) = \frac{-\mathbf{i}\mu}{2}\exp(-|\mu|)$.

   (a) Applying Proposition 18.1 on page 355, we have:

$$u(x,t) \quad = \quad \int_{-\infty}^{\infty}\widehat{f}(\mu)\cdot\exp(\mu x\mathbf{i})\cdot e^{-\mu^2t}\ d\mu \quad = \quad \boxed{\frac{-\mathbf{i}}{2}\int_{-\infty}^{\infty}\mu\exp(-|\mu|)\cdot\exp(\mu x\mathbf{i})\cdot e^{-\mu^2t}\ d\mu.}$$

   (b) By Proposition 18.9 on page 359, with $f_0(x)=0$ and $f_1(x)=\frac{-2x}{(1+x^2)^2}$, we get:

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty}\left(\widehat{f}_0(\mu)\cos(\mu t)+\frac{\widehat{f}_1(\mu)}{\mu}\sin(\mu t)\right)\cdot\exp(\mu x\mathbf{i})\ d\mu \\ &= \frac{-\mathbf{i}}{2}\int_{-\infty}^{\infty}\frac{\mu\exp(-|\mu|)}{\mu}\cdot\sin(\mu t)\cdot\exp(\mu x\mathbf{i})\ d\mu \quad = \quad \boxed{\frac{-\mathbf{i}}{2}\int_{-\infty}^{\infty}\exp(-|\mu|)\cdot\sin(\mu t)\cdot\exp(\mu x\mathbf{i})\ d\mu.} \end{aligned}$$

7. From the solution to problem # 9 on page 353 of §17.6, we know that $\widehat{f}(\mu) = \frac{e^{-\frac{1}{2}\mathbf{i}\mu}}{\pi\mu} \cdot \sin\left(\frac{9}{2}\right)$. Thus, Proposition 18.1 on page 355 says that

$$u(x;t) = \int_{-\infty}^{\infty} \widehat{f}(\mu) \cdot \exp(\mathbf{i}\mu x) \cdot e^{-\mu^2 t}\, d\mu = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-\mathbf{i}\mu/2}}{\mu} \cdot \sin\left(\frac{9}{2}\right) \cdot \exp(\mathbf{i}\mu x) \cdot e^{-\mu^2 t}\, d\mu$$
$$= \boxed{\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{\mu} \sin\left(\frac{9}{2}\right) \cdot \exp\left(\mathbf{i}\mu\left(x - \frac{1}{2}\right) - \mu^2 t\right)\, d\mu.}$$

8. By the solution to problem # 10 on page 353 of §17.6, we know that $\widehat{f}(\mu) = \begin{cases} \frac{\mathbf{i}\mu}{2} & \text{if} \quad -1 < \mu < 1 \\ 0 & \text{otherwise} \end{cases}$.

Thus, Proposition 18.1 on page 355 says that $u(x;t) = \int_{-\infty}^{\infty} \widehat{f}(\mu) \cdot \exp(\mathbf{i}\mu x) \cdot e^{-\mu^2 t}\, d\mu = \boxed{\frac{\mathbf{i}}{2}\int_{-1}^{1} \mu \exp(\mathbf{i}\mu x) \cdot e^{-\mu^2 t}\, d\mu.}$

9. (a) $u(x,t) = \int_{-\infty}^{\infty} \widehat{f}(\mu) \cdot \exp(\mathbf{i}\mu x) \cdot e^{-\mu^2 t}\, d\mu = \boxed{\int_{-\infty}^{\infty} \frac{\mu}{\mu^4+1} \cdot \exp(\mathbf{i}\mu x) \cdot e^{-\mu^2 t}\, d\mu.}$

(b) $u(x,t) = \int_{-\infty}^{\infty} \frac{\widehat{f}(\mu)}{\mu} \cdot \exp(\mathbf{i}\mu x) \cdot \sin(\mu t)\, d\mu = \boxed{\int_{-\infty}^{\infty} \frac{\exp(\mathbf{i}\mu x) \cdot \sin(\mu t)}{\mu^4+1}\, d\mu.}$

(c) $u(x,y) = \int_{-\infty}^{\infty} \widehat{f}(\mu) \cdot e^{-y|\mu|} \cdot \exp(\mathbf{i}\mu x)\, d\mu = \boxed{\int_{-\infty}^{\infty} \frac{\mu}{\mu^4+1} \cdot e^{-y|\mu|} \cdot \exp(\mathbf{i}\mu x)\, d\mu.}$

(d) First the Laplace equation. Note that, for any fixed $\mu \in \mathbb{R}$,

$$\begin{aligned}
\triangle\left(e^{-y|\mu|} \cdot \exp(\mathbf{i}\mu x)\right) &= \partial_x^2 \left(e^{-y|\mu|} \cdot \exp(\mathbf{i}\mu x)\right) + \partial_y^2 \left(e^{-y|\mu|} \cdot \exp(\mathbf{i}\mu x)\right) \\
&= \mathbf{i}\mu\partial_x \left(e^{-y|\mu|} \cdot \exp(\mathbf{i}\mu x)\right) + -|\mu| \cdot \partial_y \left(e^{-y|\mu|} \cdot \exp(\mathbf{i}\mu x)\right) \\
&= (\mathbf{i}\mu)^2 \cdot \left(e^{-y|\mu|} \cdot \exp(\mathbf{i}\mu x)\right) + |\mu|^2 \cdot \left(e^{-y|\mu|} \cdot \exp(\mathbf{i}\mu x)\right) \\
&= (-\mu^2 + \mu^2) \cdot \left(e^{-y|\mu|} \cdot \exp(\mathbf{i}\mu x)\right) = 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
\triangle u(x,y) &= \triangle \int_{-\infty}^{\infty} \frac{\mu}{\mu^4+1} \cdot e^{-y|\mu|} \cdot \exp(\mathbf{i}\mu x)\, d\mu. \\
&= \int_{-\infty}^{\infty} \frac{\mu}{\mu^4+1} \cdot \triangle\left(e^{-y|\mu|} \cdot \exp(\mathbf{i}\mu x)\right)\, d\mu = \int_{-\infty}^{\infty} \frac{\mu}{\mu^4+1} \cdot (0)\, d\mu. = 0,
\end{aligned}$$

as desired. Now the boundary conditions. Setting $y = 0$, we get:

$$u(x,0) = \int_{-\infty}^{\infty} \frac{\mu}{\mu^4+1} \cdot e^{-(0)|\mu|} \cdot \exp(\mathbf{i}\mu x)\, d\mu. = \int_{-\infty}^{\infty} \frac{\mu}{\mu^4+1} \cdot \exp(\mathbf{i}\mu x)\, d\mu. \underset{(*)}{=\!=} f(x),$$

where $(*)$ is by Fourier Inversion.

10. [omitted]

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Index

Notation

Sets:

$\mathbb{B}^2(0;1)$**:** The 2-dimensional **unit disk**; the set of all $(x, y) \in \mathbb{R}^2$ so that $x^2 + y^2 \leq 1$.

$\mathbb{B}^D(\mathbf{x}; R)$**:** The D-dimensional **ball**; of radius R around the point $\mathbf{x}$; the set of all $\mathbf{y} \in \mathbb{R}^D$ so that $\|\mathbf{x} - \mathbf{y}\| < R$.

$\mathbb{C}$**:** The set of **complex numbers** of the form $x + y\mathbf{i}$, where $x, y \in \mathbb{R}$, and $\mathbf{i}$ is the square root of -1.

$\mathbb{N}$**:** The **natural numbers** $\{0, 1, 2, 3, \ldots\}$.

$\mathbb{N}^D$**:** The set of all $\mathbf{n} = (n_1, n_2, \ldots, n_D)$, where $n_1, \ldots, n_D$ are natural numbers.

$\mathbb{R}$**:** The set of **real numbers** (eg. 2, -3, $\sqrt{7} + \pi$, etc.)

$\mathbb{R}^2$**:** The 2-dimensional infinite plane —the set of all ordered pairs (x, y), where $x, y \in \mathbb{R}$.

$\mathbb{R}^D$**:** D-dimensional space —the set of all D**-tuples** $(x_1, x_2, \ldots, x_D)$, where $x_1, x_2, \ldots, x_D \in \mathbb{R}$. Sometimes we will treat these D-tuples as **points** (representing locations in physical space); normally points will be indicated in **bold face**, eg: $\mathbf{x} = (x_1, \ldots, x_D)$. Sometimes we will treat the D-tuples as **vectors** (pointing in a particular direction); then they will be indicated with arrows, eg: $\vec{v} = (v_1, v_2, \ldots, v_D)$.

$\mathbb{R}^D \times \mathbb{R}$**:** The set of all pairs $(\mathbf{x}; t)$, where $\mathbf{x} \in \mathbb{R}^D$ is a vector, and $t \in \mathbb{R}$ is a number. (Of course, mathematically, this is the same as $\mathbb{R}^{D+1}$, but sometimes it is useful to distinguish the extra dimension as "time" or whatever.)

$\mathbb{R} \times [0, \infty)$**:** The **half-space** of all points $(x, y) \in \mathbb{R}^2$, where $y \geq 0$.

$\mathbb{S}^1(0;1)$**:** The 2-dimensional **unit circle**; the set of all $(x, y) \in \mathbb{R}^2$ so that $x^2 + y^2 = 1$.

$\mathbb{S}^{D-1}(\mathbf{x}; R)$**:** The D-dimensional **sphere**; of radius R around the point $\mathbf{x}$; the set of all $\mathbf{y} \in \mathbb{R}^D$ so that $\|\mathbf{x} - \mathbf{y}\| = R$

$\mathbb{T}^1$**:** The 1-dimensional **torus**; the interval $[0, 1]$ with the points 0 and 1 "glued together". Looks like the circle $\mathbb{S}^1$.

$\mathbb{T}^2$**:** The 2-dimensional **torus**; the square $[0, 1] \times [0, 1]$ with the top edge "glued" to the bottom edge, and the right edge glued to the left. Looks like a donut.

$\mathbb{T}^D$**:** The D-dimensional **torus**; the cube $[0, 1]^D$ with opposite faces "glued together" in every dimension.

$\mathbb{Z}$**:** The **integers** $\{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$.

$\mathbb{Z}^D$**:** The set of all $\mathbf{n} = (n_1, n_2, \ldots, n_D)$, where $n_1, \ldots, n_D$ are integers.

$[1...D] \;= \{1, 2, 3, \ldots, D\}$.

$[0,1]$**:** The (closed) **unit interval**; the set of all real numbers x where $0 \leq x \leq 1$.

$[0,1]^2$**:** The (closed) **unit square**; the set of all points $(x,y) \in \mathbb{R}^2$ where $0 \leq x, y \leq 1$.

$[0,1]^D$**:** The D-dimensional **unit cube**; the set of all points $(x_1, \ldots, x_D) \in \mathbb{R}^D$ where $0 \leq x_d \leq 1$ for all $d \in [1...D]$.

$[-L,L]$**:** The **interval** of all real numbers x with $-L \leq X \leq L$.

$[-L,L]^D$**:** The D-dimensional **cube** of all points $(x_1, \ldots, x_D) \in \mathbb{R}^D$ where $-L \leq x_d \leq L$ for all $d \in [1...D]$.

$[0,\infty)$**:** The set of all real numbers $x \geq 0$.

Spaces of Functions:

$\mathcal{C}^\infty$**:** A vector space of (infinitely) differentiable functions. Some examples:

- $\mathcal{C}^\infty[\mathbb{R}^2;\ \mathbb{R}]$: The space of differentiable scalar fields on the plane.
- $\mathcal{C}^\infty[\mathbb{R}^D;\ \mathbb{R}]$: The space of differentiable scalar fields on D-dimensional space.
- $\mathcal{C}^\infty[\mathbb{R}^2;\ \mathbb{R}^2]$: The space of differentiable vector fields on the plane.
- $\mathcal{C}^\infty[\mathbb{R}^D;\ \mathbb{R}^D]$: The space of differentiable vector fields on D-dimensional space.

$\mathcal{C}_0^\infty[0,1]^D$**:** The space of differentiable scalar fields on the cube $[0,1]^D$ satisfying **Dirichlet boundary conditions**: $f(\mathbf{x}) = 0$ for all $\mathbf{x} \in \partial[0,1]^D$.

$\mathcal{C}_\perp^\infty[0,1]^D$**:** The space of differentiable scalar fields on the cube $[0,1]^D$ satisfying **Neumann boundary conditions**: $\partial_\perp f(\mathbf{x}) = 0$ for all $\mathbf{x} \in \partial[0,1]^D$.

$\mathcal{C}_h^\infty[0,1]^D$**:** The space of differentiable scalar fields on the cube $[0,1]^D$ satisfying **mixed boundary conditions**: $\frac{\partial_\perp f}{f}(\mathbf{x}) = h(\mathbf{x})$ for all $\mathbf{x} \in \partial[0,1]^D$.

$\mathcal{C}_{\mathrm{per}}^\infty[-\pi,\pi]$**:** The space of differentiable scalar fields on the interval $[-\pi,\pi]$ satisfying **periodic boundary conditions**.

$\mathbf{L}^1(\mathbb{R})$: The set of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\int_{-\infty}^{\infty} |f(x)|\ dx\ <\ \infty$.

$\mathbf{L}^1(\mathbb{R}^2)$: The set of all functions $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x,y)|\ dx\ dy\ <\ \infty$.

$\mathbf{L}^1(\mathbb{R}^3)$: The set of all functions $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that $\int_{\mathbb{R}^3} |f(\mathbf{x})|\ d\mathbf{x}\ <\ \infty$.

$\mathbf{L}^2(\mathbb{X})$: The set of all functions $f : \mathbb{X} \longrightarrow \mathbb{R}$ so that $\|f\|_2 = \left(\int_{\mathbb{X}} |f(\mathbf{x})|^2\ d\mathbf{x}\right)^{1/2} < \infty$.

$\mathbf{L}^2(\mathbb{X};\mathbb{C})$: The set of all functions $f : \mathbb{X} \longrightarrow \mathbb{C}$ so that $\|f\|_2 = \left(\int_{\mathbb{X}} |f(\mathbf{x})|^2\ d\mathbf{x}\right)^{1/2} < \infty$.

Derivatives and Boundaries:

$\partial_k f = \frac{df}{dx_k}$.

$\nabla f = (\partial_1 f,\ \partial_2 f, \ldots, \partial_D f)$, the **gradient** of scalar field f.

$\mathbf{div}\, f = \partial_1 f_1 + \partial_2 f_2 + \ldots + \partial_D f_D$, the **divergence** of vector field f.

$\partial_\perp f$ is the derivative of f *normal* to the boundary of some region. Sometimes this is written as $\frac{\partial f}{\partial \mathbf{n}}$ or $\frac{\partial f}{\partial \nu}$, or as $\nabla f \cdot \mathbf{n}$.

$\triangle f = \partial_1^2 f + \partial_2^2 f + \ldots + \partial_D^2 f$. Sometimes this is written as $\nabla^2 f$.

$\mathsf{SL}_{s,q}(\phi) = s \cdot \partial^2 \phi + s' \cdot \partial \phi + q \cdot \phi$. Here, $s, q : [0, L] \longrightarrow \mathbb{R}$ are predetermined functions, and $\phi : [0, L] \longrightarrow \mathbb{R}$ is the function we are operating on by the **Sturm-Liouville operator** $\mathsf{SL}_{s,q}$.

$\mathsf{SL}_{s,q}(\phi) = \dfrac{s}{\rho} \cdot \partial^2 \phi + \dfrac{s'}{\rho} \cdot \partial \phi + \dfrac{q}{\rho} \cdot \phi$. Here, $s, q : [0, L] \longrightarrow \mathbb{R}$ are predetermined functions, $\rho : [0, L] \longrightarrow [0, \infty)$ is some weight function. and $\phi : [0, L] \longrightarrow \mathbb{R}$ is the function we are operating on by the **ρ-weighted Sturm-Liouville operator** $\mathsf{SL}_{s,q}$.

$\partial\, \mathbb{X}$**:** If $\mathbb{X} \subset \mathbb{R}^D$ is some region in space, then $\partial \mathbb{X}$ is the **boundary** of that region. For example:

- $\partial\, [0, 1] = \{0,\ 1\}$.
- $\partial\, \mathbb{B}^2\,(0; 1) = \mathbb{S}^2(0; 1)$.
- $\partial\, \mathbb{B}^D\,(\mathbf{x}; R) = \mathbb{S}^D(\mathbf{x}; R)$.
- $\partial\, (\mathbb{R} \times [0, \infty)) = \mathbb{R} \times \{0\}$.

Norms and Inner products:

$\|\mathbf{x}\|$**:** If $\mathbf{x} \in \mathbb{R}^D$ is a vector, then $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_D^2}$ is the **norm** (or **length**) of $\mathbf{x}$.

$\|f\|_2$**:** Let $\mathbb{X} \subset \mathbb{R}^D$ be a bounded domain, with volume $M = \int_{\mathbb{X}} 1 d\mathbf{x}$. If $f : \mathbb{X} \longrightarrow \mathbb{R}$ is an integrable function, then $\|f\|_2 = \dfrac{1}{M} \left(\int_{\mathbb{X}} |f(\mathbf{x})|\ d\mathbf{x} \right)^{1/2}$ is the **$\mathbf{L}^2$-norm** of f.

$\langle f, g \rangle$**:** If $f, g : \mathbb{X} \longrightarrow \mathbb{R}$ are integrable functions, then their **inner product** is given by: $\langle f, g \rangle = \dfrac{1}{M} \displaystyle\int_{\mathbb{X}} f(\mathbf{x}) \cdot g(\mathbf{x})\ d\mathbf{x}$.

$\langle f, g \rangle_\rho$**:** If $\rho : \mathbb{X} \longrightarrow [0, \infty)$ is some **weight function**, and $f, g : \mathbb{X} \longrightarrow \mathbb{R}$ are two other integrable functions, then their **ρ-weighted inner product** is given by: $\langle f, g \rangle_\rho = \displaystyle\int_{\mathbb{X}} f(\mathbf{x}) \cdot g(\mathbf{x}) \cdot \rho(\mathbf{x})\ d\mathbf{x}$.

$\|f\|_1$**:** Let $\mathbb{X} \subseteq \mathbb{R}^D$ be any domain. If $f : \mathbb{X} \longrightarrow \mathbb{R}$ is an integrable function, then $\|f\|_\infty = \displaystyle\int_{\mathbb{X}} |f(\mathbf{x})|\ d\mathbf{x}$ is the **$\mathbf{L}^1$-norm** of f.

$\|f\|_\infty$**:** Let $\mathbb{X} \subseteq \mathbb{R}^D$ be any domain. If $f : \mathbb{X} \longrightarrow \mathbb{R}$ is a bounded function, then $\|f\|_\infty = \sup_{\mathbf{x}\in\mathbb{X}} |f(\mathbf{x})|$ is the $\mathbf{L}^\infty$**-norm** of f.

Other Operations:

$f * g$**:** If $f, g : \mathbb{R}^D \longrightarrow \mathbb{R}$, then their **convolution** is the function $f * g : \mathbb{R}^D \longrightarrow \mathbb{R}$ defined by $(f * g)(\mathbf{x}) \;=\; \int_{\mathbb{R}^D} f(\mathbf{y}) \cdot g(\mathbf{x} - \mathbf{y}) \; d\mathbf{y}$.

$\mathbf{M}_R \, u(\mathbf{x}) \;=\; \frac{1}{4\pi R^2} \int_{\mathbb{S}(R)} f(\mathbf{x} + \mathbf{s}) \; d\mathbf{s}$ is the **spherical average** of f at $\mathbf{x}$, of radius R. Here, $\mathbf{x} \in \mathbb{R}^3$ is a point in space, $R > 0$, and $\mathbb{S}(R) = \left\{\mathbf{s} \in \mathbb{R}^3 \;;\; \|\mathbf{s}\| = R\right\}$.

Trig Functions:

Pythagorean Identities:

$$\sin^2(\theta)+\cos^2(\theta) = 1 \qquad 1+\tan^2(\theta) = \sec^2(\theta) \qquad 1+\cot^2(\theta) = \csc^2(\theta)$$

Two-angle Formulae:

$$\sin(x+y) = \sin(x)\cos(y)+\cos(x)\sin(y) \qquad \sin(x-y) = \sin(x)\cos(y)-\cos(x)\sin(y)$$

$$\cos(x+y) = \cos(x)\cos(y)-\sin(x)\sin(y) \qquad \cos(x-y) = \cos(x)\cos(y)+\sin(x)\sin(y)$$

$$2\cos(x)\cos(y) = \cos(x+y)+\cos(x-y) \qquad 2\sin(x)\sin(y) = \cos(x-y)-\cos(x+y)$$

$$2\sin(x)\cos(y) = \sin(x+y)+\sin(x-y)$$

$$\sin(x)+\sin(y) = 2\sin\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right) \qquad \sin(x)-\sin(y) = 2\cos\left(\tfrac{x+y}{2}\right)\sin\left(\tfrac{x-y}{2}\right)$$

$$\cos(x)+\cos(y) = 2\cos\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right) \qquad \cos(x)-\cos(y) = 2\sin\left(\tfrac{x+y}{2}\right)\sin\left(\tfrac{y-x}{2}\right)$$

Double-angle Formulae:

$$\sin(2x) = 2\sin(x)\cos(x) \qquad \cos(2x) = \cos^2(x)-\sin^2(x) = 2\cos^2(x)-1 = 1-2\sin^2(x)$$

Multiples of π:

$$\sin(n\pi) = 0 \qquad \cos(n\pi) = (-1)^n$$

$$\sin\left(\tfrac{n\pi}{2}\right) = \begin{cases} (-1)^k & \text{if } n=2k+1 \text{ is odd} \\ 0 & \text{if } n \text{ is even}\end{cases} \qquad \cos\left(\tfrac{n\pi}{2}\right) = \begin{cases} (-1)^k & \text{if } n=2k \text{ is even} \\ 0 & \text{if } n \text{ is odd}\end{cases}$$

Odd vs. Even:

$$\sin(-x) = -\sin(x) \qquad \cos(-x) = \cos(x) \qquad \tan(-x) = -\tan(x)$$

Phase shifting:

$$\sin(x+\tfrac{\pi}{2}) = -\cos(x) \qquad \cos(x+\tfrac{\pi}{2}) = \sin(x) \qquad \tan(x+\tfrac{\pi}{2}) = -\cot(x)$$

(anti)Derivatives:

$$\sin'(x) = \cos(x) \qquad \cos'(x) = -\sin(x) \qquad \tan'(x) = \sec^2(x)$$

$$\csc'(x) = cot(x)\csc(x) \qquad \sec'(x) = -\tan(x)\sec(x) \qquad \cot'(x) = \csc^2(x)$$

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}} \qquad \arccos'(x) = \frac{-1}{\sqrt{1-x^2}} \qquad \arctan'(x) = \frac{1}{1+x^2}$$

$$\operatorname{arc\,csc}'(x) = \frac{-1}{x\sqrt{x^2-1}} \qquad \operatorname{arc\,sec}'(x) = \frac{1}{x\sqrt{x^2-1}} \qquad \operatorname{arc\,cot}'(x) = \frac{-1}{1+x^2}$$

$$\int \sec(x)\ dx = \log\left|\sec(x)+\tan(x)\right|+c \qquad \int \csc(x)\ dx = -\log\left|\csc(x)+\cot(x)\right|+c$$

Hyperbolic Trig Functions:

Pythagorean Identities:

$$\cosh^2(\theta)-\sinh^2(\theta) = 1 \qquad 1-\tanh^2(\theta) = \operatorname{sech}^2(\theta) \qquad \coth^2(\theta)-1 = \operatorname{csch}^2(\theta)$$

Two-angle Formulae:

$$\sinh(x+y) = \sinh(x)\cosh(y)+\cosh(x)\sinh(y) \qquad \sinh(x-y) = \sinh(x)\cosh(y)-\cosh(x)\sinh(y)$$

$$\cosh(x+y) = \cosh(x)\cosh(y)+\sinh(x)\sinh(y) \qquad \cosh(x-y) = \cosh(x)\cosh(y)-\sinh(x)\sinh(y)$$

$$2\cosh(x)\cosh(y) = \cosh(x+y)+\cosh(x-y) \qquad 2\sinh(x)\sinh(y) = \cosh(x+y)-\cosh(x-y)$$

$$2\sinh(x)\cosh(y) = \sinh(x+y)+\sinh(x-y)$$

$$\sinh(x)+\sinh(y) = 2\sinh\left(\tfrac{x+y}{2}\right)\cosh\left(\tfrac{x-y}{2}\right) \qquad \sinh(x)-\sinh(y) = 2\cosh\left(\tfrac{x+y}{2}\right)\sinh\left(\tfrac{x-y}{2}\right)$$

$$\cosh(x)+\cosh(y) = 2\cosh\left(\tfrac{x+y}{2}\right)\cosh\left(\tfrac{x-y}{2}\right) \qquad \cosh(x)-\cosh(y) = 2\sinh\left(\tfrac{x+y}{2}\right)\sinh\left(\tfrac{y-x}{2}\right)$$

Double-angle Formulae:

$$\sinh(2x) = 2\sinh(x)\cosh(x) \qquad \cosh(2x) = \cosh^2(x)+\sinh^2(x)$$

Odd vs. Even:

$$\sinh(-x) = -\sinh(x) \qquad \cosh(-x) = \cosh(x) \qquad \tanh(-x) = -\tanh(x)$$

(anti)Derivatives:

$$\sinh'(x) = \cosh(x) \qquad \cosh'(x) = \sinh(x) \qquad \tanh'(x) = \operatorname{sech}^2(x)$$

$$\operatorname{csch}'(x) = -\coth(x)\operatorname{csch}(x) \qquad \operatorname{sech}'(x) = -\tanh(x)\operatorname{sech}(x) \qquad \coth'(x) = -\operatorname{csch}^2(x)$$

$$\operatorname{arcsinh}'(x) = \frac{1}{\sqrt{x^2+1}} \qquad \operatorname{arccosh}'(x) = \frac{-1}{\sqrt{x^2-1}} \qquad \operatorname{arctanh}'(x) = \frac{1}{1-x^2}$$

Exponential Functions (Euler's & de Moivre's Formulae):

$$\exp(x+y\mathbf{i}) = e^x\left(\cos(y)+\mathbf{i}\sin(y)\right) \qquad \sin(x) = \frac{e^{x\mathbf{i}}-e^{-x\mathbf{i}}}{2\mathbf{i}} \qquad \cos(x) = \frac{e^{-x\mathbf{i}}+e^{x\mathbf{i}}}{2}$$

$$[\cos(x)+\mathbf{i}\sin(x)]^n = \cos(nx)+\mathbf{i}\sin(nx) \qquad \sinh(x) = \frac{e^x-e^{-x}}{2} \qquad \cosh(x) = \frac{e^x+e^{-x}}{2}$$

Coordinate systems:

Polar Coordinates:

$$x = r\cdot\cos(\theta) \qquad y = r\cdot\sin(\theta)$$

$$r = \sqrt{x^2+y^2} \qquad \theta = \arctan(y/x) \qquad \Delta = \partial_r^2 + \tfrac{1}{r}\partial_r + \tfrac{1}{r^2}\partial_\theta^2$$

Cylindrical Coordinates:

$$x = r\cdot\cos(\theta) \qquad y = r\cdot\sin(\theta) \qquad z = z$$

$$r = \sqrt{x^2+y^2} \qquad \theta = \arctan(y/x) \qquad z = z$$

$$\Delta = \partial_z^2 + \partial_r^2 + \tfrac{1}{r}\partial_r + \tfrac{1}{r^2}\partial_\theta$$

Spherical Coordinates:

$$x = r\cdot\sin(\varphi)\cos(\theta) \qquad y = r\cdot\sin(\varphi)\sin(\theta) \qquad z = r\cdot\cos(\varphi)$$

$$r = \sqrt{x^2+y^2+z^2} \qquad \theta = \arctan(y/x) \qquad \varphi = \arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)$$

$$\triangle = \partial_r^2 + \tfrac{2}{r}\partial_r + \frac{1}{r^2\sin(\phi)}\partial_\phi^2 + \frac{\cot(\phi)}{r^2}\partial_\phi + \frac{1}{r^2\sin(\phi)^2}\partial_\theta^2.$$

Fourier Transforms:

$$\widehat{f}(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\exp(-\mathbf{i}\mu x)\ dx \qquad \mathcal{G}_t(x) = \frac{1}{2\sqrt{\pi t}}\exp\left(\frac{-x^2}{4t}\right) \qquad \widehat{f*g}(\mu) = 2\pi\cdot\widehat{f}(\mu)\cdot\widehat{g}(\mu) \qquad \widehat{(f')}(\mu) = (\mathbf{i}\mu)\cdot\widehat{f}(\mu)$$

$$f(x) = \int_{-\infty}^{\infty}\widehat{f}(\mu)\exp(\mathbf{i}\mu x)\ d\mu \qquad \widehat{\mathcal{G}}_t(\mu) = \frac{1}{2\pi}e^{-\mu^2 t} \qquad \widehat{f\cdot g}(\mu) = (\widehat{f}*\widehat{g})(\mu) \qquad \widehat{x\cdot f(x)}(\mu) = \mathbf{i}\cdot(\widehat{f})'(\mu)$$

Fourier Series:

$$A_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx \qquad A_{nm} = \frac{1/2/4}{\pi^2}\int_0^\pi\int_0^\pi f(x,y)\cos(nx)\cos(my)\,dx\,dy$$

$$f(x) = \sum_{n=0}^\infty A_n\cos(nx). \qquad f(x,y) = \sum_{n,m=0}^\infty A_{nm}\cos(nx)\cos(my).$$

$$B_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx \qquad B_{nm} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x,y)\sin(nx)\sin(my)\,dx\,dy$$

$$f(x) = \sum_{n=1}^\infty B_n\sin(nx) \qquad f(x,y) = \sum_{n,m=1}^\infty B_{nm}\sin(nx)\sin(my).$$

Important Ordinary Differential Equations:

mth order Bessel Equation: $r^2\mathcal{R}''(r) + r\cdot\mathcal{R}'(r) + (\lambda^2r^2 - m^2)\cdot\mathcal{R}(r) = 0$

Legendre Equation: $(1-x^2)\mathcal{L}''(x) - 2x\mathcal{L}'(x) + \mu\mathcal{L}(x) = 0$

Cauchy-Euler Equation: $r^2\mathcal{R}''(r) + 2r\cdot\mathcal{R}'(r) - \mu\cdot\mathcal{R}(r) = 0$

Special Functions:

mth order Bessel Function: $\mathcal{J}_m(x) = \left(\frac{x}{2}\right)^m\cdot\sum_{k=0}^\infty \frac{(-1)^k}{2^{2k}\,k!\,(m+k)!}x^{2k}$

Legendre Polynomial: $\mathcal{P}_n(x) = \frac{1}{n!\,2^n}\partial_x^n\left[(x^2-1)\right].$

Green's Functions and Solution Kernels:

One-dimensional Gauss-Weierstrass kernel: $\mathcal{G}(x;t) = \frac{1}{2\sqrt{\pi t}}\exp\left(\frac{-x^2}{4t}\right)$ for all $x\in\mathbb{R}$ and $t>0$

D-dimensional Gauss-Weierstrass kernel: $\mathcal{G}(\mathbf{x};t) = \frac{1}{(4\pi t)^{D/2}}\exp\left(\frac{-\|\mathbf{x}\|^2}{4t}\right)$ for all $\mathbf{x}\in\mathbb{R}^D$ and $t>0$

Half-plane Poisson Kernel: $\mathcal{K}_y(x) = \frac{y}{\pi(x^2+y^2)}$ for all $x\in\mathbb{R}$ and $y>0$.

Disk Poisson Kernel: $\mathcal{P}(\mathbf{x},\mathbf{s}) = \frac{R^2-\|\mathbf{x}\|^2}{\|\mathbf{x}-\mathbf{s}\|^2},$ for all $\mathbf{x}\in\mathbb{D}$ and $\mathbf{s}\in\partial\mathbb{D}$.

Disk Poisson Kernel (polar form): $\mathcal{P}_\sigma(x,y) = \frac{R^2-x^2-y^2}{(x-R\cos(\sigma))^2 + (y-R\sin(\sigma))^2}$ for all $(x,y)\in\mathbb{D}$ and $\sigma\in[-\pi,\pi)$.